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CONTENTS

IDENTITIES OF THE ROGERS-RAMANUJAN TYPE

By W. N. Bailey

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1. In a recent paper* I gave a method of obtaining transformations of basic hypergeometric series which gave the most general known transformations of such series as well as a new transformation of a nearly poised basic series. The method also led to new identities of the Rogers-Ramanujan type. This method, combined with a further study of Rogers's work, has led me to a considerable simplification in the ideas underlying the methods of finding transformations of hypergeometric series, and also identities of the Rogers-Ramanujan type, which are essentially limiting cases of transformations of basic series. The simple fundamental result is:

If
$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r},$$
and
$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n},$$
then
$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n.$$

The proof is almost trivial. In fact,

$$\begin{aligned} \sum_{n=0}^{\infty} \alpha_n \gamma_n &= \sum_{n=0}^{\infty} \sum_{r=n}^{\infty} \alpha_n \delta_r u_{r-n} v_{r+n} \\ &= \sum_{r=0}^{\infty} \sum_{n=0}^{r} \delta_r \alpha_n u_{r-n} v_{r+n} \\ &= \sum_{r=0}^{\infty} \delta_r \beta_r. \end{aligned}$$

We assume, of course, that the series converge and that the change in the order of summation is allowable. In all the cases given in this paper the justification would be simple.

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* Bailey (2).

The above result could be used to work out all the most general transformations of generalized hypergeometric series of the ordinary type. In fact, the method of finding transformations of terminating series, given in chapter IV of my tract,* is really a particular case of the above result, the numbers α_n, u_n, v_n, δ_n being chosen so that the series for β_n and γ_n can be summed by known (and comparatively elementary) results. Similarly, the method given in my paper(**2**) for finding transformations of basic series is really a particular case of the above result, the numbers α_n, u_n, v_n, δ_n being chosen so that the series for β_n and γ_n can be summed by known sums of basic hypergeometric series. In the case of series of the ordinary type it was only necessary to assume the theorems of Saalschütz and Dixon for the sums of certain series ${}_3F_2$, but in working with basic series I assumed immediately a knowledge of Jackson's basic analogue of Dougall's theorem. It may not be necessary to assume so much, but the proof of Jackson's identity is quite elementary, and it hardly seemed to be worth while to avoid assuming it, particularly as there is no exact counterpart of Dixon's theorem for basic series, and the analogue of Kummer's theorem is not nearly so well known as Kummer's theorem itself.

2. The form in which we use the result of § 1 is:

If
$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(x)_{n-r}(ax)_{n+r}},$$
and
$$\gamma_n = \sum_{r=n}^{\infty} \frac{\delta_r}{(x)_{r-n}(ax)_{r+n}},$$
then
$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n.$$

Here I use the notation of my previous paper,
$$(a)_n = (1-a)(1-ax)(1-ax^2)\ldots(1-ax^{n-1}),$$
and later I use the abbreviations
$$x_n = 1-x^n, \quad \bar{x}_n = 1+x^n, \quad x_n! = (x)_n, \quad x_n!! = x_1 x_3 x_5 \ldots x_{2n-1},$$
$$x_n^r = x_{rn}, \quad x_n^r! = x_1^r x_2^r \ldots x_n^r, \quad x_n^r!! = x_1^r x_3^r \ldots x_{2n-1}^r,$$
$$[a]_n = (1-a)(1-ax^2)\ldots(1-ax^{2n-2}), \quad \{a\}_n = (1-a)(1-ax^3)\ldots(1-ax^{3n-3}).$$

3. The formula for γ_n can be written as
$$\gamma_n = \sum_{s=0}^{\infty} \frac{\delta_{n+s}}{(x)_s (ax)_{2n+s}}.$$

* Bailey (1).

Taking $\delta_r = (\rho_1)_r(\rho_2)_r(ax/\rho_1\rho_2)^r$, we can sum the series for γ_n by the basic analogue of Gauss's theorem, and we find that

$$\gamma_n = \frac{(\rho_1)_n(\rho_2)_n}{(ax)_{2n}}\left(\frac{ax}{\rho_1\rho_2}\right)^n {}_2\Phi_1\left[\begin{matrix}\rho_1 x^n,\ \rho_2 x^n; & \dfrac{ax}{\rho_1\rho_2}\\ ax^{2n+1} & \end{matrix}\right]$$
$$= \frac{(\rho_1)_n(\rho_2)_n}{(ax/\rho_1)_n(ax/\rho_2)_n}\left(\frac{ax}{\rho_1\rho_2}\right)^n \prod_{m=1}^{\infty}\left[\frac{(1-ax^m/\rho_1)(1-ax^m/\rho_2)}{(1-ax^m)(1-ax^m/\rho_1\rho_2)}\right].$$

With these values of γ_n and δ_n, the result of § 2 gives

$$\sum_{n=0}^{\infty}(\rho_1)_n(\rho_2)_n(ax/\rho_1\rho_2)^n\beta_n$$
$$= \prod_{m=1}^{\infty}\left[\frac{(1-ax^m/\rho_1)(1-ax^m/\rho_2)}{(1-ax^m)(1-ax^m/\rho_1\rho_2)}\right]\sum_{n=0}^{\infty}\frac{(\rho_1)_n(\rho_2)_n}{(ax/\rho_1)_n(ax/\rho_2)_n}\left(\frac{ax}{\rho_1\rho_2}\right)^n\alpha_n. \quad (3\cdot1)$$

This formula has an interesting connexion with Rogers's work. Following Rogers, we define $A_n(\theta)$ by the identity

$$1+\sum_{n=1}^{\infty}\frac{A_n(\theta)}{x_n!}z^n = \prod_{n=0}^{\infty}(1-2zx^n\cos\theta+z^2x^{2n})^{-1},$$

and then

$$A_{2n}(\theta) = \frac{x_{2n}!}{(x_n!)^2}\left(1+\frac{x_n}{x_{n+1}}2\cos 2\theta+\frac{x_n x_{n-1}}{x_{n+1}x_{n+2}}2\cos 4\theta+\ldots\right),$$

$$A_{2n+1}(\theta) = \frac{x_{2n+1}!}{x_n!\,x_{n+1}!}\left(2\cos\theta+\frac{x_n}{x_{n+2}}2\cos 3\theta+\frac{x_n x_{n-1}}{x_{n+2}x_{n+3}}2\cos 5\theta+\ldots\right).$$

Thus, if we take $\alpha_0 = 1$, $\alpha_r = 2\cos 2r\theta$, and $a = 1$, then $\beta_n = A_{2n}(\theta)/x_{2n}!$, while if we take $\alpha_r = 2\cos(2r+1)\theta$ and $a = x$, then $\beta_n = A_{2n+1}(\theta)/(x^2)_{2n}$.

We therefore obtain, from (3·1), the identities

$$\sum_{n=0}^{\infty}\frac{(\rho_1)_n(\rho_2)_n}{x_{2n}!}\left(\frac{x}{\rho_1\rho_2}\right)^n A_{2n}(\theta)$$
$$= \prod_{m=1}^{\infty}\left[\frac{(1-x^m/\rho_1)(1-x^m/\rho_2)}{(1-x^m)(1-x^m/\rho_1\rho_2)}\right]\left[1+\sum_{n=0}^{\infty}\frac{(\rho_1)_n(\rho_2)_n}{(x/\rho_1)_n(x/\rho_2)_n}\left(\frac{x}{\rho_1\rho_2}\right)^n 2\cos 2n\theta\right], \quad (3\cdot2)$$

$$\sum_{n=0}^{\infty}\frac{(\rho_1)_n(\rho_2)_n}{x_{2n+1}!}\left(\frac{x^2}{\rho_1\rho_2}\right)^n A_{2n+1}(\theta)$$
$$= \prod_{m=1}^{\infty}\left[\frac{(1-x^{m+1}/\rho_1)(1-x^{m+1}/\rho_2)}{(1-x^m)(1-x^{m+1}/\rho_1\rho_2)}\right]$$
$$\times\sum_{n=0}^{\infty}\frac{(\rho_1)_n(\rho_2)_n}{(x^2/\rho_1)_n(x^2/\rho_2)_n}\left(\frac{x^2}{\rho_1\rho_2}\right)^n 2\cos(2n+1)\theta. \quad (3\cdot3)$$

If we replace ρ_1, ρ_2 by $x^{\frac{1}{2}}/u$, $x^{\frac{1}{2}}/v$ in (3·2) we get § 5 (1) of Rogers's paper (6), while if we replace ρ_1, ρ_2 by x/u, x/v in (3·3) we get § 5 (2) of the same paper.* From these two formulae Rogers derived a list of twenty-one particular cases which he used to find identities. It will be noticed that these formulae are contained in (3·1) which is obtained by a simple application of the analogue of Gauss's theorem.

4. If, instead of using the analogue of Gauss's theorem, we use the analogue of Saalschütz's theorem, we can take

$$\delta_r = \frac{(\rho_1)_r (\rho_2)_r (x^{-N})_r}{(\rho_1 \rho_2 x^{-N}/a)_r} x^r,$$

where N is a positive integer, and we find that

$$\gamma_n = \frac{(ax/\rho_1)_N (ax/\rho_2)_N}{(ax)_N (ax/\rho_1\rho_2)_N} \frac{(-1)^n (\rho_1)_n (\rho_2)_n (x^{-N})_n}{(ax/\rho_1)_n (ax/\rho_2)_n (ax^{N+1})_n} \left(\frac{ax}{\rho_1\rho_2}\right)^n x^{\frac{1}{2}n(2N-n+1)}.$$

These values of δ_r, γ_n reduce to the previous values when $N \to \infty$, and lead to more general results involving only terminating basic series. We are, however, more concerned with identities of the Rogers-Ramanujan type in this paper, as the most general formulae for basic series (apart from those already given) are too involved to be of any great interest.

5. Another useful set of values for γ_n, δ_n can be obtained by using the formula

$${}_2\Phi_1\left[\begin{matrix} f, g; & a \\ ax & \overline{fg} \end{matrix}\right] = \prod_{m=1}^{\infty}\left[\frac{(1-ax^m/f)(1-ax^m/g)}{(1-ax^m)(1-ax^m/fg)}\right] \frac{fg(1+a)-(f+g)a}{fg-a}, \tag{5·1}$$

instead of the analogue of Gauss's theorem.† This can be obtained from § 8·5 (2) of my tract by letting c, $e \to \infty$ and putting $d = 1/q$. It can also be obtained by using the simple identity

$${}_2\Phi_1\left[\begin{matrix} f, g; & a \\ ax & \overline{fg} \end{matrix}\right] = {}_2\Phi_1\left[\begin{matrix} f, g; & ax \\ ax & \overline{fg} \end{matrix}\right] + \frac{(1-f)(1-g)}{1-ax} \frac{a}{fg} {}_2\Phi_1\left[\begin{matrix} fx, gx; & a \\ ax^2 & \overline{fg} \end{matrix}\right],$$

and summing the two series on the right by the analogue of Gauss's theorem.

We thus find that if $\quad \delta_r = (\rho_1)_r (\rho_2)_r (a/\rho_1\rho_2)^r,$

then
$$\gamma_n = \frac{(\rho_1)_n (\rho_2)_n}{(ax/\rho_1)_n (ax/\rho_2)_n} \left(\frac{a}{\rho_1\rho_2}\right)^n \left[\frac{\rho_1\rho_2(1+ax^{2n}) - ax^n(\rho_1+\rho_2)}{\rho_1\rho_2 - a}\right] \times \prod_{m=1}^{\infty}\left[\frac{(1-ax^m/\rho_1)(1-ax^m/\rho_2)}{(1-ax^m)(1-ax^m/\rho_1\rho_2)}\right].$$

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* The formulae § 5 (1) and § 5 (2) of (6) are actually proved in (5).

† This formula was pointed out to me by Mr F. J. Dyson.

This simplifies when $\rho_2 \to \infty$, in which case we get the set of values

$$\left.\begin{aligned} \delta_r &= (-1)^r (\rho_1)_r \frac{a^r}{\rho_1^r} x^{\frac{1}{2}r(r-1)}, \\ \gamma_n &= \frac{(-1)^n (\rho_1)_n}{(ax/\rho_1)_n} \frac{a^n}{\rho_1^n} x^{\frac{1}{2}n(n-1)} (1 + ax^{2n} - ax^n/\rho_1) \prod_{m=1}^{\infty} \left[\frac{1 - ax^m/\rho_1}{1 - ax^m}\right]. \end{aligned}\right\} \tag{5·2}$$

6. The relation giving β_n in terms of the α's is practically one that was considered in my previous paper. We can write

$$\beta_n = \frac{1}{(x)_n (ax)_n} \sum_{r=0}^{n} \frac{(x^{-n})_r}{(ax^{n+1})_r} (-1)^r x^{\frac{1}{2}r(2n-r+1)} \alpha_r,$$

and we know from that paper that there are several ways of choosing α_r so that the series for β_n can be summed by Jackson's identity. It is perhaps worth noting that if we take

$$\alpha_r = \frac{(-1)^r (p)_r x^{\frac{1}{2}r(r-1)} y^r}{(x)_r},$$

we get

$$\beta_n = \frac{1}{(x)_n (ax)_n} {}_2\Phi_1 \left[\begin{matrix} x^{-n}, p; \ yx^n \\ ax^{n+1} \end{matrix}\right].$$

We cannot choose *y independent of n* so that the series ${}_2\Phi_1$ can be summed by the analogue of Gauss's theorem, whereas for series of the ordinary type we can use Gauss's theorem and obtain a transformation of a nearly poised ${}_4F_3$ into a Saalschützian ${}_5F_4$.* Thus, as I stated before, there appears to be a breakdown in the analogy between basic series and series of the ordinary type for nearly poised series expressed in terms of Saalschützian series.

In § 8 of my previous paper we found the following sets of values of α_n, β_n:

(i) $\alpha_0 = 1, \quad \alpha_n = (-1)^n (1 - ax^{2n}) \dfrac{(ax)_{n-1}}{x_n!} a^n x^{\frac{1}{2}n(3n-1)}, \quad \beta_n = \dfrac{1}{x_n!}.$

(ii) Changing a into a^2, x into x^2,

$$\alpha_0 = 1, \quad \alpha_n = (1 - ax^{2n}) \frac{(ax)_{n-1} (b)_n}{x_n! (ax/b)_n} \frac{a^n x^{n^2}}{b^n}, \quad \beta_n = \frac{(-ax^{n+1}/b)_n}{x_n^2! (-ax)_{2n} (ax/b)_n}.$$

(iii) Changing a into a^3, x into x^3,

$$\alpha_n = (-1)^n (1 - ax^{2n}) \frac{(ax)_{n-1}}{x_n!} a^n x^{\frac{1}{2}n(3n-1)}, \quad \beta_n = \frac{(ax)_{3n}}{x_n^3! \{a^3 x^3\}_{2n}}.$$

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* Actually § 4·6 (1) in my tract.

Also from § 10 of the same paper,

(iv) $\alpha_{2n+1} = 0, \quad \alpha_{2n} = \dfrac{[ax^2]_{n-1}\,[f]_n\,(1-ax^{4n})}{x_n^2![ax^2/f]_n}\dfrac{a^n x^{2n^2}}{f^n},$

$$\beta_n = \frac{[ax/f]_n}{x_n![ax]_n\,(ax/f)_n}.$$

(v) $\alpha_{3n\pm1} = 0, \quad \alpha_{3n} = (-1)^n \dfrac{\{ax^3\}_{n-1}}{x_n^3!}(1-ax^{6n})\,a^n x^{\frac{1}{2}n(3n-1)},$

$$\beta_n = \frac{\{ax^3\}_{n-1}}{x_n!\,(ax)_{2n-1}}.$$

From (i) and (3·1) we get merely a limiting case of Watson's transformation from which the actual Rogers-Ramanujan identities can be deduced. From the other sets of values we get,* with (3·1),

$$\sum_{n=0}^{\infty}\frac{[\rho_1]_n\,[\rho_2]_n\,(-ax/b)_{2n}}{x_n^2![a^2x^2/b^2]_n\,(-ax)_{2n}}\left(\frac{a^2x^2}{\rho_1\rho_2}\right)^n$$
$$= \prod_{m=1}^{\infty}\left[\frac{(1-a^2x^{2m}/\rho_1)\,(1-a^2x^{2m}/\rho_2)}{(1-a^2x^{2m})\,(1-a^2x^{2m}/\rho_1\rho_2)}\right]$$
$$\times\left[1+\sum_{n=1}^{\infty}\frac{(ax)_{n-1}\,(b)_n\,[\rho_1]_n\,[\rho_2]_n\,(1-ax^{2n})\,a^{3n}x^{n^2+2n}}{x_n!(ax/b)_n\,[a^2x^2/\rho_1]_n\,[a^2x^2/\rho_2]_n\,(\rho_1\rho_2 b)^n}\right]. \tag{6·1}$$

$$\sum_{n=0}^{\infty}\frac{(ax)_{3n}\,\{\rho_1\}_n\,\{\rho_2\}_n}{x_n^3!\{a^3x^3\}_{2n}}\left(\frac{a^3x^3}{\rho_1\rho_2}\right)^n$$
$$= \prod_{m=1}^{\infty}\left[\frac{(1-a^3x^{3m}/\rho_1)\,(1-a^3x^{3m}/\rho_2)}{(1-a^3x^{3m})\,(1-a^3x^{3m}/\rho_1\rho_2)}\right]$$
$$\times\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n\,(ax)_{n-1}\{\rho_1\}_n\{\rho_2\}_n}{x_n!\{a^3x^3/\rho_1\}_n\,\{a^3x^3/\rho_2\}_n}(1-ax^{2n})\,a^n x^{\frac{1}{2}(3n^2-n)}\left(\frac{a^3x^3}{\rho_1\rho_2}\right)^n\right]. \tag{6·2}$$

$$\sum_{n=0}^{\infty}\frac{(\rho_1)_n\,(\rho_2)_n\,[ax/f]_n}{x_n![ax]_n\,(ax/f)_n}\left(\frac{ax}{\rho_1\rho_2}\right)^n$$
$$= \prod_{m=1}^{\infty}\left[\frac{(1-ax^m/\rho_1)\,(1-ax^m/\rho_2)}{(1-ax^m)\,(1-ax^m/\rho_1\rho_2)}\right]$$
$$\times\left[1+\sum_{n=1}^{\infty}\frac{[ax^2]_{n-1}\,[f]_n(1-ax^{4n})\,(\rho_1)_{2n}\,(\rho_2)_{2n}}{x_n^2![ax^2/f]_n\,(ax/\rho_1)_{2n}\,(ax/\rho_2)_{2n}}\frac{a^{3n}x^{2n^2+2n}}{\rho_1^{2n}\rho_2^{2n}f^n}\right], \tag{6·3}$$

$$1+\sum_{n=1}^{\infty}\frac{(\rho_1)_n\,(\rho_2)_n\,\{ax^3\}_{n-1}}{x_n!(ax)_{2n-1}}\left(\frac{ax}{\rho_1\rho_2}\right)^n$$
$$= \prod_{m=1}^{\infty}\left[\frac{(1-ax^m/\rho_1)\,(1-ax^m/\rho_2)}{(1-ax^m)\,(1-ax^m/\rho_1\rho_2)}\right]$$
$$\times\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n\,\{ax^3\}_{n-1}\,(1-ax^{6n})\,(\rho_1)_{3n}\,(\rho_2)_{3n}}{x_n^3!(ax/\rho_1)_{3n}\,(ax/\rho_2)_{3n}}\frac{a^{4n}x^{\frac{1}{2}n(3n+1)}}{\rho_1^{3n}\rho_2^{3n}}\right]. \tag{6·4}$$

* When $\rho_1, \rho_2 \to \infty$, the formulae (6·1)–(6·4) become the same as (8·3), (8·4), (10·1) and (10·6) of (2).

Similarly, we can combine the sets of values of α_n, β_n with (5·2). It will suffice to give the result obtained from the set (v), namely,

$$1+\sum_{n=1}^{\infty}\frac{(-1)^n\{ax^3\}_{n-1}(\rho_1)_n}{x_n!(ax)_{2n-1}}\frac{a^n x^{\frac{1}{2}n(n-1)}}{\rho_1^n}$$
$$=\prod_{m=1}^{\infty}\left[\frac{1-ax^m/\rho_1}{1-ax^m}\right]\left[1+a-a/\rho_1+\sum_{n=1}^{\infty}\frac{\{ax^3\}_{n-1}(1-ax^{6n})(\rho_1)_{3n}}{x_n^3!(ax/\rho_1)_{3n}}\right.$$
$$\left.\times(1+ax^{6n}-ax^{3n}/\rho_1)\frac{a^{4n}x^{3n(3n-1)}}{\rho_1^{3n}}\right]. \quad (6{\cdot}5)$$

From these formulae many identities of the Rogers-Ramanujan type can be derived by choosing the parameters suitably. For example, from (6·1), if we let ρ_1, $b\to\infty$, and put $\rho_2=-ax$, we get

$$\sum_{n=0}^{\infty}\frac{[-ax]_n}{x_n^2!(-ax)_{2n}}a^n x^{n^2}$$
$$=\prod_{m=1}^{\infty}\left[\frac{1+ax^{2m-1}}{1-a^2x^{2m}}\right]\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n(ax)_{n-1}}{x_n!}(1-ax^{2n})a^{2n}x^{\frac{1}{2}(5n^2-n)}\right], \quad (6{\cdot}6)$$

and for $a=1$ this gives

$$\sum_{n=0}^{\infty}\frac{x^{n^2}}{x_n^4!}=\prod_{n=1}^{\infty}\left[\frac{1}{(1+x^{2n})(1-x^{5n-1})(1-x^{5n-4})}\right], \quad (6{\cdot}7)$$

which is one of the more elegant sums given by Rogers.*

The formulae (6·1) and (6·3) give many known results, and I therefore consider only the formulae (6·4) and (6·5). These give all the formulae labelled B, C, D and E in the list given in § 11 of my previous paper (and found by F. J. Dyson).

For ρ_1, $\rho_2\to\infty$ in (6·4), we get (B 4) when $a=1$, and (B 1) when $a=x^3$.

For $\rho_2\to\infty$, $\rho_1=-\sqrt{(ax)}$, we get (C 3) when $a=1$, and (D 1) when $a=x^3$.

For $\rho_1=-\sqrt{a}$, $\rho_2\to\infty$, $a=x^3$ gives (C 1), and $a=1$ gives (D 3).

For $\rho_1=\sqrt{(ax)}$, $\rho_2\to\infty$, $a=x^3$ gives (E 1).

From (6·5), for $\rho_1\to\infty$, we get (B 2) when $a=x^3$, and the sum of (B 3) and (B 4) when $a=1$.

For $\rho_1=-\sqrt{(ax)}$, we get a linear combination of (D 1) and (D 2) when $a=x^3$.

For $\rho_1=-\sqrt{a}$, we get (C 2) when $a=x^3$, and a linear combination of (D 2) and (D 3) when $a=1$.

For $\rho_1=\sqrt{(ax)}$, we get a linear combination of (E 1) and (E 2) when $a=x^3$.

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* Rogers (4, last formula on p. 330). See also Jackson (3, 175), where a different generalization of (6·7) is given.

In the course of the work we naturally obtain generalizations of these formulae involving a parameter a. The formula (B 3), namely

$$\sum_{n=0}^{\infty} \frac{x^{n^2+n} x_n^3!}{x_n!\, x_{2n+1}!} = \prod_{n=1}^{\infty} \left(\frac{1-x^{9n}}{1-x^n}\right), \tag{6·8}$$

is a particularly elegant one, and so I give an a-generalization which I found rather troublesome to obtain.

As indicated above, we let $\rho_1, \rho_2 \to \infty$ in (6·4) and (6·5) and we get

$$1+\sum_{n=1}^{\infty} \frac{\{ax^3\}_{n-1}}{x_n!(ax)_{2n-1}} a^n x^{n^2}$$
$$= \prod_{m=1}^{\infty} \left(\frac{1}{1-ax^m}\right)\left[1+\sum_{n=1}^{\infty} \frac{(-1)^n \{ax^3\}_{n-1}}{x_n^3!}(1-ax^{6n})\, a^{4n} x^{\frac{1}{2}(27n^2-3n)}\right],$$

$$1+\sum_{n=1}^{\infty} \frac{\{ax^3\}_{n-1}}{x_n!(ax)_{2n-1}} a^n x^{n^2-n}$$
$$= \prod_{m=1}^{\infty} \left(\frac{1}{1-ax^m}\right)\left[1+a+\sum_{n=1}^{\infty} \frac{(-1)^n \{ax^3\}_{n-1}}{x_n^3!}(1-a^2x^{12n})\, a^{4n} x^{\frac{1}{2}(27n^2-9n)}\right].$$

Subtracting, we get

$$\sum_{n=1}^{\infty} \frac{\{ax^3\}_{n-1}}{x_{n-1}!(ax)_{2n-1}} a^n x^{n^2-n} = \prod_{m=1}^{\infty} \left(\frac{1}{1-ax^m}\right)$$
$$\times\left[a+\sum_{n=1}^{\infty} \frac{(-1)^n \{ax^3\}_{n-1}}{x_n^3!} a^{4n} x^{\frac{1}{2}(27n^2-9n)} \{(1-x^{3n})+ax^{9n}(1-ax^{3n})\}\right]$$
$$= \prod_{m=1}^{\infty} \left(\frac{1}{1-ax^m}\right)\left[a+\sum_{n=1}^{\infty} \frac{(-1)^n \{ax^3\}_{n-1}}{x_{n-1}^3!} a^{4n} x^{\frac{1}{2}(27n^2-9n)} + a\sum_{n=1}^{\infty} \frac{(-1)^n \{ax^3\}_n}{x_n^3!} a^{4n} x^{\frac{1}{2}(27n^2+9n)}\right]$$
$$= \prod_{m=1}^{\infty} \left(\frac{1}{1-ax^m}\right)\left[a-\sum_{n=0}^{\infty} \frac{(-1)^n \{ax^3\}_n}{x_n^3!} a^{4n+4} x^{\frac{3}{2}(n+1)(3n+2)} + a\sum_{n=1}^{\infty} \frac{(-1)^n \{ax^3\}_n}{x_n^3!} a^{4n} x^{\frac{1}{2}(27n^2+9n)}\right].$$

It follows that

$$\sum_{n=0}^{\infty} \frac{\{ax^3\}_n}{x_n!(ax)_{2n+1}} a^n x^{n^2+n}$$
$$= \prod_{m=1}^{\infty} \left(\frac{1}{1-ax^m}\right)\sum_{n=0}^{\infty} \frac{(-1)^n \{ax^3\}_n}{x_n^3!} a^{4n} x^{\frac{1}{2}(27n^2+9n)}(1-a^3x^{18n+9}). \tag{6·9}$$

This gives (6·8) when $a = 1$, and it is the most elegant generalization of that formula which I have been able to find.

7. It has not been my object to find a large number of new identities of the Rogers-Ramanujan type, but it is perhaps worth while noting some which I believe are new. Thus from (6·4), by taking $\rho_1 = -\sqrt{a}$, $\rho_2 = -\sqrt{(ax)}$, $\sqrt{a} = x^{\frac{1}{2}}$ or 1, and finally changing x into x^2, we obtain the formulae

$$\sum_{n=0}^{\infty} \frac{x_n^6!\, x^n}{x_{2n+2}!\, x_n^2!} = \prod_{n=1}^{\infty} \frac{(1-x^{18n})(1-x^{18n-3})(1-x^{18n-15})}{(1-x^n)(1-x^{2n-1})}, \tag{7·1}$$

$$1+2\sum_{n=1}^{\infty} \frac{x_{n-1}^6!\, x^n}{x_{2n-1}!\, x_n^2!} = \prod_{n=1}^{\infty} \frac{(1-x^{9n})(1-x^{18n-9})}{(1-x^n)(1-x^{2n-1})}$$

$$= \prod_{n=1}^{\infty} \frac{(1+x^n)(1-x^{9n})}{(1-x^n)(1+x^{9n})}. \tag{7·2}$$

Some results from (6·2) are

$$\sum_{n=0}^{\infty} \frac{x_{3n+1}!\, x^{\frac{3}{2}(n^2+n)}}{(x_n^3!)^2\, x_{n+1}^3!!} = \prod_{n=1}^{\infty} \frac{(1-x^{6n-5})(1-x^{6n-1})}{(1-x^{6n-3})^2}, \tag{7·3}$$

$$\sum_{n=0}^{\infty} \frac{x_{3n+4}!\, x^{\frac{3}{2}(n^2+n)}}{x_n^3!\, x_{n+1}^3!\, x_{n+2}^3!!} = (1-x)\prod_{n=1}^{\infty} \frac{1-x^{2n}}{(1-x^{3n})(1-x^{6n-3})}. \tag{7·4}$$

Some further identities communicated to me by Mr Dyson are

$$\sum_{n=0}^{\infty} \frac{x^{3n^2} x_{3n}^2!}{x_n^{12}!\, x_{2n}^3!} = \prod_{n=1}^{\infty} \frac{(1-x^{12n-5})(1-x^{12n-7})}{1-x^{6n-3}}, \tag{7·5}$$

$$\sum_{n=0}^{\infty} \frac{x^{3n^2} x_{3n+1}^2!}{x_n^{12}!\, x_{2n}^3!\,(x^{6n}-x^2)} = \prod_{n=1}^{\infty} \frac{(1-x^{12n-1})(1-x^{12n-11})}{1-x^{6n-3}}, \tag{7·6}$$

$$1+\sum_{n=1}^{\infty} \frac{x^{n^2} x_{n-1}^6!}{x_n^4!\, x_{2n-1}!} = \prod_{n=1}^{\infty} \frac{(1+x^{18n-9})^2(1-x^{18n})}{(1+x^{6n-3})(1-x^{2n-1})(1-x^{4n})}, \tag{7·7}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{\frac{3}{2}(n^2+n)} x_{3n+1}!}{x_{2n+1}^3!} = \sum_{n=0}^{\infty} x^{3n^2+2n} - \sum_{n=1}^{\infty} x^{3n^2-2n}, \tag{7·8}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{\frac{3}{2}(n^2+n)} x_{3n}!(1-x^{3n+2})}{x_{2n+1}^3!} = \sum_{n=0}^{\infty} x^{3n^2+n} - \sum_{n=1}^{\infty} x^{3n^2-n}. \tag{7·9}$$

Of these, (7·5) and (7·8) can be derived from (6·2), and no doubt the others can also be obtained from the formulae of this paper.

8. I now give a final application of the simple result of § 1.

Define the differences $D^n(a_0)$ by the relations

$$D(a_0) = a_1 - a_0,$$

$$D^2(a_0) = a_2 - (1+q)a_1 + qa_0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$D^n(a_0) = \sum_{r=0}^{\infty} \frac{(-1)^r n!}{r!(n-r)!} q^{\frac{1}{2}r(r-1)} a_{n-r},$$

where $n! = (1-q)(1-q^2)\dots(1-q^n)$.

Then $D^n(a_0)/n!$ is the coefficient of x^n in the product

$$\sum_{r=0}^{\infty} \frac{(-1)^r q^{\frac{1}{2}r(r-1)} x^r}{r!} \sum_{s=0}^{\infty} \frac{a_s x^s}{s!},$$

that is in

$$\prod_{n=0}^{\infty} (1-q^n x) \sum_{s=0}^{\infty} \frac{a_s x^s}{s!}.$$

It follows that

$$\sum_{s=0}^{\infty} \frac{a_s x^s}{s!} = \prod_{n=0}^{\infty} \left(\frac{1}{1-q^n x}\right) \sum_{n=0}^{\infty} D^n(a_0) \frac{x^n}{n!}$$

$$= \sum_{m=0}^{\infty} \frac{x^m}{m!} \sum_{n=0}^{\infty} D^r(a_0) \frac{x^r}{r!},$$

and equating the coefficients of x^n on the two sides we get

$$a_n = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} D^r(a_0). \tag{8·1}$$

Now let

$$f(x) = \sum_0^{\infty} b_r x^r, \quad f'(x) = \frac{f(x)-f(qx)}{x}.$$

Taking $\alpha_r = (x^r/r!) D^r(a_0)$, $\delta_r = b_r r!$, $u_r = x^r/r!$, $v_r = 1$ in § 1, we find from (8·1) that $\beta_n = a_n x^n/n!$, and evidently $\gamma_n = f^{(n)}(x)$. It follows that

$$\sum_{n=0}^{\infty} a_n b_n x^n = \sum_{n=0}^{\infty} f^{(n)}(x) \frac{x^n}{n!} D^n(a_0). \tag{8·2}$$

This result was given by Jackson and used by him* to obtain numerous identities.

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Bedford College,
Regent's Park,
London, N.W. 1.

* Jackson (3).

SYMMETRIC OPERATORS IN HILBERT SPACE

By J. L. B. COOPER

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1. *Introduction*

In a previous paper† I gave a method by which the spectral resolution corresponding to a self-adjoint operator can be deduced from the solution of the Schrödinger equation for that operator. Unlike other proofs of the existence of the spectral resolution‡ this proof did not depend on the use of the resolvent operator or on operators which are effectively real parts of resolvent operators; but the proof of a solution of a Schrödinger equation corresponding to a self-adjoint operator, given in theorem II of my paper, does depend on the use of the resolvent operator. The first object of this present paper is to provide a proof of this existence theorem which does not depend on the resolvent operator. Taken together with the paper referred to above, this section of the present paper should, therefore, be of interest as giving a new approach to the problem of the self-adjoint operator.

The proof here given shows that a solution of the Schrödinger equation for the adjoint operator (eqn. (1)) exists with any element of the domain of the operator as initial value, and that this is true for all symmetric operators. This result can be made a basis for a discussion of the properties of general symmetric operators: but the discussion in this paper is confined to self-adjoint and maximal symmetric operators.

For self-adjoint operators, the solution of the Schrödinger equation appears as the transform of the initial value by a unitary operator: the totality of these unitary operators forms a one-parameter continuous group. This is, of course, well known: the novel feature of the treatment here is that the result is demonstrated directly, without the intervention of the spectral resolution, so that the discussion of the group of one-parameter unitary operators is the basis of the discussion of the spectral resolution, instead of *vice versa* as in previous discussions.§

For operators which are maximal but not self-adjoint, a solution of the Schrödinger equation for the operator exists for every initial value either for

† Ref. (**10**). ‡ See refs. (**1, 2, 3, 4**). § See refs. (**6, 7, 8**).

all positive or for all negative t, according to the class of the operator, but not for both. The solution is an isometric transformation of the initial values; the set of isometric transformations forms a one-parameter continuous half-group. By this we mean that the set has the multiplicative properties of a one-parameter group, but not the property that any member possesses an inverse operator with a domain dense in Hilbert space. The study of this set of isometric transformations leads to a new theory of maximal operators, and a new type of resolution of the identity is shown to correspond to these operators. The resolution of the identity given by von Neumann and Stone for self-adjoint operators has the property that if the operator has a simple spectrum the Hilbert space is mapped on $L^2(-\infty, \infty)$ in such a manner that $f(\lambda)$ of $L^2(-\infty, \infty)$ is carried by the operator into $\lambda f(\lambda)$. The resolution of the identity given here has the property that in the case of an irreducible maximal symmetric operator of deficiency index $(0, 1)$ the Hilbert space is mapped on to $L^2(0, \infty)$ in such a manner that $f(t)$ of $L^2(0, \infty)$ is carried by the operator into $i(df/dt)$.

The close association between the Schrödinger equation and the spectral theory, in this and my previous paper, has some merits from the physical point of view. Whereas the idea of a symmetric operator derives in quantum theory from a physical requirement—that the values of the corresponding observable must be real—the distinction between self-adjoint and other symmetric operators seems to have no physical meaning. On the other hand, the hypothesis of the first theorem of my last paper, that the Schrödinger equation has a solution for all values of the variable t, has the physical meaning that the system can have an infinite past and future, while the variable corresponding to the operator retains a constant probability distribution. If the Schrödinger equation has, for example, no solution for sufficiently large positive t, then the system can develop for only a finite time with the corresponding observable constant. A simple example is a particle moving with constant momentum on a semi-infinite line; if it is moving to the boundary, its motion can continue only for a finite time. The corresponding Schrödinger equation has solutions for all negative, but not all positive t.†

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† The operator here is $\frac{1}{i}\frac{\partial}{\partial x}$, defined for all absolutely continuous functions which, with their derivatives, are in $L^2(0, \infty)$ and vanish for $x = 0$. The solution of (1 A) is, for

$$\psi(0) = \phi = \phi(x), \quad \psi(t) = \psi(x, t) = \psi(x+t), \quad (t < 0).$$

For positive t a solution is possible only for those values of t with $\phi(x) = 0$ $(x < t)$. The reading of this paper will be facilitated if the reader considers the application of the arguments to this example, which is further discussed in the last section.

Through the work of Koopman,† these considerations can be extended to classical mechanics. Koopman has shown that systems with constant energy can have their motion represented by a one-parameter group of unitary operators. Here again, one-parameter half-groups of isometric operators would appear to correspond to systems which for some initial conditions can have only a finite future, or a finite past.

Just as the theory of groups of unitary operators corresponds to that of self-adjoint operators, and has been worked out as a separate problem by von Neumann and Stone,‡ so there will be a theory of half-groups of isometric operators corresponding to the theory of maximal operators in this paper. From this point of view, the theory of this paper is confined to half-groups for which the operators are differentiable functions of the parameter. It remains to be investigated how far the hypothesis of differentiability can be removed.

A final section of the paper gives some applications to the theory of Hilbert transforms and Watson transforms, etc., which should serve as examples.

Except where the contrary is stated, the terminology and notation of this paper follow that of ref. (**2**).

2. *Preliminary remarks on differential equations in Hilbert space*

In what follows we shall be concerned with equations of form

$$\frac{d\psi(t)}{dt} = \chi(t),$$

where $\psi(t)$, $\chi(t)$ are functions of a real variable t whose values are elements in a Hilbert space $\mathfrak{H}$. In the cases to be encountered, $\chi(t)$ is a linear transformation of $\psi(t)$. The left-hand side is a differential coefficient of $\psi(t)$, the limit of $[\psi(t+h)-\psi(t)]/h$. This limit may exist either

(i) in the strong sense—in which case we shall talk of the strong differential coefficient, denoted by $d\psi/dt$, and of the differential equation being obeyed in the strong sense; or

(ii) in the weak sense—in which case we talk of the weak differential coefficient, etc., $\delta\psi/\delta t$.§

† Ref. (**9**). ‡ See refs. (**6**, **7**, **8**).

§ The strong sense is the more natural one in Hilbert space theory, but the weak sense corresponds more closely to the normal interpretation of a differential equation —e.g. if we consider the Hilbert space $L^2(-\infty, \infty)$, and $\psi(t) = \psi(x, t)$ is an element of the space for each t, the equation

$$\frac{\partial\psi}{\partial t} = \chi(x, t)$$

would be interpreted to mean that the differential coefficient is equal at each point x to $\chi(x, t)$—which is practically the weak sense.

If ϕ_n is a complete normal orthogonal set in $\mathfrak{H}$, and

$$\psi_n(t) = (\psi(t), \phi_n), \quad \chi_n(t) = (\chi(t), \phi_n),$$

the weak sense of the equation is equivalent to the condition that for all n and t

$$\frac{d\psi_n}{dt} = \chi_n(t).$$

More than this is required by the strong sense of the equation, but, as we shall see from the following lemmas, the difference between the two senses is less important than might be expected.

LEMMA A. *If $\chi(t)$ is bounded, $\| \chi(t) \| \leqslant K$ for all t in (a, b), then for all t except a set of measure zero*

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \| \chi(y) - \alpha \|^2 \, dy = \| \chi(t) - \alpha \|^2$$

for every element α in $\mathfrak{H}$.

The proof of this theorem is essentially the same as that of § 11·6 of ref. (**12**). For a fixed α, we have

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \| \chi(y) - \alpha \|^2 \, dy = \| \chi(t) - \alpha \|^2,$$

except at a set of points of measure zero, by § 11·53 of ref. (**12**). Since Hilbert space is separable we can find a countable set of elements $\{\beta\}$ which are everywhere dense in $\mathfrak{H}$. The equation above is then true for all the β except at a set of points t of measure zero—the sum of the exceptional sets for individual β. Given any α in $\mathfrak{H}$ and $\epsilon > 0$, we can find a β such that $\| \alpha - \beta \| \leqslant \epsilon$. Then for all t

$$\| \chi(t) - \alpha \|^2 - \| \chi(t) - \beta \|^2 = (\chi(t) - \alpha, \beta - \alpha) + (\beta - \alpha, \chi(t) - \beta),$$

so for some K independent of t

$$\| \chi(t) - \alpha \|^2 - \| \chi(t) - \beta \|^2 \leqslant K \| \beta - \alpha \| \leqslant K\epsilon.$$

Hence $$\left| \int_t^{t+h} \| \chi(y) - \alpha \|^2 \, dy - \int_t^{t+h} \| \chi(y) - \beta \|^2 dy \right| \leqslant K\epsilon h,$$

and so, if t is not in the exceptional set for the $\{\beta\}$,

$$\frac{1}{h} \left| \int_t^{t+h} \| \chi(y) - \alpha \|^2 \, dy - \| \chi(t) - \alpha \|^2 \right| \leqslant 2K\epsilon;$$

since ϵ may be as small as we please, the result follows for any α in $\mathfrak{H}$.

COROLLARY. *In particular, for almost all t*

$$\int_t^{t+h} \| \chi(y) - \chi(t) \|^2 \, dy = o(h).$$

LEMMA B. *If for all t in* (a, b)

$$\|\chi(t)\| \leqslant K \quad \text{and} \quad \frac{\delta\psi}{\delta t} = \chi(t),$$

then the equation is satisfied in the strong sense for almost all t—in fact, for all t save those in the exceptional set of the corollary to lemma A.

We have

$$\begin{aligned}\left\|\frac{\psi(t+h)-\psi(t)}{h}-\chi(t)\right\|^2 &= \Sigma\left|\frac{\psi_n(t+h)-\psi_n(t)}{h}-\chi_n(t)\right|^2\\ &= \Sigma\left|\frac{1}{h}\int_t^{t+h}\chi_n(y)\,dy-\chi_n(t)\right|^2\\ &= \Sigma\left|\frac{1}{h}\int_t^{t+h}\{\chi_n(y)-\chi_n(t)\}\,dy\right|^2\\ &\leqslant \frac{1}{h}\Sigma\int_t^{t+h}|\chi_n(y)-\chi_n(t)|^2\,dy\\ &= \frac{1}{h}\int_t^{t+h}\|\chi(y)-\chi(t)\|^2\,dy\\ &\to 0,\end{aligned}$$

for all t save for the exceptional set of lemma A corollary.

LEMMA C. *If* $\chi(t)$ *is a continuous function of t, then the conclusions of lemmas A and B are true everywhere.*

This is a consequence of the fact that a continuous function is everywhere equal to the derivative of its integral.

LEMMA D. *If in an interval* (a, b)

$$\frac{\delta\psi_1(t)}{\delta t} = \chi_1(t), \quad \frac{\delta\psi_2(t)}{\delta t} = \chi_2(t),$$

then for all t in (a, b)

$$\frac{d}{dt}(\psi_1(t), \psi_2(t)) = (\chi_1(t), \psi_2(t)) + (\psi_1(t), \chi_2(t)).$$

We have

$$\begin{aligned}&\frac{1}{h}\{(\psi_1(t+h), \psi_2(t+h)) - (\psi_1(t), \psi_2(t))\}\\ &\quad = \frac{1}{h}\{(\psi_1(t+h)-\psi_1(t), \psi_2(t)) + (\psi_1(t), \psi_2(t+h)-\psi_2(t))\}\\ &\qquad\qquad + \frac{1}{h}\{(\psi_1(t+h)-\psi_1(t), \psi_2(t+h)-\psi_2(t))\}.\end{aligned}$$

The second bracket is less in absolute value than

$$h\left\|\frac{\psi_1(t+h)-\psi_1(t)}{h}\right\|\left\|\frac{\psi_2(t+h)-\psi_2(t)}{h}\right\| = O(h),$$

since $\dfrac{\psi_i(t+h)-\psi_i(t)}{h}$ $(i = 1, 2)$ are bounded because they are weakly convergent. The first bracket tends to

$$(\chi_1(t), \psi_2(t)) + (\psi_1(t), \chi_2(t)).$$

3. *The existence theorem for the Schrödinger equation for symmetric operators*

THEOREM I. *If H is a linear symmetric operator defined in the domain $\mathfrak{D}$ of the Hilbert space $\mathfrak{H}$, and H^* is its adjoint defined in the domain $\mathfrak{D}^*$, then for every element ϕ in $\mathfrak{D}$ there exists for all t an element $\psi(t)$ in $\mathfrak{D}^*$ such that $\delta\psi/\delta t$ exists and*

$$\frac{1}{i}\frac{\delta\psi}{\delta t} = H^*\psi(t), \quad \psi(0) = \phi. \tag{1}$$

The following lemma will be useful:

LEMMA 1. *If $\psi(t)$ and $\chi(t)$ are in the domain $\mathfrak{D}$ of a symmetric operator H for all t in a range (a, b), are weakly differentiable functions of t in that range, and satisfy*

$$\frac{1}{i}\frac{\delta\psi}{\delta t} = H\psi(t), \quad \frac{1}{i}\frac{\delta\chi}{\delta t} = H\chi(t),$$

then $(\psi(t), \chi(t+\tau))$ is independent of t as long as t and $(t+\tau)$ lie inside (a, b).

From lemma D,

$$\begin{aligned}\frac{d}{dt}(\psi(t), \chi(t+\tau)) &= (iH\psi(t), \chi(t+\tau)) + (\psi(t), iH\chi(t+\tau)) \\ &= 0.\end{aligned}$$

The lemma follows immediately.

Proof of theorem I. Let a sequence of closed linear manifolds $\mathfrak{M}_n$ be chosen so that $\mathfrak{M}_{n+1}$ contains $\mathfrak{M}_n$, so that the closed linear manifold determined by the upper limit of the $\mathfrak{M}_n$ is $\mathfrak{H}$, and so that H is bounded inside each one of the $\mathfrak{M}_n$. This is possible because the domain of H is dense in $\mathfrak{H}$.

Let E_N be the projection operator corresponding to $\mathfrak{M}_N$. Consider the equations

$$\frac{1}{i}\frac{d\psi_N}{dt} = E_N H E_N \psi_N(t), \quad \psi_N(0) = \phi. \tag{2}$$

The operator $E_N H E_N$ is bounded and self-adjoint. The solution of (2) can be written down, formally,

$$\psi_N(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} (E_N H E_N)^n \phi = e^{itE_N H E_N} \phi. \tag{3}$$

Let the bound of H in $\mathfrak{M}_N$ be K_N, i.e. in $\mathfrak{M}_N$,

$$\| H\phi \| \leqslant K_N \| \phi \|.$$

Then for all ϕ
$$\| (E_N H E_N)^r \phi \| \leqslant K_N^r \| \phi \|,$$

and the convergence of (3) follows, for

$$\begin{aligned} &\left\| \sum_{n=m}^{\infty} \frac{(it)^n}{n!} (E_N H E_N)^n \phi \right\|^2 \\ &\qquad \leqslant \left| \sum_{m}^{\infty} \sum_{m}^{\infty} \frac{(it)^r}{r!} \frac{(-it)^s}{s!} ((E_N H E_N)^r \phi, (E_N H E_N)^s \phi) \right| \\ &\qquad \leqslant \sum_{m}^{\infty} \frac{|t|^{r+s}}{r!s!} K_N^{r+s} \| \phi \| \\ &\qquad \to 0 \quad \text{as} \quad m \to \infty. \end{aligned}$$

The series
$$\sum_{n=0}^{\infty} \frac{(it)^n}{n!} (E_N H E_N)^{n+1} \phi$$

also converges uniformly over any interval in t. It is easily seen to be equal to both $\frac{1}{i}\frac{d\psi_N}{dt}$ and $(E_N H E_N)\psi$. Since, in addition, the series (3) satisfies $\psi_N(0) = \phi$, we see that the function defined by it is a solution of (2).

$d\psi_N/dt$ and $d^2\psi_N/dt^2$ are defined by similar series, and satisfy the differential equation (2) with the boundary conditions

$$\left(\frac{d\psi_N}{dt}\right)_{t=0} = i(E_N H E_N)\phi, \quad \left(\frac{d^2\psi_N}{dt^2}\right)_{t=0} = -(E_N H E_N)^2 \phi.$$

Thus, from lemma 1, we have

$$\| \psi_N(t) \| = \| \phi \|, \quad \left\| \frac{d\psi_N}{dt} \right\| = \| E_N H E_N \phi \|, \quad \left\| \frac{d^2\psi_N}{dt^2} \right\| = \| (E_N H E_N)^2 \phi \|.$$

If g is any element of $\mathfrak{D}$

$$\begin{aligned} \left| \frac{d}{dt} \left(\frac{d\psi_N}{dt}, g \right) \right| &= \left| \left(\frac{d^2\psi_N}{dt^2}, g \right) \right| \\ &= | ((E_N H E_N)^2 \psi_N, g) | \\ &= | (E_N H E_N \psi_N, E_N H E_N g) | \\ &\leqslant \| E_N H E_N \phi \| \, \| E_N H E_N g \|. \end{aligned} \tag{4}$$

Let $\{\phi_n\}$ be a set of complete normal orthogonal elements of $\mathfrak{H}$, all lying in $\mathfrak{D}$. The manifold $\mathfrak{M}_N$ may be taken to be the linear manifold determined by all ϕ_n with $n \leqslant N$. Since the ϕ_n are to a large extent arbitrary there is no loss of generality if we assume ϕ to be one of the ϕ_n, or indeed ϕ_1.

From (4)
$$\left|\frac{d}{dt}\left(\frac{d\psi_N}{dt}, \phi_n\right)\right| \leqslant \| E_N H\phi \| \, \| E_N H\phi_n \|$$
$$\leqslant \| H\phi \| \, \| H\phi_n \|,$$
if N is large enough ($N \geqslant n$). This last expression is independent of N, so for a fixed n the sequence $(d\psi_N/dt, \phi_n)$ as N varies from 1 to ∞ consists of uniformly continuous functions of t.

By a well-known theorem† on sets of uniformly continuous functions, it is possible to choose from the sequence $(d\psi_N/dt, \phi_1)$ a subsequence $(d\psi_{N_{1,m}}/dt, \phi_1)$ which tends to a continuous function of t as $N_{1,m} \to \infty$.

From the sequence $N_{1,m}$ we can choose a subsequence $N_{2,m}$ such that $(d\psi_{N_{2,m}}/dt, \phi_2)$ tends to a limit as $N_{2,m} \to \infty$, and so on for ϕ_3, etc. Then if p runs through the numbers $N_{m,m}$ the sequence $(d\psi_p/dt, \phi_n)$ tends to a continuous function of t as $p \to \infty$ for any fixed n.

Write
$$\lim_{p\to\infty}\left(\frac{d\psi_p}{dt}, \phi_n\right) = h_n(t).$$

Now
$$\left(\frac{d\psi_p}{dt}, \phi_n\right) = i(E_p H E_p \psi_p, \phi_n)$$
$$= i(E_p H \psi_p, \phi_n),$$
if p is large enough, since ϕ is one of the ϕ_n;
$$\sum_{n=1}^{\infty}\left|\left(\frac{d\psi_p}{dt}, \phi_n\right)\right|^2 \leqslant \| E_p H\psi_p \|^2$$
$$\leqslant \| H\phi \|^2,$$
i.e.
$$\sum_{n=1}^{\infty} | h_n(t) |^2 \leqslant \| H\phi \|^2;$$
so there exists a function $\chi(t)$ such that
$$\chi(t) = \Sigma h_n(t)\,\phi_n, \quad \| \chi(t) \|^2 \leqslant \| H\phi \|^2.$$

The sequence $(d\psi_p/dt, \phi_n)$ is a uniformly bounded sequence of functions of t which tend to a limit $h_n(t)$; hence the integrals of the functions of the sequence tend to the integral of their limit:
$$\lim \int_0^t \left(\frac{d\psi_p}{dt}, \phi_n\right) dt = \int_0^t h_n(t)\, dt,$$

† See Courant and Hilbert, *Methoden der Mathematischen Physik*, 1 (1931), 49.

or
$$\int_0^t h_n(t)\,dt = \lim\,(\psi_p(t)-\phi,\phi_n).$$

Since
$$\Sigma\left|\int_0^t h_n(\tau)\,d\tau\right|^2 \leqslant t\,\Sigma\int_0^t |\,h_n(\tau)\,|^2\,d\tau$$
$$= t\int_0^t \|\,\chi(\tau)\,\|^2\,d\tau,$$

there exists an element in $\mathfrak{H}$ defined by

$$\psi(t) = \phi + \sum_{n=1}^{\infty}\phi_n\int_0^t h_n(t)\,dt, \tag{5}$$

and
$$(\psi(t),\phi_n) = \lim_{p\to\infty}(\psi_p(t),\phi_n),$$

that is, $\psi(t)$ is the weak limit of $\psi_p(t)$ when p runs through the sequence of numbers N_{mm}. Clearly
$$\frac{\delta\psi}{\delta t} = \Sigma h_n(t)\,\phi_n = \chi(t).$$

Since $\psi_p(t)\to\psi(t)$ weakly, for every g in $\mathfrak{D}$
$$(\psi(t),Hg) = \lim_{p\to\infty}(\psi_p(t),Hg).$$

For all p
$$|\,(\psi_p(t),Hg)\,| = |\,(H\psi_p(t),g)\,|$$
$$\leqslant \|\,H\phi\,\|\,\|\,g\,\|,$$

so that
$$|\,(\psi(t),Hg)\,| \leqslant \|\,H\phi\,\|\,\|\,g\,\|$$

for all g in $\mathfrak{D}$. $\psi(t)$ is therefore in $\mathfrak{D}^*$ and
$$\begin{aligned}(H^*\psi(t),g) &= (\psi(t),Hg)\\ &= \lim\,(\psi_p(t),Hg)\\ &= \lim\,(H\psi_p(t),g)\\ &= -i(\chi(t),g).\end{aligned}$$

Since $\mathfrak{D}$ is dense in $\mathfrak{H}$ it follows that
$$H^*\psi(t) = -i\chi(t).$$

Hence $\psi(t)$ is a solution of (1).

THEOREM II. *If H is a self-adjoint operator, the equations*
$$\frac{1}{i}\frac{d\psi}{dt} = H\psi, \quad \psi(0) = \phi \tag{1 A}$$

have a unique solution for every ϕ in $\mathfrak{D}$ and for all t.

Since $H^* = H$, a solution of the weak form of (1 A) must exist by theorem I. We shall show that a solution of the weak form of (1 A) is a solution of the strong form, and that the solution is unique: it is of some importance to notice that the assumption that H is self-adjoint is used only to prove the existence of a solution of the weak form of (1 A), and is not made use of in the subsequent stages of this proof.

It was shown in the proof of theorem I that

$$\left\| \frac{\delta\psi}{\delta t} \right\| = \| \chi(t) \| \leqslant \| H\phi \| ;$$

hence by lemma B the solution of the weak form of (1 A) is a solution of (1 A) almost everywhere. The points at which (1 A) is satisfied are all those at which $[\psi(t+h)-\psi(t)]/h$ tends strongly to a limit, that is, those and only those points at which

$$\left\| \frac{\psi(t+h)-\psi(t)}{h} - \frac{\psi(t+h')-\psi(t)}{h'} \right\| \to 0$$

as h, $h' \to 0$ independently. The element inside the modulus sign is itself a solution of (1 A) (with a different initial value), and so by lemma 1 the value of the modulus depends only on h and h', not on t. If the differential coefficient exists in the strong sense for one t, it must therefore do so for all t: hence, since it exists for almost all t, it exists for all t, so that (1 A) is satisfied everywhere in the strong sense.

To prove the uniqueness of the solution, let $\psi_1(t)$ and $\psi_2(t)$ be two solutions of (1 A). Then $\psi_3(t) = \psi_1(t) - \psi_2(t)$ is a solution of the differential equation in (1 A) with the initial value $\psi_3(0) = 0$. By lemma 1, $\| \psi_3(t) \|$ is constant, hence $\psi_3(t) = 0$ for all t, and so the solution is unique.

In theorem I it was only possible to prove that there exists a subsequence $\psi_p(t)$ of the elements

$$\psi_N(t) = \{e^{itE_N H E_N}\}\phi \quad (N = 1, 2, 3, \ldots)$$

which tends weakly to $\psi(t)$. If H is self-adjoint, it can be shown that the sequence $\psi_N(t)$ $(N = 1, 2, 3, \ldots)$ tends strongly to $\psi(t)$.

In the first place, if $\{p\}$ is a subsequence of $\{N\}$ for which $\psi_r(t)$ tends weakly to $\psi(t)$, then since $\| \psi_N(t) \| = \| \psi(t) \|$ for all N,

$$\begin{aligned} \| \psi(t) - \psi_r(t) \|^2 &= 2 \| \psi(t) \|^2 - 2\Re(\psi(t), \psi_p(t)) \\ &\to 0, \end{aligned}$$

because of weak convergence. Thus any subsequence of the $\psi_N(t)$ which tends weakly to $\psi(t)$ tends strongly to that limit.

If, now, $\psi_N(t)$ does not tend strongly to $\psi(t)$, there is a number $\epsilon > 0$, and a subsequence $\{r\}$, $r \to \infty$, of $\{N\}$, such that

$$\| \psi(t) - \psi_r(t) \| > \epsilon$$

for all r. As in the proof of theorem I, it follows that we can choose a subsequence $\{s\}$ of the sequence $\{r\}$ for which $\psi_s(t)$ tends weakly to a solution of (1 A). Since $\psi(t)$ is the only solution of (1 A), $\psi_s(t)$ tends weakly to $\psi(t)$: hence $\psi_s(t)$ must tend strongly to $\psi(t)$, and so the inequality above cannot hold for sufficiently large values of s. It follows that there can be no subsequence of $\{N\}$ with the properties required of the subsequence $\{r\}$, and therefore that $\psi_N(t)$ tends strongly to $\psi(t)$.

This gives us a new definition of the operator e^{itH} for a self-adjoint operator:

If $\mathfrak{M}_n$ is a sequence of subspaces of $\mathfrak{H}$, such that H is bounded in $\mathfrak{M}_n$ for each n, such that $\mathfrak{M}_n \subset \mathfrak{M}_{n+1}$ for all n, and such that the least closed linear manifold which contains all the $\mathfrak{M}_n$ is $\mathfrak{H}$, and if E_n is the projection operator corresponding to $\mathfrak{M}_n$, then for every t and ϕ the elements

$$\psi_n(t) = e^{itE_nHE_n}\phi = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} (E_nHE_n)^r \phi$$

tend strongly to an element $\psi(t)$ as $n \to \infty$: we can write

$$\psi(t) = e^{itH}\phi.$$

$\| \psi(t) \| = \| \phi \|$, and e^{-itH} is the inverse of e^{itH}, so that e^{itH} is a unitary operator.

It has been shown that subspaces with the properties required of the $\mathfrak{M}_n$ exist, and the statements concerning the elements $\psi_n(t)$ have been proved for elements ϕ in $\mathfrak{D}$. The extension to any element ϕ in $\mathfrak{H}$ comes immediately by considering a sequence ϕ_r of elements of $\mathfrak{D}$ which tends to ϕ. $e^{itH}\phi$ may exist even if $H\phi$ and $H\psi(t)$ do not.

In general, whenever (1 A) has a solution, that solution is unique. For operators which are not self-adjoint, no solution of (1 A) may exist, and while there must be at least one solution of (1), this solution is not unique, in general.

If $\psi(t)$ is a solution of (1) and H is not self-adjoint,

$$\begin{aligned} \frac{d}{dt} \| \psi(t) \|^2 &= (iH^*\psi, \psi) + (\psi, iH^*\psi) \\ &= i\{(H^*\psi, \psi) - (\psi, H^*\psi)\} \\ &= -2\mathfrak{Im}(H^*\psi, \psi). \end{aligned} \tag{6}$$

Hence $\| \psi(t) \|^2$ is not necessarily constant: it will decrease for increasing t if $\mathfrak{Im}(H^*\psi, \psi)$ is positive (i.e. if ψ is in the $\mathfrak{C}^+$ class of H) and will increase if $\mathfrak{Im}(H^*\psi, \psi)$ is negative (i.e. if ψ is in the $\mathfrak{C}^-$ class of H). Thus the basis of the proof of uniqueness falls away.†

4. *The structure of maximal operators*

The properties of maximal operators, for which only one of the $\mathfrak{C}^+$ or $\mathfrak{C}^-$ classes exists, will be considered in detail. It will be assumed in what follows that only the $\mathfrak{C}^-$ class exists for the operator H. Operators for which the $\mathfrak{C}^+$ class alone exists can be brought within the scope of the reasoning by considering their negative, or by considering values of t of opposite sign to those considered in the sequel.

If only the $\mathfrak{C}^-$ class exists, then from (6)

$$\frac{d}{dt} \| \psi(t) \|^2 \geqslant 0, \tag{7}$$

the sign of equality holding if and only if ψ is in $\mathfrak{D}$. Hence $\| \psi(t) \|$ increases with t. The argument of theorem II can therefore be used to show that the solution of (1) is unique for $t < 0$.

In the proof of theorem I, a solution of (1) was constructed as the weak limit of a sequence $\{\psi_p(t)\}$, which satisfied $\| \psi_p(t) \| = \phi$ for all p; this implies that the solution $\psi(t)$ satisfies

$$\| \psi(t) \|^2 \leqslant \| \phi \|^2. \tag{8}$$

There is therefore one solution of (1) for $t > 0$ (as well as for $t < 0$) which satisfies (8). From (7), $\| \psi(t) \|$ cannot decrease as t increases, hence it must remain constant for $t \geqslant 0$. It then follows from (6) that this $\psi(t)$ is in $\mathfrak{D}$ for all t, and is therefore a solution of the weak form of (1 A). The argument of theorem II then applies to show that this solution is a solution of the strong form of (1 A), and that it is a unique solution. We therefore get the following:

THEOREM III. *If H is a maximal operator with empty $\mathfrak{C}^+$ class, a unique solution of* (1 A) *exists for all $t \geqslant 0$ for all ϕ in $\mathfrak{D}$. A unique solution of* (1) *exists for all $t < 0$ and all ϕ in $\mathfrak{D}$.*

The operator which transforms an element ϕ of $\mathfrak{D}$ into the solution $\psi(t)$ of (1 A) for $t \geqslant 0$ is a bounded, in fact an isometric, operator. Its domain can therefore be extended to include all $\mathfrak{H}$, since $\mathfrak{D}$ is dense everywhere in $\mathfrak{H}$.

† See Stone, ref. (2), Chap. IX, particularly pp. 343–344. The sign of $\mathfrak{Im}(H^*\psi, \psi)$ is used as the means of defining the $\mathfrak{C}^+$ and $\mathfrak{C}^-$ classes, in this paper, so no reference to the resolvent operator is involved.

This operator will be denoted by $U(t)$. Similarly, the operator which transforms ϕ in $\mathfrak{D}$ into the solution of (1) for $t \leqslant 0$ is bounded, and so its closed extension includes all $\mathfrak{H}$ in its range. This closed operator will be denoted by $V(t)$ $(t \leqslant 0)$.

The arguments used above in defining e^{itH} for self-adjoint H all apply here equally for $t \geqslant 0$, and we conclude that for every ϕ in $\mathfrak{H}$ the elements $e^{itE_nHE_n}\phi$ $(t \geqslant 0)$ tend strongly to $U(t)\phi$.

For $t<0$, it is obvious that the sequence $\psi_p(t)$ will converge only weakly to $\psi(t)$ (except for those elements for which it happens that $\|\psi(t)\| = \|\phi\|$ for a particular $t<0$). However, we can prove that the sequence $\psi_N(t)$ as a whole converges weakly to $\psi(t)$. For, if this is not the case, there exists an element g, an $\epsilon>0$, and a subsequence $\{r\}$ of the integers such that for every r

$$(\psi(t)-\psi_r(t), g) > \epsilon.$$

From $\{r\}$ a subsequence $\{s\}$ with $\psi_s(t)$ converging weakly to a solution of (1) can be chosen as in the proof of theorem I. Since $\psi(t)$ is the unique solution of (1), $\psi_s(t)$ must converge weakly to $\psi(t)$, and therefore no sequence with the properties required of $\{r\}$ can exist.

For any ϕ in $\mathfrak{H}$ and $t<0$, write $V(t)\phi = \psi(t)$. Choose a sequence $\{\phi_i\}$ from $\mathfrak{D}$, $\phi_i \to \phi$, and write $V(t)\phi_i = \psi_i(t)$,

$$e^{itE_nHE_n}\phi = \psi_{0,n}(t), \quad e^{itE_nHE_n}\phi_i = \psi_{i,n}(t).$$

Then

$$(\psi_{0,n}(t)-\psi(t), g) = (\psi_{0,n}(t)-\psi_{i,n}(t), g) + (\psi_{i,n}(t)-\psi_i(t), g) + (\psi_i(t)-\psi(t), g).$$

The first and third terms on the right are each less than $\|\phi-\phi_i\| \, \|g\|$, and so can be made less than $\frac{1}{3}\epsilon$ by taking i sufficiently large. The second terms tend to zero as $n \to \infty$ because of the weak convergence of the $\psi_{i,n}(t)$ to $\psi_i(t)$; having chosen i we can choose n to make this term less than $\frac{1}{3}\epsilon$. Hence there exists $n_0(\epsilon)$ such that for $n > n_0(\epsilon)$

$$(\psi_{0,n}(t)-\psi(t), g) < \epsilon,$$

so that $\psi_{0,n}(t)$ is weakly convergent to $\psi(t)$.

We get then:

If H is a maximal operator, with empty $\mathfrak{C}^+$ class, and $\mathfrak{M}_n$, E_n are defined as in theorem I, then, for every ϕ in $\mathfrak{H}$, $e^{itE_nHE_n}\phi \to U(t)\phi$ strongly, for $t \geqslant 0$, $e^{itE_nHE_n}\phi \to V(t)\phi$ weakly, for $t \leqslant 0$. We could therefore write $U(t) = e^{itH}$ for $t \geqslant 0$, $V(t) = e^{itH}$ for $t \leqslant 0$.

It is clear that for all positve t and τ

$$U(t)\,U(\tau) = U(t+\tau).$$

From lemma 1 $(U(t)f, U(t+\tau)g)$ is independent of t for $t \geqslant 0$, $t+\tau \geqslant 0$. From lemma D, if $t < 0$, $t+\tau \geqslant 0$ and f and g are in $\mathfrak{D}$,

$$\frac{d}{dt}(V(t)f, U(t+\tau)g) = i\{(H^*V(t)f, U(t+\tau)g) - (V(t)f, HU(t+\tau)g)\}$$
$$= 0.$$

By considering sequences of elements in $\mathfrak{D}$ converging to arbitrary elements in $\mathfrak{H}$, this can be extended to any elements in $\mathfrak{H}$. Thus we have

LEMMA 2. *For any elements f and g in $\mathfrak{H}$, $(U(t)f, U(t+\tau)g)$ is independent of t for $t \geqslant 0, t+\tau \geqslant 0$ and $(V(-t)f, U(-t+\tau)g)$ is independent of t for $\tau \geqslant t \geqslant 0$.*

In particular, if we put successively $t = 0$ and $t = \tau$ we get

$$(f, U(\tau)g) = (V(-\tau)f, g),$$

so that we have the corollary:

LEMMA 2A. *$V(-t)f = U^*(t)f$ for every f in $\mathfrak{H}$, $t > 0$.*

A further corollary is

LEMMA 2B. *If t, τ are positive,*

$$V(-t)\,V(-\tau) = V(-t-\tau).$$

It is required to prove that for ϕ in $\mathfrak{H}$

$$V(-t)\,V(-\tau)\phi = V(-t-\tau)\phi.$$

On applying lemma 2, we get

$$(V(-t)\,V(-\tau)\phi, g) = (V(-\tau)g, U(t)g) = (\phi, U(t+\tau)g),$$

and $$(V(-t-\tau)\phi, g) = (\phi, U(t+\tau)g),$$

if g is any element in $\mathfrak{H}$. Hence the corollary.

LEMMA 3. *H and $U(t)$, H^* and $V(-t)$ commute if $t \geqslant 0$; more precisely, if ϕ is in $\mathfrak{D}$, so is $U(t)\phi$ and*

$$HU(t)\phi = U(t)H\phi;$$

if ϕ is in $\mathfrak{D}^*$, so is $V(-t)\phi$ and

$$H^*V(-t)\phi = V(-t)H^*\phi.$$

For ϕ in $\mathfrak{D}$ the left-hand side of the first equation exists and

$$HU(t)\phi = -i\frac{d}{dt}U(t)\phi$$
$$= -i\lim_{\tau\to 0}\frac{[U(t+\tau)-U(t)]\phi}{\tau}$$
$$= -i\lim_{\tau\to 0}U(t)\,\frac{U(\tau)-1}{\tau}\phi.$$

Since
$$\lim \frac{U(\tau)-1}{\tau}\phi = iH\phi,$$
and since $U(t)$ is a continuous closed transformation,
$$\begin{aligned} HU(t)\phi &= U(t)\lim \frac{U(\tau)\phi-\phi}{i\tau} \\ &= U(t)H\phi. \end{aligned}$$

For the second part, let ϕ be in $\mathfrak{D}^*$, g in $\mathfrak{D}$. Then
$$\begin{aligned} (V(-t)\phi, Hg) &= (\phi, U(t)Hg) \\ &= (\phi, HU(t)g) \\ &= (H^*\phi, U(t)g), \end{aligned}$$
on applying successively lemma 2A and the first part of lemma 3. Hence
$$|(V(-t)\phi, Hg)| \leqslant \| H^*\phi \| \, \| U(t)g \| = \| H^*\phi \| \, \| g \|,$$
so that $V(-t)\phi$ is in $\mathfrak{D}^*$.

We can therefore put in the equation above
$$(V(-t)\phi, Hg) = (H^*V(-t)\phi, g),$$
and, by lemma 2A, $\quad (H^*\phi, U(t)g) = (V(-t)H^*\phi, g),$

and the equality of the left-hand sides of these expressions gives
$$H^*V(-t)\phi = V(-t)H^*\phi.$$

It may be mentioned that $U(t)$ does not transform an element of $\mathfrak{D}^*$ which is not in $\mathfrak{D}$ into an element of $\mathfrak{D}^*$.

LEMMA 4. *If ϕ is any element of $\mathfrak{D}^*$, $V(t)\phi$ is an element of $\mathfrak{D}^*$ and satisfies* (1) *for any $t \leqslant 0$; also*
$$\frac{d}{dt}\| V(t)\phi \|^2 = -2\mathfrak{Im}(H^*V(t)\phi, V(t)\phi).$$
In particular, for $t = 0$,
$$\frac{d}{dt}\| V(t)\phi \|^2 = -2\mathfrak{Im}(H^*\phi, \phi),$$
so that at $t = 0$, $\| V(t)\phi \|$ decreases at a non-zero rate as t decreases, unless ϕ is in $\mathfrak{D}$.

If g is any element of $\mathfrak{D}$ and $t > 0$,
$$\begin{aligned} \left(\frac{V(-t-h)-V(-t)}{h}\phi, g\right) &= \left(V(-t)\phi, \frac{U(h)g-g}{h}\right) \\ &\to (V(-t)\phi, iHg) \\ &= -(iH^*V(-t)\phi, g), \end{aligned}$$
which proves that (1) is satisfied. The rest follows immediately from lemma D.

LEMMA 5. *(a) If ϕ is in $\mathfrak{D}$, then at $t = 0$*

$$\frac{d}{dt}\| \phi - U(t)\phi \|^2 = 0, \quad \frac{d^2}{dt^2}\| \phi - U(t)\phi \|^2 = 2 \| H\phi \|^2.$$

(b) If ϕ is in $\mathfrak{D}^*$, then as $t \to 0$ through positive values,

$$\| V(-t)\phi - \phi \|^2 = O(t^2).$$

(c) For any ϕ in $\mathfrak{H}$, $\quad \| V(-t)\phi - \phi \|^2 = o(1),$

as $t \to 0$ through positive values.

(d) If ϕ is in $\mathfrak{D}^*$, $V(t)\phi$ $(t < 0)$ satisfies (1) in the strong sense everywhere.

(e) If ϕ is in $\mathfrak{D}^*$, then at $t = 0$

$$\frac{d^2}{dt^2}\| V(-t)\phi - \phi \|^2 = 2 \| H^*\phi \|^2.$$

(*a*) follows from the fact that $\{U(t)\phi - \phi\}/t$ tends strongly to $iH\phi$ as $t \to 0$, and (*b*) from the fact that $\{V(-t)\phi - \phi\}/t$ tends weakly to $-iH^*\phi$ as $t \to 0$. To prove (*c*), let $\{\phi_n\}$ be a sequence of elements of $\mathfrak{D}^*$ tending to ϕ. Then

$$\{V(-t)\phi - \phi\} = \{V(-t)\phi_n - \phi_n\} + [V(-t) - I](\phi - \phi_n).$$

We can find n such that $\| \phi - \phi_n \| \leqslant \tfrac{1}{4}\epsilon$, and then, from 5 (*b*), t such that $\| V(-t)\phi_n - \phi_n \| \leqslant \tfrac{1}{2}\epsilon$. Then

$$\| V(-t)\phi - \phi \| \leqslant \epsilon.$$

It was proved in lemma 4 that $V(t)\phi$ satisfies (1) in the weak sense: we have for all $t > 0$

$$\frac{\delta V(-t)\phi}{\delta t} = -iH^*V(-t)\phi = -iV(-t)H^*\phi,$$

and $\| V(-t)H^*\phi \| \leqslant \| H^*\phi \|$; thus by lemma C it is sufficient to show that $V(-t)H^*\phi$ is a continuous function of t. By lemma 2 (*b*)

$$\begin{aligned} \| V(-t-h)H^*\phi - V(-t)H^*\phi \| &= \| V(-t)\{V(-h)H^*\phi - H^*\phi\} \| \\ &\leqslant \| V(-h)H^*\phi - H^*\phi \| \\ &= o(1) \end{aligned}$$

from lemma 5 (*c*).

5 (*e*) follows immediately from 5 (*d*), since the latter implies that

$$\frac{V(-t)\phi - \phi}{t} \to -iH^*\phi, \quad \text{strongly}.$$

On the basis of these theorems we can determine the main properties of the operators U and V. Both have $\mathfrak{H}$ as domain. The range of $U(t)$ cannot determine $\mathfrak{H}$, for if it did $U(t)$ would be a unitary transformation, and the

inverse transformation $\{U(t)\}^{-1}$ would have $\mathfrak{H}$ as domain. If that were so, the function

$$\psi(t) = [U(-t)]^{-1}\phi \quad (t<0),$$

would give a solution of (1) for negative t and any ϕ of $\mathfrak{D}^*$ satisfying $\| \psi(t) \| = \| \phi \|$. Since (1) has a unique solution which is such that $\| \psi(t) \|$ decreases when ϕ is in $\mathfrak{C}^-$ this is impossible.

The range of $U(t)$ is a closed linear manifold, a subset of $\mathfrak{H}$ which will be denoted by $\mathfrak{R}(t)$. The elements $U(t)\phi$ for ϕ in $\mathfrak{D}$ form a linear subspace of $\mathfrak{R}(t)$ which determines $\mathfrak{R}(t)$: this linear subspace will be designated by $\mathfrak{D}(t)$. The projection operator corresponding to $\mathfrak{R}(t)$ will be designated by $E(t)$.

It is easily seen that for any ϕ $(t \geqslant \tau \geqslant 0)$

$$V(-\tau)\, U(t)\phi = U(t-\tau)\phi. \tag{9}$$

In particular,
$$V(-t)\, U(t) = I. \tag{10}$$

From (9), if $t \geqslant \tau \geqslant 0$, $\quad U(\tau)\, V(-\tau)\, U(t)\phi = U(t)\phi,$

and in particular $\quad U(t)\, V(-t)\, U(t) = U(t);$

so, for any element ϕ in $\mathfrak{R}(t)$,

$$U(t)\, V(-t)\phi = \phi. \tag{11}$$

As $[U(t)\, V(-t)]$ transforms any ϕ into an element of $\mathfrak{R}(t)$, (11) gives

$$[U(t)\, V(-t)]^2 = U(t)\, V(-t).$$

Moreover, it follows from lemma 2A that $U(t)\, V(-t)$ is symmetric, and so $[U(t)\, V(-t)]$ must be a projection operator.† Its range is included in $\mathfrak{R}(t)$, and as it transforms any element of $\mathfrak{R}(t)$ into itself, its range must coincide with $\mathfrak{R}(t)$. Hence

$$U(t)\, V(-t) = E(t). \tag{12}$$

Since $U(t)$ is isometric, $\quad \| E(t)\phi \| = \| V(-t)\phi \|. \quad (13)$

The rate of change of $\| E(t)\phi \|$ can therefore be deduced from lemma 4. In particular, if ϕ is not in $\mathfrak{D}$ but is in $\mathfrak{C}^-$,

$$\frac{d}{dt} \| E(t)\phi \|^2 < 0. \tag{14}$$

Differential independence

The following definitions will be introduced here:

Elements $f_1, f_2, \ldots, f_k$ in $\mathfrak{D}^*$ are said to be differentially independent with respect to U if there exists no set of complex numbers $a_1, \ldots, a_k$ such that, if

$$f = \Sigma a_k f_k,$$

then $\quad \| f - U(t)f \|^2 = o(t) \quad$ *as* $\quad t \to 0.$

This will be written $f_1, f_2, \ldots, f_k$ d.i. (U).

† Ref. (2), 70, def. 2·16 and theorem 2·26.

Elements $f_1, f_2, \ldots, f_k$ in $\mathfrak{D}^*$ are said to be differentially independent with respect to $E(t)$ at t if there are no complex numbers $a_1, \ldots, a_k$ such that, if

$$f = \Sigma a_k f_k,$$

then $$\| E(t)f - E(t+\Delta t)f \|^2 = o(\Delta t) \quad \textit{as} \quad \Delta t \to 0.$$

This will be written $f_1, f_2, \ldots, f_k$ d.i. $(E; t)$.

The multiplicity of U is the greatest number of f which can be d.i. (U).

The multiplicity of E at t is the greatest number of f which can be d.i. $(E; t)$.

LEMMA 6. *The multiplicity of E at $t = 0$ is equal to the multiplicity of U.*

For any f, $$U(t)f - f = \{U(t)f - E(t)f\} + \{E(t)f - f\},$$

and since the first term on the right is in $\mathfrak{R}(t)$ and the second in the manifold orthogonal to $\mathfrak{R}(t)$,

$$\| U(t)f - f \|^2 = \| U(t)f - E(t)f \|^2 + \| E(t)f - f \|^2.$$

Hence, if $\| U(t)f - f \|^2$ is $o(t)$, so is $\| E(t)f - f \|^2$. The multiplicity of $U(t)$ is therefore not less than that of $E(t)$ at 0. To prove the multiplicities equal it is sufficient to show that the first term on the right is $o(t)$ for any f in $\mathfrak{D}^*$. The term in brackets is

$$U(t)f - E(t)f = U(t)[f - V(-t)f],$$

so that $$\| U(t)f - E(t)f \|^2 = \| f - V(-t)f \|^2 = O(t^2) \tag{15}$$

from lemma 5 (b), for any f in $\mathfrak{D}^*$.

As a corollary, we see that elements d.i. (U) are d.i. $(E; 0)$, and conversely.

LEMMA 7. *The multiplicity of E at any positive t is equal to that at* 0.

Let the multiplicity of E at t be k, and let $f_1, \ldots, f_k$ be d.i. $(E; t)$. Put $f = \Sigma a_r f_r$, where the a's are arbitrary constants:

$$\begin{aligned}\{E(t)f - E(t+\Delta t)f\} &= U(t)V(-t)f - U(t+\Delta t)V(-t-\Delta t)f \\ &= U(t)[V(-t) - U(\Delta t)V(-\Delta t)V(-t)]f \\ &= U(t)[V(-t) - E(\Delta t)V(-t)]f,\end{aligned}$$

from (12). Thus

$$\| E(t)f - E(t+\Delta t)f \| = \| V(-t)f - E(\Delta t)V(-t)f \|,$$

so that if the f_r are d.i. $(E; t)$ the elements $V(-t)f_k$ are d.i. $(E; 0)$. On the other hand, if the f_r are d.i. $(E; 0)$ and $f = \Sigma a_r f_r$, we get

$$\begin{aligned}[E(t) - E(t+\Delta t)]U(t)f &= [U(t) - U(t+\Delta t)V(-t-\Delta t)U(t)]f \\ &= U(t)[I - U(\Delta t)V(-\Delta t)V(-t)U(t)]f \\ &= U(t)[f - E(\Delta t)f],\end{aligned}$$

from (11) and (12), so the $U(t)f_r$ are d.i. $(E; t)$.

This proves lemma 7. Taken together with lemma 6, it proves that the multiplicity of E at any t is equal to that of U. In future we shall speak simply of the multiplicity of E, since that has been proved independent of t.

The deficiency index of H

The definitions of the deficiency index of a symmetric operator given in the literature depend on the use of the resolvent operator. As we are not employing the theory of the resolvent operator here, it is as well to give an independent definition of deficiency index. Since we are concerned only with maximal operators, we shall consider only this case: it is possible to extend the definition to general symmetric operators.

The deficiency index of a maximal symmetric operator H is the largest number of elements of $\mathfrak{D}^*$ *which can be linearly independent* (mod $\mathfrak{D}$), *i.e. which are such that no element linearly dependent on them belongs to* $\mathfrak{D}$.

If the number of elements linearly independent (mod $\mathfrak{D}$) *is* n *(finite or countably infinite), the index is written* $(0, n)$ *if the* $\mathfrak{C}^-$ *class exists,* $(n, 0)$ *if the* $\mathfrak{C}^+$ *class exists.*

It is evident that this definition is equivalent to the standard definition, as given by von Neumann.

It will now be proved that for maximal H the deficiency index of H is equal to the multiplicity of E or U.

If $f_1, \ldots, f_k$ are d.i. (U), then we can show that $f_1, \ldots, f_k$ are linearly independent (mod $\mathfrak{D}$). For, if they are not linearly independent (mod $\mathfrak{D}$), numbers a_r exist such that $\Sigma a_r f_r$ is in $\mathfrak{D}$. In that case

$$\| f - U(t)f \|^2 = o(t)$$

by lemma 5 (a), so that the f_r are not d.i. (U).

On the other hand, with f as above,

$$\| f - E(t)f \|^2 = \| f \|^2 - \| E(t)f \|^2,$$

so that, at $t = 0$, using lemma 4 and (13),

$$\frac{d}{dt} \| f - E(t)f \|^2 = -\frac{d}{dt} \| E(t)f \|^2$$

$$= -2\mathfrak{Im}(H^*f, f),$$

and this is zero if f is in $\mathfrak{D}$. Thus

$$\| f - E(t)f \|^2 = o(t)$$

if and only if f is in $\mathfrak{D}$. Hence the following theorem.

Theorem IV. *The multiplicity of E is equal to the deficiency index of H. Elements linearly independent* (mod $\mathfrak{D}$) *are* d.i. $(E;0)$ *and conversely.*

Let the deficiency index of H be $(0,n)$ and let $f_1, \ldots, f_n$ be a set (or, if n is infinite, a sequence) of elements of $\mathfrak{D}^*$ linearly independent (mod $\mathfrak{D}$) and such that at $t=0$

$$-\frac{d}{dt}(E(t)f_k, E(t)f_l) = \delta_{kl} = 0 \quad (k \neq l),$$
$$= 1 \quad (k = l). \tag{16}$$

Such elements can be found by a process similar to the Gram-Schmidt orthogonalization process, given any set of elements linearly independent (mod $\mathfrak{D}$). Any f in $\mathfrak{D}^*$ can be put in the form

$$f = \Sigma a_k f_k + g,$$

where g is in $\mathfrak{D}$. It is easy to prove from (12) and lemma 4 that at $t=0$

$$\frac{d}{dt}(E(t)f_k, E(t)g) = 0,$$

so that
$$\| f - E(\Delta t)f \|^2 = \Delta t \Sigma |a_k|^2 + o(t), \tag{17}$$

i.e. at $t=0$,
$$\frac{d}{dt}\| E(t)f \|^2 = -\Sigma |a_k|^2. \tag{17'}$$

The expression of an arbitrary element in terms of the operators $E(t)$ and $U(t)$

The projection operators $E(t)$ correspond to manifolds $\mathfrak{R}(t)$ such that $\mathfrak{R}(t) \supseteq \mathfrak{R}(t')$ if $t < t'$. As $t \to \infty$, $\mathfrak{R}(t)$ tends to a limit, say $\mathfrak{R}(\infty)$, the space of all elements contained in every $\mathfrak{R}(t)$. $E(t)$ tends to the corresponding $E(\infty)$. $\mathfrak{R}(\infty)$ may be empty, or a Euclidean space of a finite number of dimensions, or a Hilbert space. Since

$$E(t)\,E(t') = E(t), \quad \text{if} \quad t > t',$$
$$E(t)\,[E(v) - E(u)] = 0, \quad \text{if} \quad t > u > v,$$

so that $\mathfrak{R}(t)$ is orthogonal to $[\mathfrak{R}(v) \ominus \mathfrak{R}(u)]$ if $t > u > v$. $\mathfrak{R}(\infty)$ is therefore orthogonal to $[\mathfrak{R}(v) \ominus \mathfrak{R}(u)]$ for any u and any finite v, $v < u$.

$\mathfrak{R}(u)$ is the space of all elements $U(u)\phi$, so that $U(t)$ transforms $\mathfrak{R}(u)$ into $\mathfrak{R}(u+t)$. Let ϕ be an element of $[\mathfrak{R}(v) \ominus \mathfrak{R}(u)]$ $(0 < v < u)$. $U(t)$ transforms ϕ into an element $U(t)\phi$ of $\mathfrak{R}(v+t)$. ϕ is orthogonal to $\mathfrak{R}(u)$, so that if g is any element of $\mathfrak{R}(u)$, $(\phi, g) = 0$. Hence $(U(t)\phi, U(t)g) = 0$. Any element of $\mathfrak{R}(u+t)$ can be expressed in the form $U(t)g$ with g in $\mathfrak{R}(u)$; hence $U(t)\phi$ is orthogonal to $\mathfrak{R}(u+t)$. It follows that $U(t)$ transforms $[\mathfrak{R}(v) \ominus \mathfrak{R}(u)]$ into $[\mathfrak{R}(v+t) \ominus \mathfrak{R}(u+t)]$.

If $t < u$, $V(-t)$ is isometric in $\mathfrak{R}(u)$ and transforms it into $\mathfrak{R}(u-t)$ (see (9)). Hence $U(t)$ and $V(-t)$ transform $\mathfrak{R}(\infty)$ into itself, and, in $\mathfrak{R}(\infty)$, $V(-t)$ is isometric and is the inverse of $U(t)$, so that $U(t)$ is isometric and has an isometric inverse in the space $\mathfrak{R}(\infty)$. $U(t)$ must therefore be unitary in $\mathfrak{R}(\infty)$.

For any ϕ of $\mathfrak{D}$ lying in $\mathfrak{R}(\infty)$, the equation (1)

$$\frac{1}{i}\frac{\delta\psi}{\delta t} = H^*\psi, \quad \psi(0) = \phi$$

has the solution $\psi(t) = U(t)\phi$ for $t \geqslant 0$ and $\psi(t) = V(t)\phi$ for $t \leqslant 0$. For all t, $\| \psi(t) \| = \phi$. It follows from (6) that ψ is in $\mathfrak{D}$, so that the solution is a solution of (1 A). Hence (1 A) has a solution for all t ($-\infty < t < \infty$) for all elements of $\mathfrak{D}$ in $\mathfrak{R}(\infty)$, and this solution lies in $\mathfrak{R}(\infty)$. Consequently, $\mathfrak{R}(\infty)$ reduces H, and H is a self-adjoint operator in $\mathfrak{R}(\infty)$; this is a consequence of theorem III of ref. (**10**).

Any f can be written

$$f = -\int_0^\infty dE(t)f + E(\infty)f, \tag{18}$$

and so we are led to analyse the operators $E(t)$. By (18), every element of the space $\mathfrak{H} \ominus \mathfrak{R}(\infty)$ is expressed as a sum (or integral) of elements of the spaces $[\mathfrak{R}(n\delta) \ominus \mathfrak{R}(\overline{n+1}\,\delta)]$. We have also seen that for any $\delta > 0$ the operators $U(n\delta)$, $n = 1, 2, \ldots$, transform $\mathfrak{H} \ominus \mathfrak{R}(\delta)$ into the spaces $[\mathfrak{R}(n\delta) \ominus \mathfrak{R}(\overline{n+1}\,\delta)]$. This suggests that any element of $\mathfrak{H} \ominus \mathfrak{R}(\infty)$ can be expressed as a sum or integral of elements $U(n\delta)[I - E(\delta)]f$. We shall give this idea a precise meaning, and deduce an analytic form for the expression of any element of $\mathfrak{H}$, which has interesting applications and throws considerable light on the structure of the operators H and $U(t)$.

Let $f_1, f_2, \ldots, f_k, \ldots$ be a set or sequence of elements d.i. (U). We shall suppose these to be normalized so that (16) is satisfied. With each element f_k we associate an integral, defined as follows.

Let (a, b) be any finite interval, $p(t)$ a continuous function defined in it. Let (a, b) be divided into subintervals, $a = t_0 < t_1 < \ldots < t_n = b$, with $\max(t_{r+1} - t_r) = \delta$. Write $(t_{r+1} - t_r) = \Delta t_r$, and consider the sum

$$S = \sum_{r=0}^{n-1} p(t_r)\, U(t_r)\, [I - E(\Delta t_r)] f_k.$$

The contribution to this sum from the interval (t_r, t_{r+1}) lies in the space $[\mathfrak{R}(t_r) \ominus \mathfrak{R}(t_{r+1})]$ and so is orthogonal to the contributions from the other intervals. We have, therefore,

$$\| S \|^2 = \sum_{r=0}^{n-1} | p(t_r) |^2 \| f_k - E(\Delta t_r) f_k \|^2.$$

In order to show that S tends to a limit as δ tends to 0, let us consider the effect of inserting subdivisions in the intervals. Let (t_r, t_{r+1}) be subdivided by divisions $t_r = t'_0 < t'_1 < \ldots < t'_m = t_{r+1}$, and write $(t'_{i+1} - t'_i) = \Delta t'_i$. The contribution to the new sum, S', from the interval (t_r, t_{r+1}) is

$$S'_r = \sum_{i=0}^{m-1} p(t'_i)\, U(t'_i)\, [I - E(\Delta t'_i)]\, f_k,$$

whereas the contribution to S was

$$\begin{aligned} S_r &= p(t_r)\, U(t_r)\, [I - E(\Delta t_r)]\, f_k \\ &= p(t_r)\, U(t_r) \sum_{i=0}^{m-1} [E(t'_i - t_r) - E(t'_{i+1} - t_r)]\, f_k. \end{aligned}$$

The contributions to each of these sums from the interval (t'_i, t'_{i+1}) lie in $[\mathfrak{R}(t'_i) \ominus \mathfrak{R}(t'_{i+1})]$ and so are orthogonal to the contributions from any other interval. Writing these contributions S_{ir} and S'_{ir}, we have

$$\| S_r - S'_r \|^2 = \sum_{i=0}^{m-1} \| S_{ir} - S'_{ir} \|^2,$$

where

$$\begin{aligned} S'_{ir} - S_{ir} &= p(t'_i)\, U(t'_i)\, [I - E(\Delta t'_i)]\, f_k - p(t_r)\, U(t_r)\, [E(t'_i - t_r) - E(t'_{i+1} - t_r)]\, f_k \\ &= [p(t'_i) - p(t_r)]\, U(t'_i)\, [I - E(\Delta t'_i)]\, f_k \\ &\qquad + p(t_r)\, \{U(t'_i)\, [I - E(\Delta t'_i)] - U(t_r)\, [E(t'_i - t_r) - E(t'_{i+1} - t_r)]\}\, f_k \\ &= [p(t'_i) - p(t_r)]\, \sigma_1 + p(t_r)\, \sigma_2, \end{aligned}$$

say. Next, writing $(t'_i - t_r) = u_i$,

$$\begin{aligned} \sigma_2 &= U(t_r)\, \{U(u_i)\, [I - E(\Delta t'_i)] - E(u_i) + E(u_i + \Delta t'_i)\}\, f_k \\ &= U(t'_i)\, \{I - E(\Delta t'_i) - V(-u_i) + U(\Delta t'_i)\, V(-u_i - \Delta t'_i)\}\, f_k \\ &= U(t'_i)\, \{I - E(\Delta t'_i) - V(-u_i) + E(\Delta t'_i)\, V(-u_i)\}\, f_k \\ &= U(t'_i)\, [I - E(\Delta t'_i)]\, [I - V(-u_i)]\, f_k. \end{aligned}$$

For any element ϕ of $\mathfrak{D}^*$, we have, from lemma 3, lemma 4 and (13),

$$\begin{aligned} -\frac{d}{dt} \| E(t)\phi \|^2 &= | 2\mathfrak{Im}(H^* V(-t)\phi, V(-t)\phi) | \\ &\leqslant 2\, | (H^* V(-t)\phi, V(-t)\phi) | \\ &\leqslant 2\, \| H^*\phi \|\, \| \phi \|, \end{aligned}$$

so that on integrating between 0 and $\Delta t'_i$ we get

$$\begin{aligned} & \| [I - E(\Delta t'_i)]\, [I - V(-u_i)]\, f_k \|^2 \\ &\qquad \leqslant 2\, \| H^* [I - V(-u_i)]\, f_k \|\, \| [I - V(-u_i)]\, f_k \|\, \Delta t'_i \\ &\qquad = 2 \Delta t'_i\, \| [I - V(-u_i)]\, H^* f_k \|\, \| [I - V(-u_i)]\, f_k \|, \end{aligned}$$

and by lemma 5 (c) $\qquad \| [I - V(-u_i)]\phi \| = o(1),$

as $u_i \to 0$, hence as $\Delta(t_r) \to 0$, so that $\| \sigma_2 \|^2 = o(\Delta t'_i)$ uniformly in i as $(t_{r+1} - t_r) \to 0$. As for the other terms in $S_{ir} - S'_{ir}$,

$$\begin{aligned} \| \sigma_1 \|^2 &= \| [I - E(\Delta t'_i)] f_k \|^2 \\ &= \Delta t'_i + o(\Delta t'_i), \end{aligned}$$

by (17), and $| p(t'_i) - p(t_r) | \to 0$ as $(t_{r+1} - t_r) \to 0$. Hence, altogether,

$$\| S_{ir} - S'_{ir} \|^2 = o(\Delta t'_i),$$

and therefore

$$\begin{aligned} \| S_r - S'_r \|^2 &= o(\Sigma \Delta t'_i) \\ &= o(t_{r+1} - t_r), \end{aligned}$$

as $(t_{r+1} - t_r) \to 0$. It follows, by the usual arguments, that S converges strongly to a limit as the maximum length of the intervals $(t_{r+1} - t_r)$ tends to zero. The limit of S is the integral we wish to define. It has a certain resemblance to the invariant integral for a group.

The limit of S will be written in the forms

$$\int_a^b p(t)\, d(U; k; t), \qquad \int_a^b p(t)\, d(U; f_k; t),$$

to indicate the dependence on U, $p(t)$ and f_k. In the sequel, when the f_k is obvious and no confusion will result, we shall write $(U; t)$ instead of $(U; k; t)$.

It is easy to see that $\qquad \lim \| S \|^2 = \int_a^b | p(t) |^2 \, dt,$

and hence that if

$$\phi = \int_a^b p(t)\, d(U;\, k;\, t), \quad \psi = \int_a^b q(t)\, d(U;\, k;\, t),$$

then

$$\| \phi \|^2 = \int_a^b | p(t) |^2 \, dt,$$

and

$$(\phi, \psi) = \int_a^b p(t)\, \overline{q(t)}\, dt.$$

The extension of the integral to a general function of $L^2(a, b)$ is immediate. Choose a sequence of functions $p_n(t)$ converging in mean square to any $p(t)$ of $L^2(a, b)$. Then if

$$\phi_n = \int_a^b p_n(t)\, d(U;\, k;\, t),$$

we have

$$\| \phi_m - \phi_n \|^2 = \int_a^b | p_m - p_n |^2 \, dt,$$

which tends to zero as m, n tend independently to infinity, and hence ϕ_n converges strongly to an element ϕ. This element ϕ is the meaning we attach to $\int_a^b p(t)\,d(U;k;t)$. The restriction that (a, b) be finite can be removed in the same way. If $p(t)$ is any function of $L^2(0,\infty)$, and

$$\phi(T) = \int_0^T p(t)\,d(U;\,k;\,t),$$

then

$$\| \phi(T') - \phi(T) \|^2 = \int_T^{T'} | p(t) |^2\,dt,$$

and since this tends to zero as T, $T' \to \infty$ independently, $\phi(T)$ converges strongly to a limit as $T \to \infty$. Hence for every $p(t)$ of $L^2(0,\infty)$, the integral

$$\int_0^\infty p(t)\,d(U;\,k;\,t)$$

defines a definite element ϕ of $\mathfrak{H}$, such that

$$\| \phi \|^2 = \int_0^\infty | p(t) |^2\,dt.$$

Consider now two elements

$$\phi = \int_a^b p(t)\,d(U;\,k;\,t), \quad \psi = \int_a^b q(t)\,d(U;\,l;\,t).$$

Suppose for the moment that $p(t)$ and $q(t)$ are continuous and a and b finite. Then $(\phi,\ \psi)$ is the limit of

$$\begin{aligned}\Sigma\, p(t_r)\, \overline{q(t_r)}\, (U(t_r)\,[I - E(\Delta t_r)]f_k,\ U(t_r)\,[I - E(\Delta t_r)]f_l) \\ = \Sigma\, p(t_r)\, \overline{q(t_r)}\, ([I - E(\Delta t_r)]f_k,\ [I - E(\Delta t_r)]f_l) \\ = \Sigma\, p(t_r)\, \overline{q(t_r)}\, \{(f_k, f_l) - (E(\Delta t_r)f_k,\ E(\Delta t_r)f_l)\}.\end{aligned}$$

By (16), the expression in brackets is

$$\begin{aligned} &\Delta t_r + o(\Delta t_r) \quad \text{if} \quad k = l, \\ &o(\Delta t_r) \qquad \text{if} \quad k \neq l, \end{aligned}$$

and it follows that

$$(\phi, \psi) = 0 \quad (k \neq l).$$

The extension to general p and q of $L^2(0,\infty)$ is immediate. Hence, if $p(t)$ and $q(t)$ are any elements of $L^2(0,\infty)$, and

$$\phi = \int_0^\infty p(t)\,d(U;\,k;\,t), \quad \psi = \int_0^\infty q(t)\,d(U;\,l;\,t),$$

then

$$\begin{aligned}(\phi, \psi) &= \int_0^\infty p(t)\,\overline{q(t)}\,dt \quad \text{if} \quad k = l, \\ &= 0 \qquad\qquad\qquad\quad \text{if} \quad k \neq l. \end{aligned} \tag{19}$$

The next step is to show that any element of $\mathfrak{D}^*$ can be represented in terms of these integrals. The integrand is suggested by (18). We have, for any $t>0$ and for any f in $\mathfrak{D}^*$, from (12)

$$\begin{aligned}[E(t)-E(t+\Delta t)]f &= [U(t)\,V(-t)-U(t+\Delta t)\,V(-t-\Delta t)]f \\ &= U(t)\,[I-E(\Delta t)]\,V(-t)\,f. \end{aligned} \tag{20}$$

Since $V(-t)f$ is in $\mathfrak{D}^*$, we can put

$$V(-t)f = \Sigma c_k(t) f_k + \xi(t), \tag{21}$$

where $\xi(t)$ is in $\mathfrak{D}$. Then

$$\begin{aligned}\| E(t)f \|^2 - \| E(t+\Delta t)\,f\|^2 &= \| E(t)f - E(t+\Delta t)\,f\|^2 \\ &= \| [I-E(\Delta t)]\,V(-t)f\|^2 \\ &= \Delta t \Sigma \,|\, c_k(t)\,|^2 + o(\Delta t),\end{aligned}$$

from (17), so that
$$-\frac{d}{dt}\| E(t)f\|^2 = \Sigma\,|\,c_k(t)\,|^2. \tag{22}$$

On integrating,
$$\| f\|^2 - \| E(\infty)f\|^2 = \Sigma\int_0^\infty |\,c_k(t)\,|^2\,dt, \tag{22a}$$

so that $c_k(t)$ must be $L^2(0,\infty)$ for all k, and $\sum_k \int_0^\infty |\,c_k(t)\,|^2 dt$ must be finite. These facts will be expressed in future by the phrase: *the set* $\{c_k(t)\}$ *is of integrable square*. This phrase implies that the sum of the integrals (or if they are infinite in number, for an operator of infinite deficiency index, the limit of the sum) is finite.

Let functions $\{g_k(t)\}$, $\eta(t)$ be derived in the same way for an arbitrary element g in $\mathfrak{D}^*$. From (22),

$$-\frac{d}{dt}\| E(t)\,(f+\epsilon g)\,\|^2 = \sum_k |\,c_k(t)+\epsilon g_k(t)\,|^2,$$

for any complex number ϵ, and from this we derive

$$-\frac{d}{dt}(E(t)f,\,E(t)\,g) = \sum_k c_k(t)\,\overline{g_k(t)}. \tag{23}$$

If now ϕ is the element of $\mathfrak{H}$ defined by

$$\phi = \sum_k \int_0^\infty c_k(\lambda)\,d(U;\,k;\,\lambda),$$

we shall have
$$E(t)\,\phi = \sum_k \int_t^\infty c_k(\lambda)\,d(U;\,k;\,\lambda), \tag{24}$$

because the part of the integral for ϕ over the interval $(0,t)$ lies in $\mathfrak{H}\ominus\mathfrak{R}(t)$, and that over (t,∞) in $\mathfrak{R}(t)$. Any element ϕ is the strong limit of elements

defined by equations of the same form for which the $c_k(\lambda)$ are continuous and vanish outside a finite interval of λ and for all but a finite number of k. There will therefore be no loss of generality in the following argument if we assume that the $c_k(\lambda)$ do have these properties: the extension to the general case is immediate. Under these conditions $(E(t)\phi, E(t)g)$ is the limit of the sum

$$\sum_{k=1}^{K}\sum_{i=0}^{n-1}(c_k(\lambda_i)\,U(\lambda_i)\,[I-E(\Delta\lambda_i)]\,f_k,\,[E(\lambda_i)-E(\lambda_i+\Delta\lambda_i)]\,g),$$

where $0=\lambda_0<\lambda_1<\ldots<\lambda_n=\Lambda$ is a subdivision of the interval $(0,\Lambda)$ outside which the $c_k(\lambda)=0$, and the maximum length of the intervals tends to zero. From (20) the sum is

$$\begin{aligned}\sum_k\sum_{i=0}^{n-1}&(c_k(\lambda_i)\,[I-E(\Delta\lambda_i)]f_k,\,[I-E(\Delta\lambda_i)]\,V(-\lambda_i)\,g)\\ &=\sum_k\sum_l\sum_{i=0}^{n-1}c_k(\lambda_i)\,\overline{g_k(\lambda_i)}\,([I-E(\Delta\lambda_i)]f_k,\,[I-E(\Delta\lambda_i)]f_l)\\ &\qquad+\sum_k\sum_{i=0}^{n-1}c_k(\lambda_i)\,([I-E(\Delta\lambda_i)]f_k,\,[I-E(\Delta\lambda_i)]\,\eta(\lambda_i))\quad\text{(from (21))}\\ &=\sum_k\sum_{i=0}^{n-1}c_k(\lambda_i)\,\overline{g_k(\lambda_i)}\,\Delta\lambda_i+o(1)\end{aligned}$$

from (16) and (17). Hence

$$(E(t)\phi,E(t)g)=\sum_k\int_t^\infty c_k(\lambda)\,\overline{g_k(\lambda)}\,d\lambda,$$

and therefore
$$-\frac{d}{dt}(E(t)\phi,E(t)g)=\sum_k c_k(t)\,\overline{g_k(t)}.$$

On comparing this with (23) we get

$$\frac{d}{dt}(E(t)\phi,g)=\frac{d}{dt}(E(t)f,g),$$

so that, on integrating,

$$(\phi-E(\infty)\phi,g)=(f-E(\infty)f,g).$$

This has been established for any element g of $\mathfrak{D}^*$, and hence ϕ and f differ only in their projections on $\mathfrak{R}(\infty)$. This projection is obviously zero for ϕ. We therefore get the important result that any element of $\mathfrak{D}^*$ can be expressed in the form

$$f=\sum_k\int_0^\infty c_k(t)\,d(U;\,k;\,t)+E(\infty)f, \tag{25}$$

where the coefficients $c_k(t)$ form a set of integrable square, and are determined by (21).

Any element ϕ of $\mathfrak{H}$ is the strong limit of a sequence $\{f_\nu\}$ of elements of $\mathfrak{D}^*$, and by (25) each f_ν can be expressed in the form

$$f_\nu = \sum_k \int_0^\infty c_k^\nu(t)\, d(U;\, k;\, t) + E(\infty) f_\nu,$$

where for each ν the set $\{c_k^\nu\}$ is of integrable square. Then

$$\| f_\mu - f_\nu \|^2 = \sum_k \int_0^\infty | c_k^\mu(t) - c_k^\nu(t) |^2\, dt + \| E(\infty)(f_\mu - f_\nu) \|^2.$$

Since f_ν is strongly convergent, this expression tends to zero as μ, ν tend to infinity independently. It follows that for each k the sequence $c_k^\nu(t)$ is convergent in mean square as $\nu \to \infty$. If $c_k(t)$ is the limit in mean, it is clear that the set $\{c_k(t)\}$ is of integrable square, and, if U has finite multiplicity, that

$$\lim_{\nu\to\infty} \sum_k \int_0^\infty | c_k^\nu(t) - c_k(t) |^2\, dt = 0.$$

This result is not quite obvious if the number of k is infinite. In that case, for a given ϵ, we find $\nu_0(\epsilon)$ such that $\| f_\mu - f_\nu \|^2 < \epsilon$ for all $\mu, \nu > \nu_0$, and then for a fixed ν we can find k_0 such that

$$\left\{ \sum_{k>k_0} \int_0^\infty | c_k^\nu(t) |^2\, dt \right\}^{\frac{1}{2}} < \epsilon.$$

For any $\mu > \nu_0$

$$\left\{ \sum_{k>k_0} \int_0^\infty | c_k^\mu(t) |^2\, dt \right\}^{\frac{1}{2}} < \left\{ \sum_k \int_0^\infty | c_k^\mu(t) - c_k^\nu(t) |^2\, dt \right\}^{\frac{1}{2}} + \left\{ \sum_{k>k_0} | c_k^\nu(t) |^2\, dt \right\}^{\frac{1}{2}}.$$

From this we deduce that

$$\left(\sum_{k>k_0} \int_0^\infty | c_k(t) - c_k^\mu(t) |^2\, dt \right)^{\frac{1}{2}} < 4\epsilon,$$

and the required result follows easily.

If, then,
$$\psi = \sum_k \int_0^\infty c_k(t)\, d(U;\, k;\, t) + E(\infty)\phi,$$

the sequence f_ν converges strongly to ψ, and therefore $\phi = \psi$.

It follows that every element of $\mathfrak{H}$ can be expressed in the form (25) with some set of integrable square. The converse, that to every set of integrable square corresponds an element of $\mathfrak{H}$, was proved earlier. There is a one-one correspondence between elements of $\mathfrak{H} \ominus \mathfrak{N}(\infty)$ and the sets of integrable square, provided, of course, that we treat functions of $L^2(0, \infty)$ as identical when they differ only in a set of measure zero. From (25),

$$\begin{aligned} U(t) f &= \sum_k \int_0^\infty c_k(\lambda)\, d(U;\, k;\, \lambda + t) + U(t)\, E(\infty) f \\ &= \sum_k \int_0^\infty c_k(\lambda - t)\, d(U;\, k;\, \lambda) + U(t)\, E(\infty) f. \end{aligned}$$

If f is in $\mathfrak{D}$, this is differentiable. Leaving out the projections on $\mathfrak{R}(\infty)$, which are differentiable by themselves, we get

$$U(t)f-f = \sum_k \int_0^\infty [c_k(\lambda-t)-c_k(\lambda)]\, d(U;\, k;\, \lambda).$$

Here we take $c_k(\lambda) = 0$ if $\lambda < 0$; we shall adopt the same convention in future with functions of $L^2(0,\infty)$. If f is in $\mathfrak{D}$, $\{U(t)f-f\}/t$ tends strongly to iHf as $t \to 0$; hence there must exist a set $\{g_k(\lambda)\}$ of integrable square such that

$$Hf = i\sum_k \int_0^\infty g'_k(\lambda)\, d(U;\, k;\, \lambda),$$

and then
$$\lim_{t\to 0} \sum_k \int_0^\infty \left| \frac{c_k(\lambda-t)-c_k(\lambda)}{t} + g'_k(\lambda) \right|^2 d\lambda = 0. \tag{26}$$

The existence of a set $\{g_k(\lambda)\}$ satisfying (26) is a necessary condition for f to be in $\mathfrak{D}$. We consider what sets of functions $\{c_k(\lambda)\}$ satisfy the condition (26). It is definitely satisfied by sets where the $c_k(\lambda)$ have the form

$$c_k(\lambda) = \int_0^\lambda c'_k(\tau)\, d\tau,$$

where the $\{c'_k(\lambda)\}$ are of integrable square and we take $g_k(\lambda) = c'_k(\lambda)$, for in this case the expression in (26) becomes

$$\begin{aligned}\sum_k \int_0^\infty \left| t^{-1}\int_{\lambda-t}^{\lambda} g_k(\tau)\, d\tau - g_k(\lambda)\right|^2 d\lambda &= t^{-2}\sum_k \int_0^\infty d\lambda \left| \int_0^t \{g_k(\lambda-\tau)-g_k(\lambda)\}\, d\tau \right|^2 \\ &\leqslant t^{-1}\sum_k \int_0^\infty d\lambda \int_0^t |\, g_k(\lambda-\tau)-g_k(\lambda)\,|^2\, d\tau \\ &= t^{-1}\sum_k \int_0^t d\tau \int_0^\infty |\, g_k(\lambda-\tau)-g_k(\lambda)\,|^2\, d\lambda.\end{aligned}$$

Since $\{g_k(\lambda)\}$ is of integrable square, we can find K such that

$$\sum_{k>K} \int_0^\infty |\, g_k(\lambda-\tau)-g_k(\lambda)\,|^2\, d\lambda < \epsilon$$

for all τ, and the contribution to the sum above for $k > K$ is then less than ϵ. For any particular k,

$$\int_0^\infty |\, g_k(\lambda-\tau)-g_k(\lambda)\,|^2\, d\lambda \to 0$$

as $\tau \to 0$.† Hence we can find $\tau(k,\epsilon)$ such that this expression is less than ϵ/K for $\tau < \tau(k,\epsilon)$. Then if $\tau(\epsilon) = \min_{k\leqslant K} \tau(k,\epsilon)$, we get for $t < \tau(\epsilon)$

$$t^{-1}\sum_{k\leqslant K} \int_0^t d\tau \int_0^\infty |\, g_k(\lambda-\tau)-g_k(\lambda)\,|^2\, d\lambda < \epsilon,$$

† See Wiener, *The Fourier Integral* (Cambridge, 1933), 24.

and then $$\sum_k \int_0^\infty \left| t^{-1} \int_{\lambda-t}^{\lambda} g_k(\tau)\,d\tau - g_k(\lambda) \right|^2 d\lambda < 2\epsilon,$$

so this expression tends to zero as $t \to 0$.

Now let $\{c_k(\lambda)\}$ be any set satisfying (26). Put

$$f_k(\lambda) = \int_0^\lambda g_k(\tau)\,d\tau, \quad q_k(\lambda) = c_k(\lambda) - f_k(\lambda).$$

From the result just used, and Minkowski's inequality

$$\left(\int |f+g|^2\,dx\right)^{\frac{1}{2}} \leqslant \left(\int |f|^2\,dx\right)^{\frac{1}{2}} + \left(\int |g|^2\,dx\right)^{\frac{1}{2}},$$

we see that $$\lim_{t\to 0} t^{-2} \sum_k \int_0^\infty |q_k(\lambda - t) - q_k(\lambda)|^2\,d\lambda = 0.$$

Let $r(\lambda)$, $s(\lambda)$ be functions of $L^2(0,\infty)$ such that

$$r(\lambda) = \int_0^\lambda s(\tau)\,d\tau.$$

For any particular k we must have

$$\lim t^{-1} \int_0^\infty r(\lambda)\{q_k(\lambda - t) - q_k(\lambda)\}\,d\lambda = 0.$$

The left-hand side is

$$\lim_{t\to 0} \int_0^\infty \frac{r(\lambda+t) - r(\lambda)}{t} q_k(\lambda)\,d\lambda = \int_0^\infty s(\lambda)\,q_k(\lambda)\,d\lambda.$$

This is true for every function $s(\lambda)$ for which the corresponding $r(\lambda)$ is in $L^2(0,\infty)$ and so $q_k(\lambda) = 0$ almost everywhere. It follows that the necessary and sufficient conditions that a set $\{c_k(\lambda)\}$ satisfy (26) are:

(i) $c_k(0) = 0$, *for all* k;

(ii) $c_k(\lambda)$ *is almost everywhere equal to the integral of a function of* $L^2(0,\infty)$, $c'_k(\lambda)$, *and the sets* $\{c'_k(\lambda)\}$ *are of integrable square.*

A set of functions satisfying condition (ii) will be called a set differentiable in mean square.

The conditions (i) and (ii) are necessary in order that an element defined by (25) should be in $\mathfrak{D}$, so the domain of H is included in the set of elements generated by functions satisfying (i) and (ii). For an element in $\mathfrak{D}$ given by (25) we have

$$Hf = i \sum_k \int_0^\infty c'_k(\lambda)\,d(U;\,k;\,\lambda) + HE(\infty)f. \tag{27}$$

Actually, the domain of H is identical with the set of elements generated by functions satisfying (i) and (ii): for consider the operator—$\bar{H}$ say—whose

domain is the entire set of such elements and whose value is given by (27). $\bar{H}$ is a closed symmetric operator, and an extension of H. Since H is maximal it has no true symmetric extension, hence $\bar{H} \equiv H$.

The necessary and sufficient condition that f should be in $\mathfrak{D}^*$ is, consequently, that $E(\infty)f$ be in $\mathfrak{D}$ and that for some K

$$\left| \sum_k \int_0^\infty c_k(t) \overline{g_k'(t)}\, dt \right| \leqslant K \left(\sum_k \int_0^\infty | g_k(t) |^2 dt \right)^{\frac{1}{2}}, \tag{28}$$

for every set $\{g_k(t)\}$ of integrable square satisfying (i) and (ii). It is easy to see that a sufficient condition for (28) is that the set $\{c_k(t)\}$ be differentiable in mean square: and it will now be shown that the condition is necessary.

The set of all sets of functions of integrable square, $\{g_k(t)\}$, forms a Hilbert space IS if we define the scalar product by

$$(\{g_k\}, \{h_k\}) = \sum_k \int_0^\infty g_k(t) \overline{h_k(t)}\, dt.$$

It follows from (28) that

$$L(\{g_k\}) = \sum_k \int_0^\infty \overline{c_k(t)}\, g_k'(t)\, dt$$

is a bounded linear functional in IS, with norm K. L is defined in the first place only for $\{g_k\}$ which satisfy (i) and (ii), but these are everywhere dense in the space and so L may be considered defined for all $\{g_k\}$. By the standard theorem on the form of a linear functional in a Hilbert space, there must be an element $\{y_k\}$ in IS such that for all $\{g_k\}$,

$$L(\{g_k\}) = (\{g_k\}, \{y_k\}) = \sum_k \int_0^\infty g_k(t) \overline{y_k(t)}\, dt.$$

On equating this to the original expression for L it follows that, for almost all t_1, t_2,

$$c_k(t_2) - c_k(t_1) = \int_{t_1}^{t_2} y_k(t)\, dt,$$

so that $\{c_k(t)\}$ is differentiable in mean square.†

† The necessity of this condition can be derived directly from lemma 5 (e). Omitting projections on $\mathfrak{R}(\infty)$, we have

$$V(-t)f = \sum_k \int_0^\infty c_k(\lambda + t)\, d(U; k; \lambda),$$

so that if

$$H^*f = i \sum_k \int_0^\infty g_k(\lambda)\, d(U; k; \lambda),$$

then

$$\lim_{t \to 0} \sum_k \int_0^\infty \left| \frac{c_k(\lambda + t) - c_k(\lambda)}{t} - g_k(\lambda) \right|^2 d\lambda = 0.$$

An argument similar to that carried out above for elements in $\mathfrak{D}$ shows that this is equivalent to condition (ii).

If f is given by (25) with the $\{c_k(\lambda)\}$ a set of integrable square and differentiable in mean square, H^*f is given by a formula similar to (27):

$$H^*f = i\sum_k \int_0^\infty c_k'(\lambda)\,d(U;\,k;\,\lambda) + HE(\infty)f. \tag{27a}$$

All the elements defined by putting $c_k(t) \equiv 0$ for all k except one particular value of k, say r, define a closed linear manifold, which we shall call $\mathfrak{M}_r$. $\mathfrak{M}_r$ must be a Hilbert space, since every element $c_k(t)$ in $L^2(0, \infty)$ defines a different element of $\mathfrak{M}_r$, in such a way that the relation is a homomorphism between $\mathfrak{M}_r$ and $L^2(0, \infty)$. It is plain from (19) that, if $r \neq s$, $\mathfrak{M}_r$ is orthogonal to $\mathfrak{M}_s$. The formulae (25) and (27) show that $\mathfrak{M}_1, \ldots, \mathfrak{M}_r, \ldots$ and $\mathfrak{N}(\infty)$ reduce H. Together they determine the whole of the Hilbert space $\mathfrak{H}$.

H is not reducible inside any one of the $\mathfrak{M}_r$, for if it were there would exist a closed subset $\mathfrak{N} \subset \mathfrak{M}_r$ such that if ϕ is in $\mathfrak{D} \cap \mathfrak{N}$ (the intersection of $\mathfrak{D}$ and $\mathfrak{N}$), $H\phi$ would lie in $\mathfrak{N}$. It is obvious that the solutions of (1 A) for any ϕ in $\mathfrak{D} \cap \mathfrak{N}$ would lie in $\mathfrak{N}$. For all $t > 0$, $U(t)$ would therefore transform $\mathfrak{N}$ into a subspace of itself. If, therefore, f is any element of $\mathfrak{D}^* \cap \mathfrak{N}$, the integral

$$\mathfrak{J} = \int_0^\infty p(t)\,d(U;f;\,t)$$

would give only elements of $\mathfrak{N}$. This is in contradiction with the fact that since

$$f = \int_0^\infty c_r(t)\,d(U;\,r;\,t) \quad (c_r(0) \neq 0),$$

we must have

$$\|[I - E(\Delta t)]f - [I - E(\Delta t)]\,c_r(0)f_r\|^2 = o(\Delta t),$$

and therefore the integral $\mathfrak{J}$ must give all elements of $\mathfrak{M}_r$.†

Change of elements f_k

We shall now investigate what changes are brought about in the expressions considered above if the set of elements of $\mathfrak{D}^*$, $\{f_k\}$, is changed.

If we consider the definition of the integral in $(U;f_k;\,t)$, it is obvious that if we take any f of $\mathfrak{D}^*$ in the sums S, the sum S will tend to a limit

$$\phi = \int_0^\infty p(t)\,d(U;f;\,t),$$

where

$$\|\phi\| = -\left(\frac{d\,\|E(t)f\|^2}{dt}\right)_{t=0} \int_0^\infty |\,p(t)\,|^2\,dt.$$

† This result is equivalent to the result that $i(d/dx)$ is not reducible in $L^2(0, \infty)$, and could be either deduced from that result or regarded as a new proof of it.

It is therefore obvious, from (17), that if f is in $\mathfrak{D}$, the integral has the value zero for every $p(t)$. If we substitute for f the sum of two elements g and h of $\mathfrak{D}^*$, the resulting integral is the sum of the integrals resulting from g and h alone. In particular, addition of an element of $\mathfrak{D}$ to an f makes no difference to the resulting integral.

Since the f_k are themselves elements of $\mathfrak{D}^*$ they are given by formulae of the form

$$f_k = \sum_l \int_0^\infty c_{kl}(t)\, d(U;\, l;\, t) + E(\infty) f_k,$$

where the $\{c_{kl}(t)\}$ for each k are of integrable square and differentiable in mean square, and are given by (21), from which we derive $c_{kl}(0) = \delta_{kl}$, where δ_{kl} is the Kronecker delta. We can change from the set $\{f_k\}$ to a new set $\{f'_k\}$ given by

$$f'_k = \sum_l \int_0^\infty c'_{kl}(t)\, d(U;\, l;\, t) + E(\infty) f'_k.$$

We require that the new set be such that all elements of $\mathfrak{D}^*$ can be expressed (mod $\mathfrak{D}$) linearly in terms of it—and for this it is sufficient that the f_k can be expressed in terms of the new set. It is necessary and sufficient for this that the matrix $c_{kl}(0)$ have an inverse. In addition, the set $\{f'_k\}$ should satisfy (16), so that

$$\sum_l c'_{kl}(0)\, \overline{c'_{ml}(0)} = \delta_{mk},$$

that is, the matrix $\| c'_{kl}(0) \|$ must be unitary.

In the simplest case, the $c'_{kl}(t)$ satisfy $c_{kl}(0) = \delta_{kl}$. In that case it follows from (17) that $\dfrac{d}{dt} \| E(t)\,(f_k - f'_k) \|^2 = 0$ for $t = 0$, and hence that $f_k - f'_k$ is in $\mathfrak{D}$: the change of f_k in this case does not alter any of the integrals.

In the general case, we have

$$f'_k = \sum_l c'_{kl}(0) f_k + g,$$

where g is an element of $\mathfrak{D}$. If f is any element of $\mathfrak{H}$ given by

$$f = \sum_k \int_0^\infty c_k(t)\, d(U; f_k;\, t) = \sum_l \int_0^\infty c'_l(t)\, d(U; f'_l;\, t),$$

we must therefore have $\quad c_k(t) = \sum_l c'_l(t)\, c'_{lk}(0).$

The manifolds $\mathfrak{M}'_k$ defined by the new set $\{f'_k\}$ are not the same as the manifolds $\mathfrak{M}_k$. Taken together, the $\mathfrak{M}'_k$ determine the same subspace—$\mathfrak{H} \ominus \mathfrak{N}(\infty)$—of $\mathfrak{H}$ as do the $\mathfrak{M}_k$; but, in the general case, no $\mathfrak{M}'_k$ need have any element except the zero element in common with any $\mathfrak{M}_l$. The manifolds $\mathfrak{M}_k$ are not uniquely determined by H, unless it has deficiency index 1.

The formulae will take the simplest form if the f_k' are chosen to lie in the $\mathfrak{M}_k$ which correspond to them; in particular, they could be chosen to be characteristic vectors of the transformation H^*, as will now be shown.

For let
$$f_k' = \int_0^\infty e^{-t} d(U; k; t);$$
then
$$H^* f_k' = -i \int_0^\infty e^{-t} d(U; k; t).$$
Thus $H^* f_k' + i f_k' = 0$. Obviously any element in $\mathfrak{M}_k$ which satisfies this condition is a multiple of $\mathfrak{M}_k$. There are therefore n mutually orthogonal elements satisfying $H^* f + i f = 0$, one in each of the manifolds $\mathfrak{M}_k$.

The main results reached above are summed up in the following theorem:

THEOREM V. *If H is a maximal symmetric operator, of deficiency index $(0, n)$, where n may be finite or countably infinite, then H is reduced by $(n+1)$ manifolds, $\mathfrak{M}_r$ $(r = 1, 2, \ldots, n)$ and $\mathfrak{N}(\infty)$ which together determine $\mathfrak{H}$. $\mathfrak{N}(\infty)$ may be empty, an Euclidean space of finite dimension number, or a Hilbert space. Inside $\mathfrak{N}(\infty)$, H is self-adjoint. $\mathfrak{N}(\infty)$ is determined uniquely by H.*

The manifolds $\mathfrak{M}_k$ are Hilbert spaces, and inside each one of them H is an irreducible maximal symmetric operator, of deficiency index $(0, 1)$. Elements f_r exist for $r = 1, 2, \ldots, n$ such that every f of $\mathfrak{H}$ can be put in the form (25) where the $c_k(t)$ are arbitrary functions of $L^2(0, \infty)$, the set $\{c_k(t)\}$ is of integrable square, and
$$\| f \|^2 = \| E(\infty) f \|^2 + \sum_k \int_0^\infty | c_k(t) |^2 dt.$$
Each f_r corresponds to one $\mathfrak{M}_r$, the elements of which can be put in the form
$$\int_0^\infty c_r(t)\, d(U; r; t). \tag{29}$$

The manifold determined by all the $\mathfrak{M}_k$ together is uniquely determined by H, but if $n > 1$ the individual $\mathfrak{M}_k$ can be chosen in infinitely many ways.

The necessary and sufficient conditions that f be in $\mathfrak{D}^*$ are that the set $\{c_k(t)\}$ be differentiable in mean square and $E(\infty) f$ be in $\mathfrak{D}$, when $H^* f$ is given by (27a). The additional necessary and sufficient condition for f to be in $\mathfrak{D}$ is that $c_k(0) = 0$ for all k, when (27) gives Hf.

Irreducible maximal symmetric operators

The results above show that an irreducible maximal symmetric operator $\bar{R}$ is generated by a set of isometric operators $U(t)$ of multiplicity 1, such that the corresponding $E(t)$ tend to zero as t tends to infinity. Any f in $\mathfrak{H}$ is expressible by
$$f = \int_0^\infty c(t)\, d(U; \phi; t) = \int_0^\infty c(t)\, d(U; t), \tag{30}$$

where ϕ is an element of $\mathfrak{E}^-$, arbitrary save for a multiplying constant, and $c(t)$ is $L^2(0,\infty)$. Further, f is in $\mathfrak{D}^*$ if $c(t)$ is differentiable in mean square, and in $\mathfrak{D}$ if in addition $c(0) = 0$. Then

$$\bar{R}f = i\int_0^\infty c'(t)\,d(U;\phi;t). \tag{31}$$

These formulae apply to operators with deficiency index $(0,1)$. Those with deficiency index $(1,0)$ are given by the same formulae, save for a change of sign on the right.

The inverse of $\bar{R}$ exists for a given f if the function

$$c_1(t) = \int_0^t c(\tau)\,d\tau$$

is $L^2(0,\infty)$, and in that case

$$\bar{R}^{-1}f = -i\int_0^\infty c_1(t)\,d(U;\phi;t). \tag{32}$$

This characterization of the irreducible maximal symmetric operator may be compared with that of von Neumann.† von Neumann defines the operator $\bar{R}$ by means of the isometric transformation

$$V = (H-i)(H+i)^{-1}.$$

Given a complete normal orthogonal set of elements ϕ_n, V is defined to be the operator which transforms ϕ_n into ϕ_{n+1}. If V satisfies this condition, H is the irreducible maximal operator $\bar{R}$.

In the notation of this paper, the set ϕ_n can be expressed as follows. ϕ_1 is the element satisfying

$$\bar{R}^*\phi_1 + i\phi_1 = 0,$$

and so, since $\|\phi_n\| = 1$, $\quad \phi_1 = 2\int_0^\infty e^{-t}d(U;t);$

and in general it is then easy to see that

$$\phi_n = \int_0^\infty L_n(t)\,d(U;t),$$

where $L_n(t)$ is the nth Laguerre orthogonal function.

The conjugate operator

The results given above amount to an explicit formulation of the homomorphism of $\bar{R}$ and the operator $i(d/dx)$ in $L^2(0,\infty)$. This suggests the existence of an operator $\bar{S}$ conjugate to $\bar{R}$ in the sense of quantum mechanics: that is, satisfying

$$\bar{R}\bar{S} - \bar{S}\bar{R} = iI, \tag{33}$$

† See ref. (1), or ref. (2), Chap. IX.

just as the particular realizations $\bar{R} = i(d/dx)$ and $\bar{S} = x$ satisfy

$$i\frac{d}{dx}x - ix\frac{d}{dx} = i.$$

The definition of S is obviously as follows. If

$$f = \int_0^\infty c(t)\, d(U;\, t),$$

then f is in the domain of $\bar{S}$ if and only if $tc(t)$ is $L^2(0, \infty)$ and then

$$\bar{S}f = \int_0^\infty tc(t)\, d(U;\, t). \tag{34}$$

It is easy to see that (33) is satisfied.

An operator conjugate to any maximal operator can be defined in a similar way provided $E(\infty) = 0$. If $E(\infty) \neq 0$, we can get an operator which satisfies

$$\bar{R}\bar{S} - \bar{S}\bar{R} = i(I - E(\infty)).$$

$\bar{S}$ is clearly a self-adjoint operator, with a spectrum covering the entire range from 0 to ∞.

From (18) we can write the expressions for f, $\bar{S}f$ in the form

$$f = -\int_0^\infty dE(t)f, \quad \bar{S}f = -\int_0^\infty t\, dE(t)f. \tag{35}$$

$E(t)$ are the projection operators corresponding to S, in the appropriate resolution of the identity.

The solutions of equation (1)

For any ϕ in $\mathfrak{D}$, the equations (1) have an infinite number of solutions for $t > 0$. The solutions constructed in the proof of theorem I for

$$\phi = \sum_k \int_0^\infty c_k(\lambda)\, d(U;\, k;\, \lambda) + E(\infty)\phi,$$

can, in our present notation, be written for $t > 0$

$$\psi(t) = U(t)\phi = \sum_k \int_0^\infty c_k(\lambda)\, d(U;\, k;\, \lambda + t) + U(t)\, E(\infty)\phi,$$

and for $t < 0$

$$\psi(t) = V(t)\phi = \sum_k \int_{-t}^\infty c_k(\lambda)\, d(U;\, k;\, \lambda + t) + [U(-t)]^{-1}\, E(\infty)\phi. \tag{36}$$

For $t < 0$ there is no other solution of (1), but for $t > 0$ other solutions can be obtained by addition of elements of $\mathfrak{H} \ominus \mathfrak{N}(t)$ to $U(t)\phi$. In fact, if $d_k(\lambda)$ is

any set of functions forming a set of integrable square and differentiable in mean square over any finite interval and with $d_k(0) = 0$ for all k, then the elements

$$\psi_1(t) = \sum_k \int_0^t d_k(t-\lambda)\, d(U;\, k;\, \lambda)$$

are solutions of (1) with $\psi_1(0) = 0$.

If ϕ is given by the above formula, with $c_k(0) \neq 0$, so that ϕ is in $\mathfrak{D}^*$, then $U(t)\phi$ is not in $\mathfrak{D}^*$, and so is not a solution of (1). But if we add to $U(t)\phi$ an element $\psi_1(t)$ defined as above, except that we must have $d_k(0) = c_k(0)$, instead of $d_k(0) = 0$, then $U(t)\phi + \psi_1(t)$ is a solution of (1) with $\psi(0) = \phi$. Hence the enunciation of theorem I is also true for elements in $\mathfrak{D}^*$.

Analogue of the spectral resolution of self-adjoint operators

It is impossible to give a representation of an operator which is maximal but not self-adjoint similar to that which exists for self-adjoint operators. However, a representation, analogous to the spectral resolution of self-adjoint operators in one respect, can be found: it is possible to give a representation in which, when f is represented by a given integral, Hf is represented by a similar integral with the integrand multiplied by the independent variable. It will be sufficient to prove this for the irreducible operator $\bar{R}$.

In (30), $c(t)$ is $L^2(0,\infty)$, so that if

$$C(u) = \frac{1}{\sqrt{(2\pi)}} \int_0^\infty c(t)\, e^{iut}\, dt,$$

then

$$c(t) = \text{l.i.m.} \frac{1}{\sqrt{(2\pi)}} \int_{-\lambda}^{\lambda} C(u)\, e^{-iut}\, du,$$

by the well-known theorem on Fourier transforms in $L^2(0,\infty)$,† and if

$$c(t) = \int_0^t c'(t)\, dt, \text{ where } c'(t) \text{ is } L^2(0,\infty),$$

$$c'(t) = \text{l.i.m.} \frac{1}{\sqrt{(2\pi)}} \int_{-\lambda}^{\lambda} (-iu)\, C(u)\, e^{-iut}\, du$$

almost everywhere. Thus (31) gives

$$f = \frac{1}{\sqrt{(2\pi)}} \int_0^\infty d(U;\, 1;\, t) \int_{-\infty}^\infty C(u)\, e^{iut}\, du,$$

$$\bar{R}f = \frac{1}{\sqrt{(2\pi)}} \int_0^\infty d(U;\, 1;\, t) \int_{-\infty}^\infty uC(u)\, e^{iut}\, du.$$

---

† See ref. (5), theorem 48.

It is easy to show that the order of integration can be inverted, using the facts that $c(t)$ and $c'(t)$ are $L^2(0,\infty)$, while $C(u)$ and $uC(u)$ are $L^2(-\infty,\infty)$. Then

$$f=\frac{1}{\sqrt{(2\pi)}}\lim_{T\to\infty}\int_{-\infty}^{\infty}C(u)\,du\int_0^T e^{-iut}d(U;t),$$

where the integral over $(-\infty,\infty)$ means the limit of the integral over $(-\lambda,\lambda)$. The inner integral is equal to

$$\frac{d}{du}\int_0^T\frac{e^{-iut}-1}{-it}d(U;t).$$

From (22), we have for all T

$$\left\|\int_0^T\frac{e^{-iut}-e^{-ivt}}{-it}d(U;t)\right\|^2=\int_0^T\left|\frac{e^{-it(u-v)}-1}{-it}\right|^2dt$$
$$\leqslant\tfrac{1}{2}\pi\,|u-v|.$$

The total variation over an interval (u,v) of

$$\int_0^T\frac{e^{-iut}-1}{-it}d(U;t)$$

is therefore less than $\frac{1}{2}\pi\,|u-v|$. This function therefore tends to a function of bounded variation as T tends to ∞, and we can show, as for functions of a real variable (to which case this is indeed reduced by considering (f,g) for an arbitrary g), that we can write

$$f=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}C(u)\,d(W;u),\tag{37}$$

where

$$(W;u)=\int_0^{\infty}\frac{e^{-iut}-1}{-it}d(U;t).$$

If Hf exists, it corresponds to the function $ic'(t)$ which is then $L^2(0,\infty)$, satisfies $c(0)=0$, and has Fourier transform $uC(u)$. Thus if f is given by (37)

$$Hf=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}uC(u)\,d(W;u).\tag{38}$$

Clearly, for $\mathfrak{D}$, $\int_{-\infty}^{\infty}C(u)\,du=0$. If f is in $\mathfrak{D}^*$, and $c(0)\neq 0$, the Fourier transform of $ic'(t)$ is $ic(0)+uC(u)$.

A formula for $(H-z)^{-1}$ can be derived from this, or verified independently. If $\mathfrak{I}(z)<0$, we get for any f in $\mathfrak{H}$

$$(H-z)^{-1}f=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\frac{C(u)}{u-z}d(W;u).$$

The expression for this directly in terms of $c(t)$ is

$$(H-z)^{-1}f=\frac{1}{i\sqrt{(2\pi)}}\int_0^{\infty}d(U;t)\int_0^t e^{iz(t-\tau)}c(\tau)\,d\tau.$$

The representations (37) and (38) cannot be spectral resolutions, for only a self-adjoint operator can have a spectral resolution. We shall discuss this in more detail shortly.

The $C(u)$ are restricted to belong to the class of functions of $L^2(-\infty,\infty)$ which are transforms of functions of $L^2(0,\infty)$; this set of $C(u)$ gives the Hilbert space $\mathfrak{H}$ by the representation (37). If we consider the set of all $C(u)$ in $L^2(-\infty,\infty)$, and for each $C(u)$ define an element f by the formula (37), the set of elements so defined constitutes a Hilbert space $\mathfrak{H}_1$ if we define the scalar product by

$$(f_1,f_2) = \int_{-\infty}^{\infty} C_1(u)\,\overline{C_2(u)}\,du.$$

This space $\mathfrak{H}_1$ contains the space $\mathfrak{H}$: for if $C_1(u)$, $C_2(u)$ are transforms of functions of $L^2(0,\infty)$, then

$$(f_1,f_2) = \int_0^{\infty} c_1(t)\,\overline{c_2(t)}\,dt = \int_{-\infty}^{\infty} C_1(u)\,\overline{C_2(u)}\,du$$

by Parseval's theorem. In $\mathfrak{H}_1$ the formula (38) defines a transformation H_1 which is equal to H for elements in $\mathfrak{D}$ but is defined in a set $\mathfrak{D}_1$ everywhere dense in $\mathfrak{H}_1$: the set with $uC(u)$ in $L^2(-\infty,\infty)$. This operator H_1 is obviously self-adjoint in $\mathfrak{H}_1$.

It is obvious that the formulae (37) and (38) give the spectral resolution formula for H_1 in $\mathfrak{H}_1$.

That (37) and (38) are not spectral resolution formulae in $\mathfrak{H}$ can be seen as follows. If $c(t)$ is any element of $L^2(0,\infty)$, $C(u)$ its transform, then if $C_{\lambda\mu}(u)$ is the function equal to $C(u)$ in an interval (λ,μ), the formula (37) applied to $C_{\lambda\mu}(u)$ defines an element of $\mathfrak{H}_1$, not in $\mathfrak{H}$; for the transform of $C_{\lambda\mu}(u)$ must be an analytic function of t, since $C_{\lambda\mu}(u)$ vanishes outside a finite interval, and can therefore not vanish for all $t<0$. If the transform is $c_{\lambda\mu}(t)$ (a function of $L^2(-\infty,\infty)$), we have

$$\int_{-\infty}^{\infty} c_{\lambda\mu}(t)\,\overline{c_{\lambda'\mu'}(t)}\,dt = \int_{-\infty}^{\infty} C_{\lambda\mu}(u)\,\overline{C_{\lambda'\mu'}(u)}\,du$$

for any two intervals (λ,μ), (λ',μ'). This shows that the components corresponding to non-overlapping intervals are orthogonal in $\mathfrak{H}_1$; but the projections on to $\mathfrak{H}$ of elements generated by non-overlapping intervals are not necessarily orthogonal. The scalar product of these projections is in fact

$$\int_0^{\infty} c_{\lambda\mu}(t)\,\overline{c_{\lambda'\mu'}(t)}\,dt,$$

which need not be zero for non-overlapping intervals.

These results can be extended to general maximal operators: this merely involves replacing the integrals (37) and (38) by sums of integrals over the $\mathfrak{M}_k$ and adding projections on $\mathfrak{N}(\infty)$, $E(\infty)f$ and $HE(\infty)f$ to them. We thus get

THEOREM VI. *If H is a maximal operator in a Hilbert space $\mathfrak{H}$, then $\mathfrak{H}$ can be imbedded in a Hilbert space in which there exists a self-adjoint operator equal to H for elements lying in the domain $\mathfrak{D}$ of H.*

The integrals in (37) and (38) are analogous to ordinary Stieltjes integrals, and to the integrals in the ordinary spectral resolution theory, in that $(W; u)$ is an element for every u and has bounded variation over every finite interval. They are unlike the integrals in $(U;\, k;\, t)$ in this, for these are not integrals in terms of the variation of elements of $\mathfrak{H}$. (37) and (38) do express H in terms of projection operators: for the operator which transforms f given by (37) into the projection on to $\mathfrak{H}$ of the element of $\mathfrak{H}_1$

$$\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\lambda} C(u)\, d(W;\, u)$$

is a projection operator: but the manifolds corresponding to non-overlapping intervals are not orthogonal (in $\mathfrak{H}$).

5. *Applications*

Theorems I and II have an application to the method adopted by Professor Titchmarsh for the proof of the expressions of functions in terms of eigen-functions of Sturm-Liouville differential equations. In his paper† Professor Titchmarsh starts with the Schrödinger equation corresponding to the Sturm-Liouville operator, but uses this equation only in a heuristic fashion: for the deduction of the expansions he does not actually make use of any assumption as to the existence of a solution of the Schrödinger equation, but works with its Fourier transform in the complex plane—or, what comes to the same thing, with the resolvent operator of the Sturm-Liouville operator. The method of this paper allows of a rigorous use of the Schrödinger equation: in fact, it gives immediately the result that if H has the discrete spectrum $\{\lambda_n\}$ with characteristic functions ϕ_n, then if $(\phi, \phi_n) = c_n$ the series $\Sigma c_n e^{i\lambda_n t}\phi_n$ tends to ϕ as t tends to 0, if ϕ is a continuous function.

Theorems I and II can also be used to prove the existence of solutions of differential equations which are not of Schrödinger type. Consider, for example, the Hilbert space consisting of elements $\{f_1, f_2, f_3, f_4\} = f$, where f_1, f_2, f_3, f_4 are functions of x, y, z and of integrable square over all values of

† See ref. (11).

(x, y, z). Sum of two elements, multiplication by a complex number, and inner product are defined by taking the Hilbert space of the f as the direct sum of that of f_1, f_2, f_3, f_4; e.g.,

$$(\{f_1, f_2, f_3, f_4\}, \{g_1, g_2, g_3, g_4\}) = (f_1, g_1)+(f_2, g_2)+(f_3, g_3)+(f_4, g_4).$$

In this space, the operator

$$H = \begin{pmatrix} 0 & 0 & 0 & \dfrac{1}{i}\dfrac{\partial}{\partial x} \\ 0 & 0 & 0 & \dfrac{1}{i}\dfrac{\partial}{\partial y} \\ 0 & 0 & 0 & \dfrac{1}{i}\dfrac{\partial}{\partial z} \\ \dfrac{1}{i}\dfrac{\partial}{\partial x} & \dfrac{1}{i}\dfrac{\partial}{\partial y} & \dfrac{1}{i}\dfrac{\partial}{\partial z} & 0 \end{pmatrix}$$

is easily seen to be self-adjoint. The equation

$$H\psi = \frac{1}{i}\frac{\partial \psi}{\partial t}$$

with $\quad \psi(0) = \{f_1, f_2, f_3, f_4\}, \quad \psi(t) = \{\psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t)\},$

has therefore a solution for any set of $\{f_1, f_2, f_3, f_4\} = f$ whose differential coefficients are of integrable square, and the solution is unique. The equations mean

$$\frac{\partial \psi_4}{\partial x} = \frac{\partial \psi_1}{\partial t}, \quad \frac{\partial \psi_4}{\partial y} = \frac{\partial \psi_2}{\partial t}, \quad \frac{\partial \psi_4}{\partial z} = \frac{\partial \psi_3}{\partial t},$$

$$\frac{\partial \psi_1}{\partial x}+\frac{\partial \psi_2}{\partial y}+\frac{\partial \psi_3}{\partial z} = \frac{\partial \psi_4}{\partial t},$$

hence

$$\frac{\partial^2 \psi_4}{\partial x^2}+\frac{\partial^2 \psi_4}{\partial y^2}+\frac{\partial^2 \psi_4}{\partial z^2} = \frac{\partial^2 \psi_4}{\partial t^2},$$

so that ψ_4 satisfies the wave equation. On choosing $f_4 = \phi$ and f_1, f_2, f_3 such that

$$\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z} = \chi,$$

the existence of a solution of a wave equation with initial conditions $\psi_4 = \phi$, $\partial \psi_4/\partial t = \chi$ at $t = 0$ is demonstrated. The restrictions that ϕ and χ and their differential coefficients must be of integrable square are easily removed.

Some examples of the representations (30), (31) and (32) may be given. If H is the operator $i(d/dx)$ in $L^2(0, \infty)$, defined for functions of the form

$$f(x) = \int_0^x g(\tau)\, d\tau$$

with $g(\tau)$ in $L^2(0,\infty)$, the results are obvious. Here for t negative

$$V(t)\,f(x) = f(x-t),$$

for t positive $U(t)\,f(x) = f(x-t)$, where $f(x) = 0$ for $x<0$. $E(t)\,f(x)$ is the function equal to zero for $0 \leqslant x \leqslant t$, and to $f(x)$ for $x \geqslant t$. If we choose $f(x)$ with $f(0) = 1$,

$$\int_0^\infty c(t)\,d(U; f;\, t) = c(x).$$

Formulae (37) and (38) reduce to the Fourier integral formulae.

A realization of $\bar{R}$ given by von Neumann (ref. (**1**), Anhang II) is as follows. The functions of the complex variable z regular for $|z| \leqslant 1$ and with the integral

$$\int_C \frac{|F(z)|^2}{z}\,dz,$$

bounded for any circle inside the unit circle, form a Hilbert space. $\bar{R}$ is realized by taking

$$V = z, \quad \bar{R} = \frac{1}{i}\frac{1+z}{1-z}.$$

$U(t)$ is then the operator which multiplies a function by $\exp\left(-t\dfrac{1+z}{1-z}\right)$.

The resolution of a function (30) becomes

$$F(z) = \frac{i}{\pi}\int_0^\infty \exp\left(-t\frac{1+z}{1-z}\right)dt \int_C F(u)\exp\left(t\frac{1+u}{1-u}\right)\frac{du}{(u-1)^2}.$$

This formula can be deduced from the Fourier inversion formula for functions in $L^2(0,\infty)$ by putting $w = i(z+1)/(z-1)$.

Some interesting realizations of $\bar{R}$ can be got in $L^2(0,\infty)$. Starting with Fourier's cosine formula as a particular case of (30), we proceed to find the corresponding $\bar{R}$. We have

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty F(u)\cos ux\,du,$$

where

$$F(u) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty f(x)\cos ux\,dx.$$

Then (eqn. (35)),

$$U(t)\,f(x) = \sqrt{\left(\frac{2}{\pi}\right)}\int_t^\infty F(u-t)\cos ux\,du,$$

and so from (31) $\bar{R}$ exists if $F(0) = 0$, and $F'(u)$ is $L^2(0,\infty)$, and then

$$\begin{aligned}\bar{R}f(x) &= i\int_0^\infty F'(u)\cos ux\,du \\ &= \frac{1}{\pi i}\int_0^\infty \xi f(\xi)\left[\frac{1}{\xi+x}+\frac{1}{\xi-x}\right]d\xi, \qquad (39)\end{aligned}$$

under suitable conditions. Similarly, from (32),

$$\bar{R}^{-1}f(x) = \frac{1}{\pi i}\int_0^\infty \frac{f(\xi)}{\xi}\left[\frac{1}{\xi+x}+\frac{1}{\xi-x}\right]d\xi.$$

This formula, with (39), leads to inversion formulae equivalent to those for the Hilbert transform: the theory of the Hilbert transform in $L^2(-\infty,\infty)$ can be deduced from the present theory.

The same method can be applied more generally on the basis of the theory of Watson transforms and general transforms (see ref. (5), chap. VIII). Let

$$f(x) = \int_0^\infty h(xy)\,g(y)\,dy, \quad g(x) = \int_0^\infty k(xy)\,f(y)\,dy, \tag{40}$$

where $$k(x) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\mathfrak{K}(s)\,x^{-s}\,ds, \quad h(x) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\mathfrak{H}(s)\,x^{-s}\,ds, \tag{41}$$

with $$\mathfrak{H}(s)\,\mathfrak{K}(1-s) = 1.$$

If, now, $g(0) = 0$ we can put

$$\begin{aligned} f_1(x) = \bar{R}f(x) &= i\int_0^\infty h(xy)\frac{d}{dy}g(y)\,dy \\ &= i\int_0^\infty h(xy)\left(\frac{d}{dy}\int_0^\infty k(y\xi)\,f(\xi)\,d\xi\right)dy \\ &= i\lim_{\delta\to 0}\int_0^\infty y^{-\delta}h(xy)\left(\frac{d}{dy}\int_0^\infty k(y\xi)\,f(\xi)\,d\xi\right)dy \\ &= i\int_0^\infty k_1(\xi/x)\,f(\xi)\,d\xi, \end{aligned} \tag{42}$$

where $$k_1(\xi) = \frac{1}{2\pi i}\int \mathfrak{K}(s)\,\mathfrak{H}(-s)\,(-s)\,\xi^{-s}\,ds.$$

The analysis is easily justified for a suitably restricted class of functions f, $\mathfrak{H}$, $\mathfrak{K}$, and the restrictions can be removed by considerations of continuity of the operators. Then

$$f_2(x) = \bar{R}^{-1}f(x) = -i\int_0^\infty h(xy)\,dy\int_0^y d\tau\int_0^\infty k(\tau\xi)\,f(\xi)\,d\xi,$$

leading to $$x^2 f_2(x) = -i\int_0^\infty f(x)\,k_2(\xi/x)\,d\xi,$$

where $$k_2(\xi) = \frac{1}{2\pi i}\int \mathfrak{K}(s)\,\mathfrak{H}(2-s)\,\xi^{-s}\frac{ds}{1-s}.$$

Since $\bar{R}^{-1}\bar{R} = I$, we get the inversion formula

$$\begin{aligned} f(x) &= \frac{1}{x^2}\int_0^\infty f_1(\xi)\,k_2(\xi/x)\,d\xi \\ &= \int_0^\infty g_1(\eta)\,k_2(\eta^{-1}x^{-1})\,\eta^{-2}x^{-2}\,d\eta \\ &= \int_0^\infty g_1(\eta)\,h_1(x\eta)\,d\eta, \end{aligned}$$

where

$$\begin{aligned} g_1(\eta) &= f_1(\eta^{-1}), \\ h_1(\eta) &= \eta^{-2}k_2(\eta^{-1}) \\ &= \frac{1}{2\pi i}\int \mathfrak{K}(2-s)\,\mathfrak{H}(s)\,x^{-s}\frac{ds}{1+s}. \end{aligned}$$

Since $\mathfrak{K}(s)\,\mathfrak{H}(1-s) = 1$, it is easy to see that the new kernels

$$\mathfrak{K}_1(s) = \mathfrak{K}(s)\,\mathfrak{H}(1-s)\,(-s), \quad \mathfrak{H}_1(s) = \mathfrak{K}(2-s)\,\mathfrak{H}(s)/(1+s),$$

satisfy

$$\mathfrak{K}_1(s)\,\mathfrak{H}_1(1-s) = 1.$$

These inversion formulae may be considered generalizations of the Hilbert transforms, related to the general transformations in the same manner as Hilbert transform to cosine transform.

Further interesting results are obtainable from the formulae (33) and (34) for the conjugate operator $\bar{S}$.

If the realization of $\bar{R}$ is as in (39), $\bar{S}$ is given by

$$\bar{S}f(x) = s(x) = \frac{2}{\pi}\int_0^\infty u\cos ux F(u)\,du,$$

so that

$$\begin{aligned} \int_0^x s(\tau)\,d\tau &= \frac{2}{\pi}\int_0^\infty \sin ux du \int_0^\infty f(\xi)\cos u\xi d\xi \\ &= \frac{1}{\pi}\int_0^\infty f(t)\left[\frac{1}{t+x}+\frac{1}{t-x}\right]dt. \end{aligned}$$

The last expression is the Hilbert transform of f. If we write Tf for the Hilbert transform of f, then

$$\bar{S}f = \frac{d}{dx}Tf,$$

and from (39)

$$\bar{R}f = \frac{1}{i}T(xf),$$

so that (33) leads to the formula

$$\frac{1}{i}T\left[x\frac{d}{dx}(Tf)\right] - \frac{1}{i}\frac{d}{dx}TTxf = if,$$

or
$$T\left(x\frac{d}{dx}Tf\right)-\frac{d}{dx}T^2xf=-f,$$

or, since $T^2=-I$,
$$T\left(x\frac{d}{dx}Tf\right)+\frac{d}{dx}xf=-f.$$

If we start with the Watson transforms, as in (40) and (42), we get

$$\begin{aligned}\bar{S}f &= \int_0^\infty yh(xy)\,g(y)\,dy\\ &= \int_0^\infty yh(xy)\,dy\int_0^\infty k(y\xi)\,f(\xi)\,d\xi\\ &= \int_0^\infty f(\xi)\,d\xi\int_0^\infty yh(xy)\,k(y\xi)\,d\xi.\end{aligned}$$

Then
$$\int_0^\infty yh(xy)\,k(y\xi)\,dy = x^{-2}l(\xi x^{-1}),$$

where
$$l(\eta)=\frac{1}{2\pi i}\int\mathfrak{K}(s)\,\mathfrak{H}(2-s)\,\eta^{-s}\,ds.$$

Thus
$$Sf = x^{-2}\int_0^\infty f(\xi)\,l(\xi x^{-1})\,d\xi = f_2(x),$$

say. The formula (33) now gives, with (42),

$$\begin{aligned}\bar{R}\bar{S}f &= i\int_0^\infty f_2(\xi)\,k_1(\xi x^{-1})\,d\xi\\ &= i\int_0^\infty f_2(\eta^{-1})\,k_1(x^{-1}\eta^{-1})\,\eta^{-2}\,d\eta\\ &= i\int_0^\infty k_1(x^{-1}\eta^{-1})\,d\eta\int_0^\infty f(\xi)\,l(\xi\eta)\,d\xi,\\ \bar{S}\bar{R}f &= ix^{-2}\int_0^\infty l(\eta x^{-1})\,d\eta\int_0^\infty f(\xi)\,k_1(\xi\eta^{-1})\,d\xi\\ &= i\int_0^\infty l(\eta^{-1}x^{-1})\,x^{-2}\eta^{-2}\,d\eta\int_0^\infty f(\xi)\,k_1(\xi\eta)\,d\xi,\end{aligned}$$

and so

$$\int_0^\infty k_1(x^{-1}\eta^{-1})\,d\eta\int_0^\infty f(\xi)\,l(\xi\eta)\,d\xi-\int_0^\infty l(\eta^{-1}x^{-1})\frac{d\eta}{x^2\eta^2}\int_0^\infty f(\xi)\,k_1(\xi\eta)\,d\xi=f(x). \tag{43}$$

The analysis leading to this expression has been purely formal: but it can be made rigorous easily enough if suitable restrictions are laid on the

functions involved. In any case, (43) can be proved by observing that the Mellin transform of the first term on the left-hand side of (43) is

$$\begin{aligned}\mathfrak{F}(s)\,\mathfrak{L}(1-s)\,\mathfrak{K}_1(-s) &= \mathfrak{F}(s)\,\mathfrak{K}(1-s)\,\mathfrak{H}(1+s)\,\mathfrak{K}(-s)\,\mathfrak{H}(s)\,s \\ &= s\mathfrak{F}(s),\end{aligned}$$

while the Mellin transform of the second term is

$$\begin{aligned}\mathfrak{F}(s)\,\mathfrak{K}_1(1-s)\,\mathfrak{L}(2-s) &= \mathfrak{F}(s)\,\mathfrak{K}(1-s)\,\mathfrak{H}(s-1)\,(s-1)\,\mathfrak{K}(2-s)\,\mathfrak{H}(s) \\ &= (s-1)\,\mathfrak{F}(s).\end{aligned}$$

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Birkbeck College
University of London

THE HYPERGEOMETRIC IDENTITIES OF CAYLEY, ORR, AND BAILEY

By J. L. BURCHNALL AND T. W. CHAUNDY

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1. *Introduction*

Cayley* in 1858 stated that, if

$$(1-x)^{a+b-c} F(2a, 2b;\, 2c;\, x) \equiv \sum_{n=0}^{\infty} A_n x^n,$$

then $$F(a,b;\, c+\tfrac{1}{2};\, x)\, F(c-a, c-b;\, c+\tfrac{1}{2};\, x) = \sum_{n=0}^{\infty} \frac{(c)_n}{(c+\frac{1}{2})_n} A_n x^n. \qquad (1)$$

Using the notation† $$\Delta(h) \equiv \Gamma(\delta+h)/\Gamma(h), \qquad (2)$$

we can write this identity more conveniently as

$$(1-x)^{a+b-c} F\left[\begin{matrix}2a, 2b; \\ 2c\end{matrix}\; x\right] = \frac{\Delta(c+\frac{1}{2})}{\Delta(c)} F\left[\begin{matrix}a, b; \\ c+\frac{1}{2}\end{matrix}\; x\right] F\left[\begin{matrix}c-a, c-b; \\ c+\frac{1}{2}\end{matrix}\; x\right]; \qquad (3)$$

or again, in a notation we have used elsewhere for double hypergeometric functions,‡

$$(1-x)^{a+b-c} F\left[\begin{matrix}2a, 2b; \\ 2c\end{matrix}\; x\right] = F\left[\begin{matrix}c+\frac{1}{2};\; a, b;\; c-a, c-b; \\ c;\; c+\frac{1}{2};\; c+\frac{1}{2};\end{matrix}\; x, x\right]. \qquad (4)$$

Orr,§ discussing the differential equations satisfied by the product of a pair of hypergeometric functions, gave a proof of Cayley's identity and added the two analogous results

$$(1-x)^{a+b-c-\frac{1}{2}} F\left[\begin{matrix}2a, 2b; \\ 2c\end{matrix}\; x\right] = \frac{\Delta(c+1)}{\Delta(c+\frac{1}{2})} F\left[\begin{matrix}a, b; \\ c\end{matrix}\; x\right] F\left[\begin{matrix}c-a+\frac{1}{2}, c-b+\frac{1}{2}; \\ c+1\end{matrix}\; x\right], \qquad (5)$$

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* (5), (3), 268–269; quoted by Bailey (1), 84.

† This is at variance with the meaning we have given to $\Delta(h)$ in (3), 250 (5), but confusion is not likely to arise here.

‡ See, for instance (4), § 1.

§ (10), 1–15 [10 (48‴), 9 (42′)]; quoted by Bailey, (1), 85.

$$(1-x)^{a+b-c-\frac{1}{2}} F\left[\begin{matrix}2a-1, 2b; \\ 2c-1\end{matrix} x\right] = \frac{\Delta(c)}{\Delta(c-\frac{1}{2})} F\left[\begin{matrix}a, b; \\ c\end{matrix} x\right] F\left[\begin{matrix}c-a+\frac{1}{2}, c-b-\frac{1}{2}; \\ c\end{matrix} x\right]. \tag{6}$$

With these we may associate the identity given by Bailey,*

$$(1-x)^{a+b-c-\frac{1}{2}} {}_3F_2\left[\begin{matrix}2a, 2b, c; \\ 2c, a+b+\frac{1}{2}\end{matrix} x\right]$$
$$= \frac{\Delta(2c-a-b+\frac{1}{2})}{\Delta(c+\frac{1}{2})} F\left[\begin{matrix}a, b; \\ a+b+\frac{1}{2}\end{matrix} x\right] F\left[\begin{matrix}c-a+\frac{1}{2}, c-b+\frac{1}{2}; \\ 2c-a-b+\frac{1}{2}\end{matrix} x\right], \tag{7}$$

which is identical with (5) when $c = a+b+\frac{1}{2}$.

In this note the differential equations satisfied by the subjects of these identities are again considered though with a technique which seems simpler than that employed by Orr. We obtain some additional identities of similar character and then show that these identities are closely linked with certain expansions of hypergeometric functions that may be thought of, loosely, as "duplication formulae".

2. *The product* $(1-x)^{-d} F(a, b; c; x)$

The product
$$y = (1-x)^{-d} F(a, b; c; x) \tag{8}$$
satisfies the differential equation

$$[\delta(\delta+c-1) - x\{2\delta^2 + (a+b+c+2d-1)\delta + ab + cd\}$$
$$+ x^2(\delta+a+d)(\delta+b+d)]\, y = 0.$$

This is (7), (21) or (49), but direct proof is simple, since $(1-x)^d y$ satisfies the ordinary hypergeometric differential equation. The equation is conveniently "factorized" as

$$[\delta+c-1-x(\delta+a+d-1)]\,[\delta-x(\delta+b+d)]\, y = (a-c)\, bxy. \tag{9}$$

Now, in the general equation of rank 2

$$[(\delta+c-1) f(\delta) - x\, g(\delta) + x^2(\delta+a)\, h(\delta)]\, y = 0,$$

the substitution
$$\Gamma(\delta+c)\, y = \Gamma(\delta+a)\, z \tag{10}$$
"exchanges" the end-factors $\delta+c-1$, $\delta+a$ to give the differential equation

$$[(\delta+a-1) f(\delta) - x\, g(\delta) + x^2(\delta+c)\, h(\delta)]\, z = 0.$$

This is readily seen, for

$$\Gamma(\delta+c-1)\, [(\delta+c-1) f(\delta) - x\, g(\delta) + x^2(\delta+a)\, h(\delta)]\, y$$
$$= [f(\delta) - x\, g(\delta) + x^2(\delta+a)(\delta+c)\, h(\delta)]\, \Gamma(\delta+c)\, y.$$

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* (2), 378, theorem I; quoted in (1), 86.

From the symmetry on the right and (10), it is clear that interchange of c, a interchanges y, z. This is essentially the transformation referred to in (7), 59 as "augmenting (varying) the coefficients". Thus, if in (9) we write

$$\Gamma(\delta+c)\,y = \Gamma(\delta+b+d)\,z,$$

we get the equation

$$[\delta+b+d-1-x(\delta+a+d-1)]\,[\delta-x(\delta+c)]\,z = (a-c)\,bxz. \tag{11}$$

This differs (formally) from (9) only in the change of parameters

$$a,\, b,\, c,\, d \to d,\, c-a,\, b+d,\, a.$$

Hence $$\Delta(c)\,(1-x)^{-d}\,F\begin{bmatrix} a, b; & x \\ c & \end{bmatrix} = \Delta(b+d)\,(1-x)^{-a}\,F\begin{bmatrix} d, c-a; & x \\ b+d & \end{bmatrix}. \tag{12}$$

We replace the Γ-operators by Δ-operators, since we have now to secure the identity of the *functions*, not merely of the differential equations. We get the more symmetrical form of (12)

$$\Delta(c)\,(1-x)^{b-d}\,F\begin{bmatrix} a+c, b; & x \\ c & \end{bmatrix} = \Delta(d)\,(1-x)^{b-c}\,F\begin{bmatrix} a+d, b; & x \\ d & \end{bmatrix}, \tag{13}$$

if we make the change $a, d \to a+c, d-b$ and use Euler's identity (25) below.*

Remembering that, if the values of the parameters are suitably restricted, we have

$$\frac{\Gamma(\delta+c)\,\Gamma(d-c)}{\Gamma(\delta+d)} f(x) = \int_0^1 u^{c-1}(1-u)^{d-c-1} f(xu)\,du,$$

we can rewrite (13) as

$$\int_0^1 u^{c-1}(1-u)^{d-c-1}\,(1-xu)^{b-d}\,F\begin{bmatrix} a+c, b; & xu \\ c & \end{bmatrix} du$$
$$= \mathrm{B}(c, d-c)\,(1-x)^{b-c}\,F\begin{bmatrix} a+d, b; & x \\ d & \end{bmatrix}. \tag{14}$$

Replacing the hypergeometric functions by their Eulerian integrals we get the identity of double integrals

$$\int_0^1\!\!\int_0^1 u^{c-1}(1-u)^{d-c-1}\,(1-xu)^{b-d}\,v^{b-1}(1-v)^{c-b-1}\,(1-xuv)^{-(a+c)}\,du\,dv$$
$$= (1-x)^{b-c}\int_0^1\!\!\int_0^1 U^{c-b-1}(1-U)^{d-c-1}\,V^{b-1}(1-V)^{d-b-1}\,(1-Vx)^{-(a+d)}\,dU\,dV. \tag{15}$$

This is established directly by the substitution

$$U = \frac{u(1-v)\,(1-x)}{(1-uv)\,(1-ux)}, \quad V = uv.$$

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* Direct identification of coefficients in (13) gives an identity between ${}_3F_2$'s of unit argument, which, in Whipple's notation, is $Fp(0;\,4,5) = Fp(1;\,4,5)$; cf. Bailey (1), 18, Table II A.

We can give an independent proof of (14) by expanding the hypergeometric function on the left so as to express the left-hand side (formally) as

$$\sum_{n=0}^{\infty} \frac{(a+c)_n\,(b)_n}{n!(c)_n} x^n \int_0^1 u^{c+n-1}(1-u)^{d-c-1}(1-xu)^{b-d}\,du$$
$$= \mathrm{B}(c,d-c)\sum_{n=0}^{\infty} \frac{(a+c)_n\,(b)_n}{n!(d)_n} x^n F\binom{d-b,c+n;}{d+n}\,x\Big)$$
$$= \mathrm{B}(c,d-c)\,(1-x)^{b-c}\sum_{n=0}^{\infty} \frac{(a+c)_n\,(b)_n}{n!(d)_n} x^n F\binom{b+n,d-c;}{d+n}\,x\Big),$$

by Euler's identity. Finally, we have by Vandermonde's theorem, as in (6), (7), that

$$\sum_{n=0}^{\infty} \frac{(a+c)_n\,(b)_n}{n!(d)_n} x^n F\left(\begin{matrix}b+n,d-c;\\ d+n\end{matrix}\;x\right) = F\left(\begin{matrix}b,a+d;\\ d\end{matrix}\;x\right).$$

3. *The Saalschützian product* $(1-x)^{-f}\,{}_3F_2$

The product
$$y = (1-x)^{-f}\,{}_3F_2\left[\begin{matrix}a,b,c;\\ d,e\end{matrix}\;x\right] \tag{16}$$
can be expanded as
$$y = \sum_{n=0}^{\infty} \frac{(f)_n x^n}{n!}\,{}_4F_3\left[\begin{matrix}a,b,c, & -n\\ d,e, & 1-f-n\end{matrix}\right].$$
If
$$a+b+c+f = d+e, \tag{17}$$
the ${}_4F_3$ is Saalschützian and we then extend the term to the product (16) itself. Under this Saalschützian condition (17) the differential equation satisfied by (16) is of rank 2 only. By (7), 68, (59) this equation, after "factorization", is

$$[\delta+d-1-x(\delta+a+f-1)]\,[\delta(\delta+e-1)-x(\delta+b+f)(\delta+c+f)]\,y$$
$$= -(d-a)\,x[(d-a+1)\,\delta+bc+df]\,y. \tag{18}$$

The substitution $\qquad \Gamma(\delta+d)\,y = \Gamma(\delta+b+f)\,z$

"exchanges" the factors $\delta+d-1$, $\delta+b+f$ to give

$$[\delta+b+f-1-x(\delta+a+f-1)]\,[\delta(\delta+e-1)-x(\delta+d)(\delta+c+f)]\,z$$
$$= -(d-a)\,x[(d-a+1)\,\delta+bc+df]\,z.$$

In form this can be got from (18) by the change of parameters

$$a,\ b,\ c,\ d,\ e,\ f \to e-c,\ b,\ e-a,\ b+f,\ e,\ d-b.$$

Hence we have

$$\Delta(d)\,(1-x)^{-f}\,{}_3F_2\left[\begin{matrix}a,b,c;\\ d,e\end{matrix}\;x\right] = \Delta(b+f)\,(1-x)^{b-d}\,{}_3F_2\left[\begin{matrix}e-c,b,e-a;\\ b+f,e\end{matrix}\;x\right]. \tag{19}$$

This can be written more symmetrically with changed parameters as

$$\Delta(d)\,(1-x)^{b-f}\,{}_3F_2\begin{bmatrix} a, b, e-c; \\ d, e \end{bmatrix} x = \Delta(f)\,(1-x)^{b-d}\,{}_3F_2\begin{bmatrix} c, b, e-a; \\ f, e \end{bmatrix} x, \tag{20}$$

where the Saalschützian condition is now $a+f = c+d$.

As in § 2 above we can express this as an identity connecting definite integrals of which independent proofs can similarly be given. The identities (12), (19) are possibly new.

4. *The Saalschützian product* ${}_2F_1 \times {}_2F_1$

We can expand the product

$$y = {}_2F_1\begin{bmatrix} a, b; \\ c \end{bmatrix} x\,{}_2F_1\begin{bmatrix} a', b'; \\ c' \end{bmatrix} x \tag{21}$$

in the form $$y = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{n!(c)_n} x^n\,{}_4F_3\begin{bmatrix} a', b', 1-c-n, -n \\ c', 1-a-n, 1-b-n \end{bmatrix}.$$

When $$a+a'+b+b' = c+c' \equiv 2k \text{ (say)}, \tag{22}$$

the ${}_4F_3$ is Saalschützian, a term that again we extend to the product itself. With this condition (22) the product satisfies a differential equation of rank 2, given in (7), 75, (91) as

$$\begin{aligned}[] [(\delta+c-1)\,(\delta+k-2)\,(\delta+2k-2) - x(\delta+k)\,(\delta+a+a'-1)\,(\delta+a+b'-1)] \\ \times [\delta(\delta+c'-1) - x(\delta+a'+b)\,(\delta+b'+b)]\,y \\ = 4(a-c)\,bx(\delta+k-1)\,(\delta+k-\tfrac{1}{2})\,(\delta+k)\,y. \end{aligned} \tag{23}$$

We interchange the pair of factors $\delta+c-1$, $\delta+b+b'$ by the substitution

$$\Gamma(\delta+c)\,y = \Gamma(\delta+b+b')\,y_1,$$

and the pair of factors $\delta+c'-1$, $\delta+a+a'-1$ by the substitution

$$\Gamma(\delta+c')\,y_1 = \Gamma(\delta+a+a')\,z.$$

Employing these substitutions simultaneously, so that

$$\Gamma(\delta+c)\,\Gamma(\delta+c')\,y = \Gamma(\delta+a+a')\,\Gamma(\delta+b+b')\,z,$$

we get the equation

$$\begin{aligned}[] [(\delta+b+b'-1)\,(\delta+k-2)\,(\delta+2k-2) - x(\delta+k)\,(\delta+c'-1)\,(\delta+a+b'-1)] \\ \times [\delta(\delta+a+a'-1) - x(\delta+a'+b)\,(\delta+c)]\,z \\ = 4(a-c)\,bx(\delta+k-1)\,(\delta+k-\tfrac{1}{2})\,(\delta+k)\,z. \end{aligned}$$

It is soon seen that this equation is obtained from (23) by the change of parameters

$$a, b, c;\ a', b', c' \to b', c-a, b+b';\ c'-b', a, a+a',$$

which leaves unaltered the Saalschützian condition (22). We thus have the identity

$$\Delta(c)\,\Delta(c')\,{}_2F_1\left[\begin{matrix}a, b;\\ c\end{matrix}\ x\right]{}_2F_1\left[\begin{matrix}a', b';\\ c'\end{matrix}\ x\right]$$
$$= \Delta(a+a')\,\Delta(b+b')\,{}_2F_1\left[\begin{matrix}b', c-a;\\ b+b'\end{matrix}\ x\right]{}_2F_1\left[\begin{matrix}c'-b', a;\\ a+a'\end{matrix}\ x\right], \qquad (24)$$

subject, of course, to the Saalschützian condition

$$a+a'+b+b' = c+c'.$$

The symmetry latent in this result can be brought out by use of Euler's identity

$$F(\alpha, \beta; \gamma; x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta; \gamma; x). \qquad (25)$$

Applied, for instance, to the right-hand side of (24) it turns the $F \times F$ into

$${}_2F_1\left[\begin{matrix}b, c'-a';\\ b+b'\end{matrix}\ x\right]{}_2F_1\left[\begin{matrix}c'-b', a;\\ a+a'\end{matrix}\ x\right]$$

and so on.

If we pick out the coefficients of x^n on the two sides of (24), we get the identity between terminated Saalschützian ${}_4F_3$'s with unit argument:*

$${}_4F_3\left[\begin{matrix}a', b', 1-c-n, -n\\ c', 1-a-n, 1-b-n\end{matrix}\right]$$
$$= \frac{(b')_n\,(c-a)_n\,(a+a')_n}{(a)_n\,(b)_n\,(c')_n}\,{}_4F_3\left[\begin{matrix}c'-b', a, 1-b-b'-n, -n\\ a+a', 1-b'-n, 1-c+a-n\end{matrix}\right]. \qquad (26)$$

We can express (24) as an identity between triple integrals in the form

$$\int_0^1\int_0^1\int_0^1 t^{c-1}(1-t)^{a+a'-c-1}\,u^{a-1}(1-u)^{c-a-1}\,v^{a'-1}(1-v)^{c'-a'-1}$$
$$\times (1-tux)^{-b}\,(1-tvx)^{-b'}\,dt\,du\,dv$$
$$= \int_0^1\int_0^1\int_0^1 T^{b+b'-1}(1-T)^{c'-b-b'-1}\,U^{c-a-1}(1-U)^{c'-a'-1}\,V^{a-1}(1-V)^{a'-1}$$
$$\times (1-TUx)^{-b'}\,(1-TVx)^{b'-c'}\,dT\,dU\,dV. \qquad (27)$$

This can be verified directly by the substitution

$$T = \frac{1-tu-v(1-t)}{1-tu-tuv(1-t)\,x}, \quad U = \frac{tv(1-u)}{1-tu-v(1-t)}, \quad V = tu.$$

* In Whipple's notation this is the identity $S(1, 2, 3) = S(2, 4, 6)$; see Bailey (1), § 7·3.

5. *Bailey's identity*

If in the equation (23) we write $a+a' = b+b'$, so that with the Saalschützian relation (22) we have

$$a+a' = b+b' = k, \quad c+c' = 2k, \tag{28}$$

we can remove a left-hand operator $\delta+k-2$ and rewrite the equation (now of the fourth order) as

$$\begin{aligned}[][(\delta+c-1)(\delta+2k-2)-x(\delta+k)(\delta+k+a-b-1)] \\ \times [\delta(\delta+2k-c-1)-x(\delta+k+b-a)(\delta+k)]\,y \\ = 4(a-c)\,bx(\delta+k)(\delta+k-\tfrac{1}{2})\,y.\end{aligned}$$

If now on the left we exchange the last operator $\delta+k$ with one or other of the operators $\delta+c-1$, $\delta+2k-2$, we can remove a further operator $\delta+k-1$ from the left. That is to say, if we write

$$\text{(i)}\ \Gamma(\delta+c)\,y = \Gamma(\delta+k)\,z_1, \qquad \text{(ii)}\ \Gamma(\delta+2k-1)\,y = \Gamma(\delta+k)\,z_2,$$

we get the respective third-order equations

$$\begin{aligned}\text{(i)}\quad & [\delta+2k-2-x(\delta+k+a-b-1)] \\ & \times [\delta(\delta+2k-c-1)-x(\delta+c)(\delta+k-a+b)]\,z_1 \\ & = 4(a-c)\,bx(\delta+k-\tfrac{1}{2})\,z_1,\end{aligned} \tag{29}$$

$$\begin{aligned}\text{(ii)}\quad & [\delta+c-1-x(\delta+k+a-b-1)] \\ & \times [\delta(\delta+2k-c-1)-x(\delta+2k-1)(\delta+k-a+b)]\,z_2 \\ & = 4(a-c)\,bx(\delta+k-\tfrac{1}{2})\,z_2.\end{aligned} \tag{30}$$

We can identify the expressions on the left of these equations with the expression on the left of (18) above if, in (18), we make the respective changes of parameter

$$\begin{array}{lll} & \text{(i)} & \text{(ii)} \\ a \to & 2k+a-b-c-\tfrac{1}{2}, & a-b+\tfrac{1}{2} \\ b \to & 2k+b-a-c-\tfrac{1}{2}, & b-a+\tfrac{1}{2} \\ c \to & k-\tfrac{1}{2}, & k-\tfrac{1}{2} \end{array}\Bigg\}, \qquad \begin{array}{lll} & \text{(i)} & \text{(ii)} \\ d \to & 2k-1, & c \\ e \to & 2k-c, & 2k-c \\ f \to & c-k+\tfrac{1}{2}, & k-\tfrac{1}{2} \end{array}\Bigg\}.$$

Each substitution alike gives on the right of (18)

$$-x\{(b+c-a)^2-\tfrac{1}{4}\}(\delta+k-\tfrac{1}{2})\,y,$$

so that to complete the identification of (18) with (29) or (30) we need further

$$(b+c-a)^2-\tfrac{1}{4} = -4(a-c)\,b,$$

i.e.
$$c = a+b\pm\tfrac{1}{2}.$$

We therefore have the identities

$$\text{(i)}\quad (1-x)^{k-c-\frac{1}{2}}\,{}_3F_2\left[\begin{matrix}2k+a-b-c-\frac{1}{2},\,2k+b-a-c-\frac{1}{2},\,k-\frac{1}{2};\\ 2k-1\qquad\qquad 2k-c\end{matrix}\;x\right]$$
$$=\frac{\Delta(c)}{\Delta(k)}\,{}_2F_1\left[\begin{matrix}a,b;\\ c\end{matrix}\;x\right]{}_2F_1\left[\begin{matrix}k-a,k-b;\\ 2k-c\end{matrix}\;x\right],$$

$$\text{(ii)}\quad (1-x)^{-(k-\frac{1}{2})}\,{}_3F_2\left[\begin{matrix}a-b+\frac{1}{2},\,b-a+\frac{1}{2},\,k-\frac{1}{2};\\ c,\,2k-c\end{matrix}\;x\right]$$
$$=\frac{\Delta(2k-1)}{\Delta(k)}\,{}_2F_1\left[\begin{matrix}a,b;\\ c\end{matrix}\;x\right]{}_2F_1\left[\begin{matrix}k-a,k-b;\\ 2k-c\end{matrix}\;x\right],$$

when c is given one of the values $a+b\pm\frac{1}{2}$. These two identities, we note in passing, are identical when $2k = c+1$. If we equate in them the two expressions for ${}_2F_1\times{}_2F_1$, we get, not unnaturally, an identity included in (19).

In (i) choose the value $c = a+b-\frac{1}{2}$ and make the change of parameters

$$a,\,b,\,k\to c-a+\tfrac{1}{2},\,c-b+\tfrac{1}{2},\,c+\tfrac{1}{2},$$

so that consequently $c\to 2c-a-b+\frac{1}{2}$. Then we at once have Bailey's identity (7).

If in (ii) we choose the same value of c and make the same change of parameters, we get the identity (which may be new)

$$(1-x)^{-c}\,{}_3F_2\left[\begin{matrix}a-b+\frac{1}{2},\,b-a+\frac{1}{2},\,c;\\ 2c-a-b+\frac{1}{2},\,a+b+\frac{1}{2}\end{matrix}\;x\right]$$
$$=\frac{\Delta(2c)}{\Delta(c+\frac{1}{2})}\,{}_2F_1\left[\begin{matrix}a,b;\\ a+b+\frac{1}{2}\end{matrix}\;x\right]{}_2F_1\left[\begin{matrix}c-a+\frac{1}{2},c-b+\frac{1}{2};\\ 2c-a-b+\frac{1}{2}\end{matrix}\;x\right]. \quad (31)$$

Choice of the alternative value $a+b+\frac{1}{2}$ for c gives essentially the same identities in changed parameters. It is, in fact, equivalent to a transformation of the products ${}_2F_1\times{}_2F_1$ by Euler's identity, for, under the Saalschützian condition $a+a'+b+b' = c+c'$, we have

$${}_2F_1\left[\begin{matrix}a,b;\\ c\end{matrix}\;x\right]{}_2F_1\left[\begin{matrix}a',b';\\ c'\end{matrix}\;x\right]={}_2F_1\left[\begin{matrix}c-a,c-b;\\ c\end{matrix}\;x\right]{}_2F_1\left[\begin{matrix}c'-a',c'-b';\\ c'\end{matrix}\;x\right]. \quad (32)$$

Identification of the coefficients of x^n on the two sides of (31) gives the identity of terminated ${}_4F_3$'s of unit argument

$${}_4F_3\left[\begin{matrix}a-b+\frac{1}{2},\,b-a+\frac{1}{2},\,c,\,-n\\ 2c-a-b+\frac{1}{2},\,a+b+\frac{1}{2},\,1-c-n\end{matrix}\right]$$
$$=\frac{(a)_n\,(b)_n\,(2c)_n}{(a+b+\frac{1}{2})_n\,(c)_n\,(c+\frac{1}{2})_n}\,{}_4F_3\left[\begin{matrix}c-a+\frac{1}{2},\,c-b+\frac{1}{2},\,\frac{1}{2}-a-b-n,\,-n\\ 2c-a-b+\frac{1}{2},\,1-a-n,\,1-b-n\end{matrix}\right]. \quad (33)$$

6. *Orr's identities*

Orr's first identity (5) comes at once from (29) by writing

$$k = c+\tfrac{1}{2}, \quad u = (\delta+c)\,z_1.$$

For we then get

$$[\delta+2c-1-x(\delta+c+a-b-\tfrac{1}{2})]\,[\delta-x(\delta+c-a+b+\tfrac{1}{2})]\,u = 4(a-c)\,bxu.$$

But this is the equation we get from (9) by the change of parameters

$$a,\, b,\, c,\, d \to 2a,\, 2b,\, 2c,\, c-a-b+\tfrac{1}{2}.$$

This proves (5).

We can deduce (6) from (5) by using (24). For, in (24), make the Eulerian transformation (32) on the right and put $c' = c+1$, $a+a' = b+b' = c+\tfrac{1}{2}$. This gives us

$$\Delta(c)\,\Delta(c+1)\,{}_2F_1\begin{bmatrix} a, b; & x \\ c & \end{bmatrix} {}_2F_1\begin{bmatrix} c-a+\tfrac{1}{2}, c-b+\tfrac{1}{2}; & x \\ c+1 & \end{bmatrix}$$
$$= \Delta(c+\tfrac{1}{2})\,\Delta(c+\tfrac{1}{2})\,{}_2F_1\begin{bmatrix} a+\tfrac{1}{2}, b; & x \\ c+\tfrac{1}{2} & \end{bmatrix} {}_2F_1\begin{bmatrix} c-a+\tfrac{1}{2}, c-b; & x \\ c+\tfrac{1}{2} & \end{bmatrix}.$$

With (5) we get

$$(1-x)^{a+b-c-\frac{1}{2}}\,{}_2F_1\begin{bmatrix} 2a, 2b; & x \\ 2c & \end{bmatrix} = \frac{\Delta(c+\tfrac{1}{2})}{\Delta(c)}\,{}_2F_1\begin{bmatrix} a+\tfrac{1}{2}, b; & x \\ c+\tfrac{1}{2} & \end{bmatrix} {}_2F_1\begin{bmatrix} c-a+\tfrac{1}{2}, c-b; & x \\ c+\tfrac{1}{2} & \end{bmatrix},$$

which is (6) with $a+\tfrac{1}{2}$, $c+\tfrac{1}{2}$ written for a, c.

We can put this analysis in another way in deriving (6) directly from (23). For put

$$a+a' = c+\tfrac{1}{2}, \quad b+b' = c-\tfrac{1}{2}, \quad c = c' = k.$$

Then (23) is

$$(\delta+c-1)\,[(\delta+c-2)\,(\delta+2c-2)-x(\delta+c-\tfrac{1}{2})\,(\delta+a-b+c-\tfrac{3}{2})]$$
$$\times\,[\delta(\delta+c-1)-x(\delta+c-\tfrac{1}{2})\,(\delta-a+b+c+\tfrac{1}{2})]\,y$$
$$= 4(a-c)\,bx\,(\delta+c)\,(\delta+c-\tfrac{1}{2})\,(\delta+c-1)\,y.$$

Remove the left-hand factor $\delta+c-1$, operate with $\Gamma(\delta+c-2)$, and write

$$\Gamma(\delta+c-\tfrac{1}{2})\,z = \Gamma(\delta+c)\,y. \tag{34}$$

We can then remove a left-hand operator $\Gamma(\delta+c-\tfrac{1}{2})$, leaving

$$[\delta+2c-2-x(\delta+a-b+c-\tfrac{3}{2})]\,[\delta-x(\delta-a+b+c+\tfrac{1}{2})]\,z = 4(a-c)\,bxz.$$

Reference to (9) above shows that this is a differential equation satisfied by

$$z = (1-x)^{a+b-c-\frac{1}{2}}\,{}_2F_1(2a-1, 2b;\, 2c-1;\, x).$$

This with (34) gives (6).

7. *Cayley's identity*

In Cayley's identity the product ${}_2F_1 \times {}_2F_1$ is no longer Saalschützian and it therefore escapes the net of the foregoing analysis. It is, however, deducible from the pair of Orr's identities.* We can, in fact, go a little further and make it depend on either of Orr's identities by a suitable operation of the type $\delta + h$ that increases by one the parameter difference

$$\text{``}a + a' + b + b' - c - c'\text{''}$$

and restores the Saalschützian condition. Operation, for instance, with $\delta + a - b + c$ will give, on the right of (3),

$$\begin{aligned}\frac{\Delta(c+\frac{1}{2})}{\Delta(c)}\Bigg\{F\left[\begin{matrix}a, b; \\ c+\frac{1}{2}\end{matrix}\ x\right](\delta + c - b)\, F\left[\begin{matrix}c-a, c-b; \\ c+\frac{1}{2}\end{matrix}\ x\right] \\ + F\left[\begin{matrix}c-a, c-b; \\ c+\frac{1}{2}\end{matrix}\ x\right](\delta + a)\, F\left[\begin{matrix}a, b; \\ c+\frac{1}{2}\end{matrix}\ x\right]\Bigg\} \\ = \frac{\Delta(c+\frac{1}{2})}{\Delta(c)}\Bigg\{(c-b)\, F\left[\begin{matrix}a, b; \\ c+\frac{1}{2}\end{matrix}\ x\right] F\left[\begin{matrix}c-a, c-b+1; \\ c+\frac{1}{2}\end{matrix}\ x\right] \\ + a\, F\left[\begin{matrix}a+1, b; \\ c+\frac{1}{2}\end{matrix}\ x\right] F\left[\begin{matrix}c-a, c-b; \\ c+\frac{1}{2}\end{matrix}\ x\right]\Bigg\}; \qquad (35)\end{aligned}$$

and, on the left of (3),

$$\begin{aligned}(1-x)^{a+b-c}(\delta + 2c)\, F\left[\begin{matrix}2a, 2b; \\ 2c\end{matrix}\ x\right] + F\left[\begin{matrix}2a, 2b; \\ 2c\end{matrix}\ x\right](\delta - a - b + c)(1-x)^{a+b-c} \\ = 2a(1-x)^{a+b-c}\, F\left[\begin{matrix}2a+1, 2b; \\ 2c\end{matrix}\ x\right] + (c-a-b)(1-x)^{a+b-c-1}\, F\left[\begin{matrix}2a, 2b; \\ 2c\end{matrix}\ x\right]. \qquad (36)\end{aligned}$$

Using (6) we can equate (35) to the expression

$$(c-b)(1-x)^{a+b-c-1}\, F\left[\begin{matrix}2a, 2b-1; \\ 2c\end{matrix}\ x\right] + a(1-x)^{a+b-c}\, F\left[\begin{matrix}2a+1, 2b; \\ 2c\end{matrix}\ x\right],$$

and a little manipulation then identifies this with (36). Operation with δ or with $\delta + 2c + 1$ similarly contrived will bring into play Orr's other identity (5).

From its simplicity and symmetry of form Cayley's identity seems to deserve a better status than that of a corollary,† but much fruitless effort has driven us to see in it an exceptional theorem dependent on the precise values of the parameters and accordingly without obvious extensions.

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* See, for instance, Bailey (1), § 10·1, 86.

† Moreover, in the duplication formulae that follow, it is the series derived from Orr's identities that appear as corollaries to the series derived from Cayley's identity.

The following direct proof may be of interest.*

Write

$$y = \frac{\Delta(c+\frac{1}{2})}{\Delta(c)} F(a,b;\, c+\tfrac{1}{2};\, x)\, F(c-a, c-b;\, c+\tfrac{1}{2};\, x)$$

$$= F\begin{bmatrix} c+\frac{1}{2};\ a,b;\ c-a, c-b; \\ c;\ \ c+\frac{1}{2};\ \ \ c+\frac{1}{2} \end{bmatrix} x, x \Bigg],$$

so that y satisfies the differential equations

$$\Theta y = 0, \quad \Phi y = 0,$$

where $\Theta \equiv \theta(\theta+c-\frac{1}{2})(\delta+c-1) - x(\theta+a)(\theta+b)(\delta+c+\frac{1}{2}),$

$$\Phi \equiv \phi(\phi+c-\tfrac{1}{2})(\delta+c-1) - x(\phi+c-a)(\phi+c-b)(\delta+c+\tfrac{1}{2}).$$

Here θ is δ operating on the first factor only in y, ϕ is δ operating on the second factor only, and therefore $\theta+\phi$ is δ without restriction. We now have to prove that

$$y = (1-x)^{a+b-c} F(2a, 2b;\, 2c;\, x).$$

Subtraction and reduction gives

$$\Theta - \Phi = (\delta+c-\tfrac{1}{2})(\delta+c-1)[\theta-\phi-x(\theta-\phi+a+b-c)]. \tag{37}$$

Hence $$[\theta-\phi-x(\theta-\phi+a+b-c)]\, y = 0,$$

since y contains only positive integral powers of x. Thus

$$(1-x)\, 2\theta y = [\delta - x(\delta+a+b-c)]\, y. \tag{38}$$

Write $$y = (1-x)^{a+b-c} z, \tag{39}$$

so that we want $$z = F(2a, 2b;\, 2c;\, x),$$

and remember that always

$$[\delta - x(\delta+h)]\,(1-x)^h w = (1-x)^{h+1} \delta w. \tag{40}$$

Then (38) gives $$2\theta y = (1-x)^{a+b-c} \delta z. \tag{41}$$

Recalling that $\theta x = x(\theta+1)$ in Θ, but $\theta x = x\theta$ in Φ, while $\delta x = x(\delta+1)$ in both, we can write

$$(2\delta+2c-1-\theta)\,\Theta + \theta\Phi$$

$$= (2\delta+2c-1)\,\theta(\theta+c-\tfrac{1}{2})(\delta+c-1)$$

$$- x(2\delta+2c)(\theta+a)(\theta+b)(\delta+c+\tfrac{1}{2}) - (\Theta-\Phi)\,\theta$$

$$= (\delta+c-\tfrac{1}{2})(\delta+c-1)\{2[\theta(\theta+c-\tfrac{1}{2}) - x(\theta+a)(\theta+b)]$$

$$- [(\theta-\phi) - x(\theta-\phi+a+b-c)]\,\theta\}, \quad \text{by (37)}.$$

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* This (received 28 February 1945) replaces the much more cumbrous original analysis. We are indebted to the referee for suggesting its omission, and for thus spurring us to a further investigation.

Hence again

$$2[\theta(\theta+c-\tfrac{1}{2})-x(\theta+a)(\theta+b)]y = [(\theta-\phi)-x(\theta-\phi+a+b-c)]\theta y,$$

i.e.

$$[(\delta+2c-1)-x(\delta+a+b+c)]\theta y = 2abxy.$$

Thus, from (39), (41),

$$[\delta+2c-1-x(\delta+a+b+c)](1-x)^{a+b-c}\delta z = 4abx(1-x)^{a+b-c}z,$$

and so, with (40),

$$[\delta+2c-1-x(\delta+2a+2b)]\delta z = 4abxz,$$

i.e.

$$[\delta(\delta+2c-1)-x(\delta+2a)(\delta+2b)]z = 0.$$

Thus $z = F(2a, 2b; 2c; x)$ and the proof is complete.

8. *Some hypergeometric expansions*

Most of the foregoing identities can be made to yield expansions of a certain type by use of the following lemma:

LEMMA. *If* $F\begin{bmatrix} a, \ldots; & x \\ c, \ldots; & \end{bmatrix}$, $F\begin{bmatrix} a', \ldots; & x \\ c', \ldots; & \end{bmatrix}$ *are any two hypergeometric functions (of any order) and* h, k *any two suitable constants, then*

$$\frac{\Delta(h)}{\Delta(k)} F\begin{bmatrix} a, \ldots; & x \\ c, \ldots; & \end{bmatrix} F\begin{bmatrix} a', \ldots; & x \\ c', \ldots; & \end{bmatrix} = \sum_{r=0}^{\infty} \frac{(h)_r (k-h)_r (a)_r \ldots (a')_r \ldots}{r!(k+r-1)_r (k)_{2r} (c)_r \ldots (c')_r \ldots} x^{2r}$$
$$\times F\begin{bmatrix} h+r, a+r, \ldots; & x \\ k+2r, c+r, \ldots; & \end{bmatrix} F\begin{bmatrix} h+r, a'+r, \ldots; & x \\ k+2r, c'+r, \ldots; & \end{bmatrix},$$

where

$$\Delta(h) \equiv \Gamma(\delta+h)/\Gamma(h), \quad etc. \tag{42}$$

Proof. We have

$$\frac{\Delta(h)}{\Delta(k)} F\begin{bmatrix} a, \ldots; & x \\ c, \ldots; & \end{bmatrix} = F\begin{bmatrix} h, a, \ldots; & x \\ k, c, \ldots; & \end{bmatrix}$$

and so, as usual,*

$$\frac{(-\delta)_r \Delta(h)}{(\delta+k)_r \Delta(k)} F\begin{bmatrix} a, \ldots; & x \\ c, \ldots; & \end{bmatrix} = (-)^r \frac{(h)_r (a)_r \ldots}{(k)_{2r} (c)_r \ldots} x^r F\begin{bmatrix} h+r, a+r, \ldots; & x \\ k+2r, c+r, \ldots; & \end{bmatrix}.$$

Thus the series on the right of (42) can be written

$$\frac{\Gamma(\theta+h)\Gamma(\phi+h)\{\Gamma(k)\}^2}{\Gamma(\theta+k)\Gamma(\phi+k)\{\Gamma(h)\}^2} \sum_{r=0}^{\infty} \frac{(k-h)_r (k)_{2r} (-\theta)_r (-\phi)_r}{r!(h)_r (k+r-1)_r (\theta+k)_r (\phi+k)_r}$$
$$\times F\begin{bmatrix} a, \ldots; & x \\ c, \ldots; & \end{bmatrix} F\begin{bmatrix} a', \ldots; & x \\ c', \ldots; & \end{bmatrix},$$

where (as in § 7 above) θ, ϕ act respectively on the first F alone and on the second F alone.

* Cf., for instance, (3), § 1, 250.

We have therefore to prove that

$$\frac{\Gamma(\theta+\phi+h)\,\Gamma(h)\,\Gamma(\theta+k)\,\Gamma(\phi+k)}{\Gamma(\theta+\phi+k)\,\Gamma(k)\,\Gamma(\theta+h)\,\Gamma(\phi+h)} = {}_5F_4\left[\begin{matrix} k-1, \tfrac{1}{2}k+\tfrac{1}{2}, k-h, & -\theta, & -\phi \\ \tfrac{1}{2}k-\tfrac{1}{2}, \quad h, & \theta+k, & \phi+k \end{matrix}\right].$$

But this is a corollary of Dougall's theorem.* As quoted by Bailey we make the change of parameters

$$a, c, d, e \rightarrow k-1, k-h, -\theta, -\phi,$$

and this completes the proof of the lemma.

Applying the lemma to (12) above in the form

$$(1-x)^{-d}\,{}_2F_1\left[\begin{matrix} a, b; \\ c \end{matrix}\; x\right] = \frac{\Delta(b+d)}{\Delta(c)}(1-x)^{-a}\,{}_2F_1\left[\begin{matrix} d, c-a; \\ b+d \end{matrix}\; x\right],$$

we get (after a little cancelling)

$$(1-x)^{-d}\,{}_2F_1\left[\begin{matrix} a, b; \\ c \end{matrix}\; x\right] = \sum_{r=0}^{\infty} \frac{(c-b-d)_r\,(a)_r\,(d)_r\,(c-a)_r}{r!(c+r-1)_r\,(c)_{2r}}\,x^{2r}$$
$$\times\,{}_2F_1\left[\begin{matrix} b+d+r, a+r; \\ c+2r \end{matrix}\; x\right]{}_2F_1\left[\begin{matrix} d+r, c-a+r; \\ c+2r \end{matrix}\; x\right].$$

Euler's transformation performed on the first two F's reduces this to

$$ {}_2F_1\left[\begin{matrix} c-a, c-b; \\ c \end{matrix}\; x\right] = \sum_{r=0}^{\infty} \frac{(c-a)_r\,(a)_r\,(d)_r\,(c-b-d)_r}{r!(c+r-1)_r\,(c)_{2r}}\,x^{2r}$$
$$\times\,{}_2F_1\left[\begin{matrix} c-a+r, c-b-d+r; \\ c+2r \end{matrix}\; x\right]{}_2F_1\left[\begin{matrix} c-a+r, d+r; \\ c+2r \end{matrix}\; x\right], \quad (43)$$

which is exactly our (4), 114, (16) in different parameters.

We can similarly apply the lemma to (19) having first moved $\Delta(d)$ to the right. The factor $(1-x)^{b-d}$ on the right gives rise to the hypergeometric function $F(b+f+r, d-b+r; d+2r; x)$ which by Euler's identity is equal to

$$(1-x)^{-f}\,F(b+r, d-b-f+r; d+2r; x).$$

Cancelling the $(1-x)^{-f}$ on the two sides and noting that $d-b-f = a+c-e$ by the Saalschützian condition (17), we have finally

$$ {}_3F_2\left[\begin{matrix} a, b, c; \\ d, e \end{matrix}\; x\right] = \sum_{r=0}^{\infty} \frac{(b)_r\,(d-b)_r\,(a+c-e)_r\,(e-a)_r\,(e-c)_r}{r!(e)_r\,(d+r-1)_r\,(d)_{2r}}\,x^{2r}$$
$$\times\,{}_2F_1\left[\begin{matrix} b+r, a+c-e+r; \\ d+2r \end{matrix}\; x\right]{}_3F_2\left[\begin{matrix} b+r, e-a+r, e-c+r; \\ d+2r, e+r \end{matrix}\; x\right]. \quad (44)$$

---

* (9), 122 (9); or Bailey (1), 27, § 4·4 (1).

Applying similar technique to Bailey's identity (7) we get

$$
{}_3F_2\left[\begin{matrix}2a, 2b, c;\\ 2c, a+b+\frac{1}{2}\end{matrix}\, x\right] = \sum_{r=0}^{\infty} \frac{(a+b-c)_r\,(a)_r\,(b)_r\,(c-a+\frac{1}{2})_r\,(c-b+\frac{1}{2})_r}{r!(a+b+\frac{1}{2})_r\,(c+r-\frac{1}{2})_r\,(c+\frac{1}{2})_{2r}}\, x^{2r}
$$

$$
\times\, {}_2F_1\left[\begin{matrix}a+r, b+r;\\ c+2r+\frac{1}{2}\end{matrix}\, x\right] {}_3F_2\left[\begin{matrix}2c-a-b+r+\frac{1}{2}, a+r, b+r;\\ c+2r+\frac{1}{2}, a+b+r+\frac{1}{2}\end{matrix}\, x\right]. \tag{45}
$$

The analogous identity (31) does not yield a result of interest. We may note that, if in (45) we make $c \to \infty$, then, formally at any rate,

$$
{}_2F_1\left[\begin{matrix}2a, 2b;\\ a+b+\frac{1}{2}\end{matrix}\, \tfrac{1}{2}x\right] = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r\,(b)_r}{r!(a+b+\frac{1}{2})_r}\, x^{2r}\, {}_2F_1\left[\begin{matrix}a+r, b+r;\\ a+b+r+\frac{1}{2}\end{matrix}\, 2x\right]. \tag{46}
$$

The coefficients here can be verified by use of Saalschütz's identity and the duplication formula of the gamma function.

9. *Duplication formulae*

The method of the preceding section applied to Cayley's identity (3) gives the expansion

$$
{}_2F_1\left[\begin{matrix}2a, 2b;\\ 2c\end{matrix}\, x\right] = \sum_{r=0}^{\infty} \frac{(-\frac{1}{2})_r\,(a)_r\,(b)_r\,(c-a)_r\,(c-b)_r}{r!(c+\frac{1}{2})_r\,(c+r-1)_r\,(c)_{2r}}\, x^{2r} \left\{ {}_2F_1\left[\begin{matrix}a+r, b+r;\\ c+2r\end{matrix}\, x\right]\right\}^2, \tag{47}
$$

which may be regarded as a "duplication formula" of the hypergeometric function. It should be compared for contrast and similarity with our "duplication formula" (**4**), 115, (24):

$$
{}_2F_1\left[\begin{matrix}a, b;\\ c\end{matrix}\, x^2\right] = \sum_{r=0}^{\infty} \frac{(a)_r\,(b)_r\,(c-a)_r\,(c-b)_r}{r!(c+r-1)_r\,(c)_{2r}}\, x^{2r}
$$

$$
\times\, {}_2F_1\left[\begin{matrix}a+r, b+r;\\ c+2r\end{matrix}\, x\right] {}_2F_1\left[\begin{matrix}a+r, b+r;\\ c+2r\end{matrix}\, -x\right].
$$

Operation on (47) with $1+\delta/(2c-1)$ and with $1+\delta/2a$ gives respectively

$$
{}_2F_1\left[\begin{matrix}2a, 2b;\\ 2c-1\end{matrix}\, x\right] = \sum_{r=0}^{\infty} \frac{(-\frac{1}{2})_r\,(a)_r\,(b)_r\,(c-a)_r\,(c-b)_r}{r!(c-\frac{1}{2})_r\,(c+r-1)_r\,(c)_{2r}}\, x^{2r}
$$

$$
\times\, {}_2F_1\left[\begin{matrix}a+r, b+r;\\ c+2r\end{matrix}\, x\right] {}_3F_2\left[\begin{matrix}a+r, b+r, c+r+\frac{1}{2};\\ c+2r, c+r-\frac{1}{2}\end{matrix}\, x\right], \tag{48}
$$

$$
{}_2F_1\left[\begin{matrix}2a+1, 2b;\\ 2c\end{matrix}\, x\right] = \sum_{r=0}^{\infty} \frac{(-\frac{1}{2})_r\,(a+1)_r\,(b)_r\,(c-a)_r\,(c-b)_r}{r!(c+\frac{1}{2})_r\,(c+r-1)_r\,(c)_{2r}}\, x^{2r}
$$

$$
\times\, {}_2F_1\left[\begin{matrix}a+r, b+r;\\ c+2r\end{matrix}\, x\right] {}_2F_1\left[\begin{matrix}a+r+1, b+r;\\ c+2r\end{matrix}\, x\right]. \tag{49}
$$

These are the series derived from Orr's identities (5), (6) by the method of § 8.* Here it is to be noted that the formulae derived from Orr's identities appear as subsidiary to those derived from Cayley's identity. The series (47) converges when $|x| < 1$, since by (**3**), 267, (77),

$$F(a+r, b+r; c+2r; x) < \frac{(c)_{2r}}{(a)_r (b)_r} F(a, b; c; x),$$

where we sufficiently regard a, b, c, x as positive.

Expansions, reciprocal to (47), (48), (49), can be given as follows:

$$\left\{ {}_2F_1\left[\begin{matrix} a, b; \\ c \end{matrix}\; x\right]\right\}^2 = \sum_{r=0}^{\infty} \frac{(\frac{1}{2})_r (a)_r (b)_r (c-a)_r (c-b)_r}{r!(c)_r (c)_{2r} (c+r-\frac{1}{2})_r} x^{2r}\, {}_2F_1\left[\begin{matrix} 2a+2r, 2b+2r; \\ 2c+4r \end{matrix}\; x\right], \tag{50}$$

$${}_2F_1\left[\begin{matrix} a, b; \\ c \end{matrix}\; x\right] {}_3F_2\left[\begin{matrix} a, b, c+\frac{1}{2}; \\ c-\frac{1}{2}, c \end{matrix}\; x\right]$$
$$= \sum_{r=0}^{\infty} \frac{(\frac{1}{2})_r (a)_r (b)_r (c-a)_r (c-b)_r (c+\frac{1}{2})_r}{r!(c)_r (c)_{2r} (c-\frac{1}{2})_{2r}} x^{2r}\, {}_3F_2\left[\begin{matrix} 2a+2r, 2b+2r, 2c+2r; \\ 2c+4r, 2c+2r-1 \end{matrix}\; x\right], \tag{51}$$

$${}_2F_1\left[\begin{matrix} a, b; \\ c \end{matrix}\; x\right] {}_2F_1\left[\begin{matrix} a+1, b; \\ c \end{matrix}\; x\right]$$
$$= \sum_{r=0}^{\infty} \frac{(\frac{1}{2})_r (a+1)_r (b)_r (c-a)_r (c-b)_r}{r!(c)_r (c+r-\frac{1}{2})_r (c)_{2r}} x^{2r}\, {}_2F_1\left[\begin{matrix} 2a+2r+1, 2b+2r; \\ 2c+4r \end{matrix}\; x\right]. \tag{52}$$

Of these (51), (52) can be obtained from (50) by operation (once more) with $1+\delta/(2c-1)$, $1+\delta/2a$, respectively.

We can deduce (50) from (47). For, if on the right of (50) we replace $F(2a+2r, 2b+2r; 2c+4r; x)$ by the series in (47), taking $N-r$ as the parameter of summation, we get after some rearrangement

$$\sum_{N=0}^{\infty} \frac{(-\frac{1}{2})_N (a)_N (b)_N (c-a)_N (c-b)_N}{N!(c+\frac{1}{2})_N (c+N-1)_N (c)_{2N}} x^{2N} \left\{ {}_2F_1\left[\begin{matrix} a+N, b+N; \\ c+2N \end{matrix}\; x\right]\right\}^2$$
$$\times\, {}_5F_4\left[\begin{matrix} c-\frac{1}{2}, \frac{1}{2}c+\frac{3}{4}, \frac{1}{2}, c+N-1, & -N \\ \frac{1}{2}c-\frac{1}{4}, c, \quad \frac{3}{2}-N, & c+N+\frac{1}{2} \end{matrix}\right].$$

By the corollary to Dougall's theorem already quoted, this ${}_5F_4$ can be expressed as a product of gamma functions which includes $\Gamma(1-N)$ in the denominator. The ${}_5F_4$ thus vanishes unless $N = 0$. The outer sum therefore reduces to its first term $\{F(a, b; c; x)\}^2$, and the proof of (50) is complete. Conversely we can deduce (47) from (50) by a similar argument.

---

* We have to make the changes $c \to c-\frac{1}{2}$ in (48) and $a, c \to a+1, c+\frac{1}{2}$ in (49).

With the foregoing "duplication formulae" we may associate the somewhat similar pairs of reciprocal series

$$ {}_2F_1\left[\begin{matrix} a, b; \\ c \end{matrix}\ 2x-x^2\right] = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r (b)_r (c-a)_r (c-b)_r}{r!(c+r-1)_r (c)_{2r}} x^{2r} \left\{ {}_2F_1\left[\begin{matrix} a+r, b+r; \\ c+2r \end{matrix}\ x\right]\right\}^2, \tag{53} $$

$$ \left\{ {}_2F_1\left[\begin{matrix} a, b; \\ c \end{matrix}\ x\right]\right\}^2 = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (c-a)_r (c-b)_r}{r!(c)_r (c)_{2r}} x^{2r}\, {}_2F_1\left[\begin{matrix} a+r, b+r; \\ c+2r \end{matrix}\ 2x-x^2\right], \tag{54} $$

$$ {}_2F_1\left[\begin{matrix} a, b; \\ c \end{matrix}\ 2x-x^2\right] = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r (b)_r (c-a)_r (c-b)_r}{r!(c+r-\frac{1}{2})_r (c)_{2r}} x^{2r}\, {}_2F_1\left[\begin{matrix} 2a+2r, 2b+2r; \\ 2c+4r \end{matrix}\ x\right], \tag{55} $$

$$ {}_2F_1\left[\begin{matrix} 2a, 2b; \\ 2c \end{matrix}\ x\right] = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (c-a)_r (c-b)_r}{r!(c+\frac{1}{2})_r (c)_{2r}} x^{2r}\, {}_2F_1\left[\begin{matrix} a+r, b+r; \\ c+2r \end{matrix}\ 2x-x^2\right]. \tag{56} $$

Of these (53), (54) are merely our (3), (50), (51) with y put equal to x. We can deduce (56) from (47) by substituting for $({}_2F_1)^2$ from (54) in (47), rearranging as in the proof of (50) above, and using a corollary of Dougall's theorem.* Similarly, (55) can be deduced from (56) or from (53), (50).

Lastly we note the identity

$$ {}_2F_1\left[\begin{matrix} a, b; \\ c \end{matrix}\ 4x-4x^2\right] = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (a+b-c+\frac{1}{2})_r}{r!(c)_{2r}} (4x^2)^r\, {}_2F_1\left[\begin{matrix} 2a+2r, 2b+2r; \\ c+2r \end{matrix}\ x\right], \tag{57} $$

which reduces, when $c = a+b+\frac{1}{2}$, to Gauss's quadratic identity. To prove (57) we quote (4), 115, (21), (23) in the forms

$$ {}_2F_1\left[\begin{matrix} a, b; \\ c \end{matrix}\ 2x\right] = \sum_{s=0}^{\infty} \frac{(a)_s (b)_{2s}}{s!(c)_{2s}} x^{2s}\, {}_2F_1\left[\begin{matrix} 2a+2s, b+2s; \\ c+2s \end{matrix}\ x\right], \tag{58} $$

$$ {}_2F_1\left[\begin{matrix} a, b; \\ c \end{matrix}\ 4x-4x^2\right] = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (c-a)_r}{r!(c)_{2r}} (-4x^2)^r\, {}_2F_1\left[\begin{matrix} a+r, 2b+2r; \\ c+2r \end{matrix}\ 2x\right], \tag{59} $$

and substitute from (58) for the ${}_2F_1$ on the right of (59). Writing $N-r$ for s and rearranging, we then have

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} a, b; \\ c \end{matrix}\ 4x-4x^2\right] &= \sum_{N=0}^{\infty} \frac{(a)_N (2b)_{2N}}{N!(c)_{2N}} x^{2N}\, {}_2F_1\left[\begin{matrix} 2a+2N, 2b+2N; \\ c+2N \end{matrix}\ x\right] {}_2F_1\left[\begin{matrix} c-a, -N \\ b+\frac{1}{2} \end{matrix}\right] \\ &= \sum_{N=0}^{\infty} \frac{(a)_N (b)_N (a+b-c+\frac{1}{2})_N}{N!(c)_{2N}} (2x)^{2N}\, {}_2F_1\left[\begin{matrix} 2a+2N, 2b+2N; \\ c+2N \end{matrix}\ x\right], \end{aligned} $$

by Vandermonde's theorem.

---

* Precisely it is the one given by Bailey as (1), 28, (3).

10. *A multiplication formula*

We conclude with the "multiplication formula"

$$
{}_2F_1\left[\begin{matrix}a,b;\\c\end{matrix}\;x\right]{}_2F_1\left[\begin{matrix}\alpha,\beta;\\\gamma\end{matrix}\;x\right]
$$
$$
=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r(\gamma)_r}{r!(c)_r(c+\gamma+r-1)_r}\,{}_3F_2\left[\begin{matrix}\alpha,1-c-r,-r\\\gamma,1-a-r\end{matrix}\right]
$$
$$
\times\,{}_3F_2\left[\begin{matrix}\beta,1-c-r,-r\\\gamma,1-b-r\end{matrix}\right]x^r\,{}_2F_1\left[\begin{matrix}a+\alpha+r,b+\beta+r;\\c+\gamma+2r\end{matrix}\;x\right]. \tag{60}
$$

By comparing coefficients of x^n on the two sides we see that it is equivalent to the identity

$$
\frac{(a)_n(b)_n(c+\gamma)_n}{(c)_n(a+\alpha)_n(b+\beta)_n}\,{}_4F_3\left[\begin{matrix}\alpha,\beta,1-c-n,&-n\\\gamma,\;1-a-n,1-b-n\end{matrix}\right]
$$
$$
=\sum_{r=0}^{n}\binom{n}{r}\frac{(a)_r(b)_r(\gamma)_r(c+\gamma)_{2r}}{(a+\alpha)_r(b+\beta)_r(c)_r(c+\gamma+n)_r(c+\gamma+r-1)_r}
$$
$$
\times\,{}_3F_2\left[\begin{matrix}\alpha,1-c-r,-r\\\gamma,1-a-r\end{matrix}\right]{}_3F_2\left[\begin{matrix}\beta,1-c-r,-r\\\gamma,1-b-r\end{matrix}\right]. \tag{61}
$$

To prove this pair of formulae notice first that operation with $\delta+a+\alpha$ on (60) gives

$$
a\,{}_2F_1\left[\begin{matrix}a+1,b;\\c\end{matrix}\;x\right]{}_2F_1\left[\begin{matrix}\alpha,\beta;\\\gamma\end{matrix}\;x\right]+\alpha\,{}_2F_1\left[\begin{matrix}a,b;\\c\end{matrix}\;x\right]{}_2F_1\left[\begin{matrix}\alpha+1,\beta;\\\gamma\end{matrix}\;x\right]
$$
$$
=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r(\gamma)_r}{r!(c)_r(c+\gamma+r-1)_r}\,{}_3F_2\left[\begin{matrix}\alpha,1-c-r,-r\\\gamma,1-a-r\end{matrix}\right]{}_3F_2\left[\begin{matrix}\beta,1-c-r,-r\\\gamma,1-b-r\end{matrix}\right]
$$
$$
\times\,(a+\alpha+r)\,x^r\,{}_2F_1\left[\begin{matrix}a+\alpha+r+1,b+\beta+r;\\c+\gamma+2r\end{matrix}\;x\right].
$$

If we substitute from (60) for the two products ${}_2F_1\times{}_2F_1$ on the left of this, we get an identity so long as

$$
(a+r)\,{}_3F_2\left[\begin{matrix}\alpha,1-c-r,-r\\\gamma,-a-r\end{matrix}\right]+\alpha\,{}_3F_2\left[\begin{matrix}\alpha+1,1-c-r,-r\\\gamma,1-a-r\end{matrix}\right]
$$
$$
=(a+\alpha+r)\,{}_3F_2\left[\begin{matrix}\alpha,1-c-r,-r\\\gamma,1-a-r\end{matrix}\right],
$$

which is just the identity

$$
\{-(\delta+c_2-1)+(\delta+a_1)\}\,{}_3F_2\left[\begin{matrix}a_1,a_2,a_3;\\c_1,c_2\end{matrix}\;x\right]=(a_1-c_2+1)\,{}_3F_2\left[\begin{matrix}a_1,a_2,a_3;\\c_1,c_2\end{matrix}\;x\right]
$$

when $x = 1$. Rearranging the argument we see that, if (60) is true for some $b, c, \alpha, \beta, \gamma$ and all a, it is true for $b, c, \alpha+1, \beta, \gamma$ and so on. The operation $\delta+a+\alpha$ merely multiplies a term x^n by $n+a+\alpha$; we can therefore transfer the argument to (61) for some fixed n. If we multiply (61) through by $(a+\alpha)_n$, we get a (presumed) identity in α of order not exceeding n. Thus, by the argument above, if it is true for some α (and all a), it is true by induction for $\alpha+1, \alpha+2, \ldots, \alpha+n, \ldots$, i.e. for more than n values of α, and so for all α.

Now, if in (61) we put $\alpha = 0$, $\beta = 0$, we get

$$\frac{(c+\gamma)_n}{(c)_r} = \sum_{r=0}^{\infty} \binom{n}{r} \frac{(\gamma)_r (c+\gamma)_{2r}}{(c)_r (c+\gamma+n)_r (c+\gamma+r-1)_r}$$

$$= {}_4F_3\left[\begin{matrix} c+\gamma-1, \tfrac{1}{2}c+\tfrac{1}{2}\gamma+\tfrac{1}{2}, \gamma, \quad -n; \\ \tfrac{1}{2}c+\tfrac{1}{2}\gamma-\tfrac{1}{2}, c, c+\gamma+n; \end{matrix} -1\right],$$

and this is true by the corollary of Dougall's theorem just quoted.* Thus (61) is true for $\alpha = 0$, $\beta = 0$, all γ, a, b, c and any fixed (positive integer) n. Hence by our argument it is true for $\beta = 0$, all α, γ, a, b, c and fixed n. There is symmetry between α, β and we can therefore argue similarly in β, so that now (61) is proved for all $\alpha, \beta, \gamma, a, b, c$ and every positive integer n. The proof of (61), and consequently of (60), is then complete.

In (60) put $(\alpha, \beta, \gamma) = (a, b, c)$. The first ${}_3F_2$ on the right is then

$${}_3F_2\left[\begin{matrix} -r, a, 1-c-r \\ 1-a-r, c \end{matrix}\right].$$

By Dixon's theorem† this reduces to

$$\frac{\Gamma(1-\tfrac{1}{2}r)\,\Gamma(1-r-a)\,\Gamma(c)\,\Gamma(\tfrac{1}{2}r-a+c)}{\Gamma(1-r)\,\Gamma(1-\tfrac{1}{2}r-a)\,\Gamma(\tfrac{1}{2}r+c)\,\Gamma(c-a)}. \qquad (62)$$

This vanishes, if r is odd, because of the infinite factor $\Gamma(1-r)$ in the denominator (or we can change the sign of the ${}_3F_2$ by reversing the order of its terms). When r is an even integer $2n$, we can rewrite (62) as

$$\frac{(n)_n (c-a)_n}{(a+n)_n (c)_n}.$$

Dealing similarly with the other ${}_3F_2$ and cancelling factors on the right, we again get the "duplication formula" (50).

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* (1), 28, (3).

† (8), 289; quoted by Bailey (1), 13, (1).

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The University, Durham

Christ Church, Oxford

THE SELF-POLAR RIEMANN COMPLEX FOR A V_4

By H. S. RUSE

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This paper follows three others (Ruse, 1944*a*, *b*, 1946).† It deals with the case when the Riemann tensor at a point P, (x^i), of V_4 defines in the associated projective space S_3 a quadratic complex of lines which is self-polar with respect to the fundamental quadric.

Some of the results in the present paper are given in one form or another by Struik (1927–8), Lamson (1930) and Churchill (1932). The whole theory is contained explicitly or implicitly in works on classical projective geometry, and it has, moreover, close connexions with spinor analysis and the theory of groups. For the latter facts no apology is made, the whole purpose of the paper being to obtain results in Riemannian geometry by making as wide an appeal as possible to classical projective geometry and group theory.

1. *Notation and preliminaries*‡

Let T_4 be the affine tangent space at P, (x^i), of V_4, and S_3 the projective 3-space at infinity in T_4. Then the equation of the *Riemann complex* in S_3 is

$$R_{ijkl}\,p^{ij}p^{kl} = 0. \tag{1·1}$$

Here $p^{ij} \equiv X^iY^j - Y^iX^j$ are the Plücker coordinates of the line of S_3 joining the points X^i, Y^i, and are the components in V_4 of the simple bivector defined by the contravariant vectors X^i, Y^i at P.

The dual equation of the complex is

$${}^\circ R^{ijkl}\,{}^\circ p_{ij}\,{}^\circ p_{kl} = 0,$$

where

$${}^\circ p_{ij} = \tfrac{1}{2}\epsilon_{ijkl}\,p^{kl}, \tag{1·2}$$

$${}^\circ R^{ijkl} = \tfrac{1}{4}\epsilon^{ijmn}\epsilon^{klpq}R_{mnpq}, \tag{1·3}$$

† These papers will be referred to as **1**, **2**, **3** respectively.

‡ A more detailed account of the notation, etc., is given in papers **1**, **3**.

the ϵ's being the dualizing tensors of non-zero components $\pm\sqrt{g}$, $\pm 1/\sqrt{g}$ respectively. The normalizing factors $\frac{1}{2}$, $\frac{1}{4}$ are inserted for convenience.

The fundamental tensor defines in S_3 the *fundamental quadric* of point- and tangential-equations

$$g_{ij}X^iX^j = 0, \quad g^{ij}u_iu_j = 0$$

respectively. The Riemann complex is self-polar with respect to this quadric (**1**, § 6) if

$$\kappa R^{ijkl} = {}^\circ R^{ijkl},$$

where, as shown in **1**, the scalar κ must be equal to ± 1. If $\kappa = +1$, then

$$R^{ijkl} = {}^\circ R^{ijkl}, \tag{1·4}$$

whence

$$R_{ij} = \tfrac{1}{4}g_{ij}R, \tag{1·5}$$

so that V_4 satisfies the *Einstein condition* at (x^i). Here R_{ij} is the Ricci tensor defined by $R_{ij} = g^{mn}R_{imnj}$. If $\kappa = -1$, then

$$R^{ijkl} = -{}^\circ R^{ijkl}, \tag{1·6}$$

whence

$$R_{ijkl} = -\tfrac{1}{2}(g_{ik}R_{jl}+g_{jl}R_{ik}-g_{il}R_{jk}-g_{jk}R_{il}), \tag{1·7}$$

and also

$$R = 0. \tag{1·8}$$

It is easy to see that (1·8) is a necessary consequence of (1·7).

When the Riemann complex satisfies (1·4), or equivalently (1·5), it is called *self-polar of the first kind*; and when it satisfies (1·6), or equivalently (1·7) and (1·8), it is called *self-polar of the second kind*. The former is by far the more interesting case, and it is mainly with it that the present paper deals.

Even when the Riemann complex is not itself self-polar, the tensor can be expressed as a sum of two parts which respectively represent self-polar complexes of the first and second kinds. Thus, if we put

$$R_{ijkl} = G_{ijkl} + E_{ijkl},$$

where

$$G_{ijkl} = \tfrac{1}{2}(R_{ijkl}+{}^\circ R_{ijkl}), \quad E_{ijkl} = \tfrac{1}{2}(R_{ijkl}-{}^\circ R_{ijkl}),$$

following Churchill (1932), then G_{ijkl} satisfies (1·4) and E_{ijkl} satisfies (1·6).

Raising and lowering suffixes by means of the fundamental tensor corresponds geometrically to taking the polar reciprocal in S_3 with respect to the fundamental quadric; raising and lowering pairs of skew suffixes by means of the ϵ-tensors (indicated notationally by $^\circ$) corresponds to taking the dual coordinates of the geometric object represented by the operand. The two operations are commutative.

A skew tensor (bivector) a_{ij} represents in S_3 the linear complex of dual equations†

$$a_{ij}p^{ij} = 0, \quad {}^\circ a^{ij\circ}p_{ij} = 0.$$

A covariant vector r_i represents the plane of equation $r_i X^i = 0$.

Ennuplet coordinates X^a in S_3 are defined by taking at P, (x^i), in V_4 an arbitrary orthogonal ennuple $h_i^a \equiv (h_i^1, \ldots, h_i^4)$, $a, b, \ldots, g$ being used for ennuplet suffixes and $i, j, \ldots$ for tensor suffixes. The h's satisfy the usual relations

$$g_{ab}h_i^a h_j^b = g_{ij}, \quad h_a^i h_j^a = \delta_j^i, \quad g^{ij}h_i^a h_j^b = g^{ab},$$

etc., where g_{ab}, g^{ab} *are both equal to the Kronecker delta* δ_{ab}. When the h's define real directions of V_4 and the metric is not positive-definite at P, some of them are purely imaginary; for example, if the signature is $(+ + + -)$, then the h_i^a are real for $a = 1, 2, 3$ and imaginary for $a = 4$.

The ennuplet coordinates of a point and of a plane in S_3 are defined in the usual way by the equations

$$X^a = h_i^a X^i, \quad r_a = h_a^i r_i,$$

and are scalars for transformations of the coordinates (x^i) in V_4, but transform linearly under rotations of the ennuple. The points whose coordinates in the X^i-system are $h_1^i, \ldots, h_4^i$ are the vertices of the tetrahedron of reference of the ennuplet system X^a. The non-zero components of the ennuplet dualisors ϵ_{abcd}, ϵ^{abcd} are ± 1, and the ennuplet equation of the fundamental quadric is

$$g_{ab}X^a X^b \equiv (X^1)^2 + \ldots + (X^4)^2 = 0. \tag{1·9}$$

A linear complex of ennuplet coordinates q^{ab} may be represented in the usual way by a point $(q^1, \ldots, q^6)$ of a projective S_5 by writing

$$\left.\begin{aligned} q^1 = q^{23}, \quad q^2 = q^{31}, \quad q^3 = q^{12}, \\ q^4 = q^{14}, \quad q^5 = q^{24}, \quad q^6 = q^{34}, \end{aligned}\right\} \tag{1·10}$$

or, as a single equation,

$$\text{either} \quad q^\alpha = \tfrac{1}{2}\gamma_{ab}^\alpha q^{ab} \quad \text{or} \quad q^{ab} = \gamma_\alpha^{ab} q^\alpha \tag{1·11}$$

(see **3**, § 3). Greek suffixes will always run from 1 to 6 and will refer to S_5, while Latin suffixes will continue to run from 1 to 4 and will refer to S_3 or to V_4. As it stands, the transformation amounts simply to the replacement of pairs of skew Latin suffixes by single Greek suffixes according to the scheme

$$\left.\begin{matrix} ab = & 23 & 31 & 12 & 14 & 24 & 34 \\ \alpha = & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}\right\}. \tag{1·12}$$

† In discussing an arbitrary bivector, it is a matter of taste or convenience whether the covariant tensor be regarded as primary and the contravariant as the dual, or vice versa.

It is to be noticed that, if γ^α_{ab} and γ^{ab}_α in (1·11) are regarded as tensors of the types indicated by their suffixes, then, if desired, the coordinates of S_3 and S_5 may be transformed independently of one another. In particular, (1·11) may be written

$$q^\alpha = \tfrac{1}{2}\gamma^\alpha_{ij}q^{ij}, \quad q^{ij} = \gamma^{ij}_\alpha q^\alpha, \tag{1·13}$$

where

$$\gamma^\alpha_{ij} = \gamma^\alpha_{ab}h^a_i h^b_j, \quad \gamma^{ij}_\alpha = \gamma^{ab}_\alpha h^i_a h^j_b.$$

When the transformation has the simple form (1·10) the coordinates q^α are called *basic*, the orthogonal ennuple h^i_a being the *basis*.

The lines (or special complexes) p^{ij} of S_3 satisfy the identity

$$\tfrac{1}{4}\epsilon_{ijkl}\,p^{ij}p^{kl} = 0 \tag{1·14}$$

and so correspond in S_5 to the points of the 4-quadric

$$\epsilon_{\alpha\beta}p^\alpha p^\beta = 0,$$

where

$$\epsilon_{\alpha\beta} = \epsilon_{\beta\alpha} = \tfrac{1}{4}\epsilon_{abcd}\gamma^{ab}_\alpha\gamma^{cd}_\beta. \tag{1·15}$$

This will be called the ϵ-quadric. In the basic coordinate-system,†

$$\epsilon_{\alpha\beta} = \begin{bmatrix} O & I \\ I & O \end{bmatrix} = \epsilon^{\alpha\beta} \quad (\alpha \text{ row, } \beta \text{ column}), \tag{1·16}$$

O and I being the null and unit 3×3 matrices respectively. The $\epsilon^{\alpha\beta}$ are the coefficients in the tangential equation of the same 4-quadric, so $\epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} = \delta^\alpha_\beta$, the tangential equation being

$$\epsilon^{\alpha\beta}\,{}^\circ p_\alpha{}^\circ p_\beta = 0. \tag{1·17}$$

To the Riemann complex $R_{ijkl}\,p^{ij}p^{kl} = 0$ and to the special complex of lines in S_3 which touch the fundamental quadric, namely

$$g_{ijkl}\,p^{ij}p^{kl} = 0, \tag{1·18}$$

where

$$g_{ijkl} \equiv g_{ik}\,g_{jl} - g_{il}\,g_{jk}, \tag{1·19}$$

correspond in S_5 the 4-quadrics

$$R_{\alpha\beta}p^\alpha p^\beta = 0, \quad g_{\alpha\beta}p^\alpha p^\beta = 0,$$

—or rather not the complete 4-quadrics themselves but their intersections with the ϵ-quadric. Here

$$R_{\alpha\beta} = \tfrac{1}{4}R_{abcd}\gamma^{ab}_\alpha\gamma^{cd}_\beta, \quad g_{\alpha\beta} = \tfrac{1}{4}g_{abcd}\gamma^{ab}_\alpha\gamma^{cd}_\beta. \tag{1·20}$$

In the basic system the matrices $R_{\alpha\beta}$, $g_{\alpha\beta}$ have the forms

$$R_{\alpha\beta} = R_{\beta\alpha} = \begin{bmatrix} P & S \\ S^t & Q \end{bmatrix}, \quad g_{\alpha\beta} = g_{\beta\alpha} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}, \tag{1·21}$$

† Equations (1·16) are of course non-covariantive. In a basic coordinate-system the covariant and contravariant components of the ϵ-tensors happen to be the same.

P and Q being symmetric 3×3 matrices and S^t the transpose of the 3×3 matrix S [**3**, (3·13), (3·28), (3·29)]. The tangential equation of the g-quadric is $g^{\alpha\beta}p_\alpha p_\beta = 0$, $g^{\alpha\beta}$ also being given in the basic system by the second of the matrices (1·21). It should be noted that $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are both equal to the unit 6×6 matrix, and that $g_{\alpha\beta}p^\alpha p^\beta = \sum_{\alpha=1}^{6}(p^\alpha)^2$.

To the dual $^\circ R^{ijkl}$ of the Riemann tensor corresponds in S_5 the 4-quadric (envelope) of equation

$$^\circ R^{\alpha\beta}\,{}^\circ p_\alpha\,{}^\circ p_\beta = 0,$$

where

$$^\circ R^{\alpha\beta} = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}R_{\gamma\delta}.$$

This is the polar of the 4-quadric $R_{\alpha\beta}$ with respect to the ϵ-quadric. Thus the processes in S_3 of taking dual coordinates and of taking polars with respect to the fundamental quadric correspond in S_5 to taking polars with respect to the ϵ- and g-quadrics. Each of the latter quadrics is self-polar with respect to the other (**3**, (3·21) *et seq.*).

Thus, as stated in **3**, we have in S_5 the 4-quadrics $\epsilon_{\alpha\beta}$, $g_{\alpha\beta}$, $R_{\alpha\beta}$, from the last of which we obtain:

(i) its ϵ-polar (envelope) $^\circ R^{\alpha\beta}$;

(ii) its g-polar (envelope) $R^{\alpha\beta}$;

(iii) the g-polar $^\circ R_{\alpha\beta}$ of $^\circ R^{\alpha\beta}$, which is the same as the ϵ-polar of $R^{\alpha\beta}$. That is,

$$^\circ R_{\alpha\beta} = \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}R^{\gamma\delta} = g_{\alpha\gamma}g_{\beta\delta}\,{}^\circ R^{\gamma\delta}.$$

The Riemann complex may be classified in the usual way by means of the Segrè characteristic of the matrix pencil $[R_{\alpha\beta}-\lambda\epsilon_{\alpha\beta}]$ (Jessop, 1903, pp. 230–232; Zindler, 1922, pp. 1128–1131, 1133). Also, as pointed out in § 4 of **3**, the Segrè characteristic of the pencil $[R_{\alpha\beta}-\mu g_{\alpha\beta}]$ has a similar significance in the present theory. The characteristics are called the ϵ- and g-characteristics respectively, and the roots of the corresponding characteristic equations the ϵ- and g-roots.

2. *Change of basis*

Let $p^\alpha \equiv (p^1, \ldots, p^6)$ continue to represent the basic coordinate-system in S_5 obtained by an arbitrary choice of orthogonal ennuple h^i_a in V_4, the p^α thereby being equal to the ennuplet coordinates $(p^{23}, \ldots, p^{34})$ of a line, or more generally of a linear complex, in S_3. Take in S_5 the new coordinate-system defined by

$$\left.\begin{aligned} \chi^1 &= \frac{1}{\sqrt{2}}(p^1+p^4), & \chi^2 &= \frac{1}{\sqrt{2}}(p^2+p^5), & \chi^3 &= \frac{1}{\sqrt{2}}(p^3+p^6),\\ \chi^4 &= \frac{1}{\sqrt{2}}(p^1-p^4), & \chi^5 &= \frac{1}{\sqrt{2}}(p^2-p^5), & \chi^6 &= \frac{1}{\sqrt{2}}(p^3-p^6),\end{aligned}\right\} \tag{2·1}$$

or, as one equation, $$\chi^\alpha = K^\alpha_\beta p^\beta, \tag{2·2}$$
where the 6×6 matrix $[K^\alpha_\beta]$ is given by
$$K \equiv [K^\alpha_\beta] = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}. \tag{2·3}$$
By (1·11) the χ's are then related to the ennuplet coordinates in S_3 by the equations
$$p^\alpha = \tfrac{1}{2}\gamma'^\alpha_{ab} p^{ab},$$
where $$\gamma'^\alpha_{ab} = K^\alpha_\beta \gamma^\beta_{ab}.$$
Evidently $$K = K^{-1} = K^t, \tag{2·4}$$
so the transformation inverse to (2·2) is
$$p^\alpha = K^\alpha_\beta \chi^\beta.$$
Under this transformation the tensor $R_{\alpha\beta}$ becomes†
$$\begin{aligned} {}_{(\chi)}R_{\alpha\beta} &= R_{\gamma\delta}\frac{\partial p^\gamma}{\partial \chi^\alpha}\frac{\partial p^\delta}{\partial \chi^\beta} \\ &= R_{\gamma\delta} K^\gamma_\alpha K^\delta_\beta \\ &= K^t R K \quad \text{in matrix notation,} \\ &= \tfrac{1}{2}\begin{bmatrix} P+Q+S+S^t & P-Q-S+S^t \\ P-Q+S-S^t & P+Q-S-S^t \end{bmatrix} \end{aligned} \tag{2·5}$$
by (1·21) and (2·3). Likewise, by (1·16) and (2·21), $\epsilon_{\alpha\beta}$ and $g_{\alpha\beta}$ become
$${}_{(\chi)}\epsilon_{\alpha\beta} = \begin{bmatrix} I & O \\ O & -I \end{bmatrix} = {}_{(\chi)}\epsilon^{\alpha\beta}, \quad {}_{(\chi)}g_{\alpha\beta} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} = {}_{(\chi)}g^{\alpha\beta}. \tag{2·6}$$
Thus $g_{\alpha\beta}$ is invariant under the transformation, and we have
$$\begin{aligned} {}_{(\chi)}\epsilon_{\alpha\beta}\chi^\alpha\chi^\beta &= (\chi^1)^2+(\chi^2)^2+(\chi^3)^2-(\chi^4)^2-(\chi^5)^2-(\chi^6)^2 \\ &= \sum_{\kappa=1}^{6}(\pm)_\alpha(\chi^\alpha)^2, \end{aligned} \tag{2·7}$$
$${}_{(\chi)}g_{\alpha\beta}\chi^\alpha\chi^\beta = \sum_{\alpha=1}^{6}(\chi^\alpha)^2 \tag{2·8}$$
for the ϵ- and g-quadrics. Here, by definition,
$$\left.\begin{aligned} (\pm)_\alpha &= +1 \quad \text{when} \quad \alpha = 1, 2 \text{ or } 3, \\ &= -1 \quad \text{when} \quad \alpha = 4, 5 \text{ or } 6. \end{aligned}\right\} \tag{2·9}$$
The suffix of $(\pm)_\alpha$ will never sum unless preceded by Σ.

† For the moment the new coordinate-system will be indicated by a bracketed suffix preceding the tensor. This cumbersome notation will be discarded later.

From the point of view here adopted, the χ's form a new coordinate-system in S_5, but they may also be regarded as pseudo-Klein coordinates of a line in S_3 [Jessop, 1903, p. 20; Hudson, 1905, p. 46. The proper Klein coordinates are $(x^\alpha) \equiv (\chi^1, \chi^2, \chi^3, -i\chi^4, -i\chi^5, -i\chi^6)$].

A change of basic reference-system is brought about by a rotation of the ennuple h_a^i in V_4. This leads to an orthogonal transformation of coordinates in S_3, which, as is well known, induces a particular kind of linear transformation of the coordinates χ^α of S_5. The following is an exact formulation of this transformation-theory in the form needed for present purposes. But a word of caution is required. As remarked above, if the fundamental quadratic form is not positive-definite at (x^i), some vectors of the basic ennuple h_a^i are purely imaginary if the vectors define real directions of V_4. This concealment of the signature of g_{ij} is made in order that all cases may be treated together without the use of the indicators e_a (Eisenhart, 1926, pp. 36, 37) which specify the signature. The non-zero ennuplet components g_{ab} of the fundamental tensor are then all $+1$. To ignore the signature in this way may lead to mistakes unless care is exercised, but the question of signature fortunately does not seriously arise until reality-conditions are considered in § 4. For the moment, therefore, it suffices to say that the symbols used below in general represent complex numbers, and that for the time being we are dealing with the complex rotation group (cf. Cartan, 1938, vol. 1, p. 11, § 8).

If, then, h_a^i, h_i^a are the respective contravariant and covariant components of an arbitrary orthogonal ennuple at (x^i) in V_4 (hereafter called the *initial ennuple*), the equation in ennuplet coordinates $X^a = h_i^a X^i$ of the null cone at (x^i), or of the fundamental quadric in S_3, is as usual

$$g_{ab} X^a X^b \equiv (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = 0. \tag{2·10}$$

A rotation of the ennuple in V_4 is defined by equations

$$'h_a^i = l_a^b h_b^i, \quad 'h_i^a = L_b^a h_i^b, \tag{2·11}$$

where the coefficients l_a^b are any that satisfy the equations

$$g_{cd} l_a^c l_b^d = g_{ab}, \quad \det | l_b^a | = +1, \tag{2·12}$$

and $[L_b^a]$ is the matrix reciprocal to $[l_b^a]$ (cf. Ruse, 1937, p. 102). Thus $[l_b^a]$ is a 4×4 orthogonal matrix and its reciprocal $[L_b^a]$ is its transpose. It is mainly characterized by the fact that, if

$$\begin{aligned} 'X^a &= 'h_i^a X^i \\ &= L_b^a X^b, \end{aligned} \tag{2·13}$$

then $'g_{ab} = g_{ab} = \delta_{ab}$, and therefore

$$'g_{ab}\,'X^a\,'X^b = \Sigma('X^a)^2,$$

that is, the quadratic form (2·10) is invariant. In S_3 the $'X^a$ are coordinates having as tetrad of reference the points whose coordinates in the non-ennuplet system are $'h^i_a$.

The matrices $[l^a_b]$, $[L^a_b]$ are further characterized by the requirement, indicated in (2·12), that they be unimodular. We are thus concerned with the group of rotations in the strict sense, and not with the wider group of rotations-and-reflexions in which the determinant of the transformation may have either of the values ± 1.

In matrix notation, (2·13) is

$$'X = LX, \tag{2·14}$$

where $L \equiv [L^a_b]$ (a row, b column).

Instead of regarding this as defining a change of coordinates in S_3, we may regard it as defining a direct collineation which leaves the fundamental quadric invariant. Now (Veblen and Young, 1918, p. 336; Klein, 1928, p. 111 *et seq.*) any such collineation transforms each regulus of the quadric into itself, and is the product of two collineations, each of which transforms one of the reguli into itself while leaving the other invariant. Thus L in (2·14) is expressible in the form

$$L = {}_1L(\alpha)\,.\,{}_2L(\beta) = {}_2L(\beta)\,.\,{}_1L(\alpha), \tag{2·15}$$

or, in tensor notation, $$L^a_b = {}_1L^a_c\,.\,{}_2L^c_b = {}_2L^a_c\,.\,{}_1L^c_b, \tag{2·16}$$

where $${}_1L(\alpha) \equiv [{}_1L^a_b] = \begin{bmatrix} \alpha_4 & \alpha_3 & -\alpha_2 & \alpha_1 \\ -\alpha_3 & \alpha_4 & \alpha_1 & \alpha_2 \\ \alpha_2 & -\alpha_1 & \alpha_4 & \alpha_3 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & \alpha_4 \end{bmatrix}, \tag{2·17}$$

$${}_2L(\beta) = [{}_2L^a_b] = \begin{bmatrix} \beta_4 & \beta_3 & -\beta_2 & -\beta_1 \\ -\beta_3 & \beta_4 & \beta_1 & -\beta_2 \\ \beta_2 & -\beta_1 & \beta_4 & -\beta_3 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix}, \tag{2·18}$$

the α's and β's being numbers such that

$$\Sigma\alpha^2 = 1, \quad \Sigma\beta^2 = 1. \tag{2·19}$$

The determinants of these matrices are respectively equal to $(\Sigma\alpha^2)^2$, $(\Sigma\beta^2)^2$, so both have the value unity. Any transformation of type (2·17) leaves one

regulus invariant, and any of type (2·18) leaves the other invariant.† That any matrix ${}_1L(\alpha)$ is commutative with any matrix ${}_2L(\beta)$ is geometrically obvious.

An ${}_1L$-matrix becomes an ${}_2L$-matrix, and *vice versa*, if the signs of all the elements in the last row and column are altered, the common element α_4 (or β_4) thereby remaining unchanged in sign.

Suppose in particular that L is itself of the form ${}_1L(\alpha)$. Then (2·15) still holds, ${}_2L(\beta)$ now being the unit matrix. If p^{ab} is a line (or more generally any linear complex) in the initial ennuplet system X^a, it transforms into

$$'p^{ab} = {}_1L^a_c \cdot {}_1L^b_d\, p^{cd},$$

or, in matrix notation, $$'p = {}_1L(\alpha) \, . \, p \, . \, {}_1L^t(\alpha), \tag{2·20}$$

t as usual denoting the transpose. Substituting from (2·17) and then introducing the variables χ^α defined by (2·1), we find after some calculation that, *when the coordinates X^a in S_3 undergo the transformation ${}_1L(\alpha)$, the variables χ^α in S_5 undergo the transformation*

$$'\chi = \begin{bmatrix} \Lambda(\alpha) & O \\ O & I \end{bmatrix} \chi, \tag{2·21}$$

where χ is the column-matrix $\{\chi^1, \ldots, \chi^6\}$, O, I are as usual the null and unit 3×3 matrices, and

$$\Lambda(\alpha) = \begin{bmatrix} \alpha_4^2+\alpha_1^2-\alpha_2^2-\alpha_3^2 & 2(\alpha_1\alpha_2+\alpha_3\alpha_4) & 2(\alpha_1\alpha_3-\alpha_2\alpha_4) \\ 2(\alpha_1\alpha_2-\alpha_3\alpha_4) & \alpha_4^2+\alpha_2^2-\alpha_3^2-\alpha_1^2 & 2(\alpha_2\alpha_3+\alpha_1\alpha_4) \\ 2(\alpha_1\alpha_3+\alpha_2\alpha_4) & 2(\alpha_2\alpha_3-\alpha_1\alpha_4) & \alpha_4^2+\alpha_3^2-\alpha_1^2-\alpha_2^2 \end{bmatrix}. \tag{2·22}$$

Thus χ^1, χ^2, χ^3 undergo a linear transformation, while $'\chi^4 = \chi^4$, $'\chi^5 = \chi^5$, $'\chi^6 = \chi^6$. The invariance of χ^4, χ^5, χ^6 shows that, in S_5, the conic of intersection of the 2-plane $\chi^4 = 0 = \chi^5 = \chi^6$ with the ϵ-quadric is invariant under the transformation. This conic is the one whose points correspond to the lines of the regulus in S_3 which is transformed into itself by the collineation ${}_1L(\alpha)$, and its invariance was therefore to be expected.

If we put $$\chi_{123} \equiv \{\chi^1, \chi^2, \chi^3\}, \quad \chi_{456} \equiv \{\chi^4, \chi^5, \chi^6\}, \tag{2·23}$$

† This well-known result follows immediately from the theory of two-component spinors (or *semi-spinors* in the terminology of Cartan, 1938). If the coordinates of a point on the fundamental quadric are expressed in terms of such spinors (cf. Ruse, 1937, pp. 103, 104), then linear transformations applied to each spinor separately lead to transformations of types (2·17), (2·18) respectively. The possibility of factorizing a proper rotation, or rather a proper Lorentz transformation, in this way, was made the basis of the definition of *semi-vectors* by Einstein and Mayer (1932; also Murnaghan, 1938, p. 358).

curly brackets denoting column-matrices, then (2·21) may be written

$$'\chi_{123} = \Lambda(\alpha)\,\chi_{123}, \quad '\chi_{456} = \chi_{456}. \tag{2·23}'$$

$\Lambda(\alpha)$ is an orthogonal matrix. Thus

$$\Lambda(\alpha)\,.\,\Lambda^t(\alpha) = I$$

and

$$('\chi^1)^2+('\chi^2)^2+('\chi^3)^2 = (\chi^1)^2+(\chi^2)^2+(\chi^3)^2.$$

Its determinant has the value $(\Sigma\alpha^2)^3$, and this is equal to unity, by (2·19). It is the well-known matrix arising in the theory of rotations in a Euclidean 3-space, the α's being Euler parameters (see, e.g., Whittaker, 1927, p. 8; Darboux, 1896, Note v, p. 435; Klein, 1928, p. 102; or Murnaghan, 1938, p. 328). In the language of Cartan (1928, vol. 2, p. 61), χ_{123} and χ_{456} are *semi-bivectors*.

If instead of imposing the transformation ${}_1L(\alpha)$ in S_3 we now impose the transformation ${}_2L(\beta)$, so that (2·19) is replaced by

$$'p = {}_2L(\beta)\,.\,p\,.\,{}_2L^t(\beta),$$

we find that the χ's undergo the transformation

$$'\chi = \begin{bmatrix} I & O \\ O & \Lambda(\beta) \end{bmatrix} \chi \tag{2·24}$$

instead of (2·21), $\Lambda(\beta)$ *being the same function of the* β*'s as* $\Lambda(\alpha)$ *is of the* α*'s.* In this case, therefore,

$$'\chi_{123} = \chi_{123}, \quad '\chi_{456} = \Lambda(\beta)\,\chi_{456}, \tag{2·24}'$$

so, as we should expect, χ_{123} is invariant while χ_{456} undergoes a linear (orthogonal) transformation.

From (2·21) and (2·24) it follows that any transformation $L \equiv {}_1L(\alpha)\,.\,{}_2L(\beta)$ in S_3 which transforms the reguli simultaneously, each into itself, brings about in S_5 the transformation

$$'\chi = \begin{bmatrix} \Lambda(\alpha) & O \\ O & \Lambda(\beta) \end{bmatrix} \chi \tag{2·25}$$

(cf. Cartan, 1938, vol. 2, p. 64, (17)). The quadratic forms $(\chi^1)^2+(\chi^2)^2+(\chi^3)^2$ and $(\chi^4)^2+(\chi^5)^2+(\chi^6)^2$ are separately invariant, and hence, by (2·7) and (2·8), the ϵ- and g-quadrics are invariant. This again was to be expected, the $'\chi$'s being pseudo-Klein coordinates associated with the new ennuple defined by (2·11).

Conversely, since any 3×3 unimodular orthogonal matrix can be thrown into the form (2·22), the α's being determinate but for the sign of the set as a whole, it follows that, if we are given in S_5 a transformation which subjects χ_{123} and χ_{456} to separate unimodular orthogonal transformations, then it can be expressed in the form (2·25) and the corresponding trans-

formation ${}_1L(\alpha)\,.\,{}_2L(\beta)$ in S_3 can be determined but for sign. The corresponding new basic ennuple $'h^a_i$ is therefore also determined except for sign, that is, except for the orientation of the ennuple as a whole.

A simple type of orthogonal transformation to which χ_{123} or χ_{456} may be subjected, and one which will be of importance below, is that obtained by permuting the χ_{123} or the χ_{456} among themselves (cf. Hudson, 1905, p. 118). For present purposes, as will shortly be seen, it suffices to consider permutations of χ_{456} only, which lead in V_4 to rotations of type ${}_2L(\beta)$. Consider, for example, the transformation

$$'\chi_{123} = \chi_{123}, \quad '\chi^4 = -\chi^4, \quad '\chi^5 = -\chi^6, \quad '\chi^6 = -\chi^5, \tag{2·26}$$

which amounts to an interchange of χ^5, χ^6. (The minus signs are inserted to make the transformation unimodular, the permutation 465 of 456 being odd.†) Symbolically,

$$'\chi_{123} = \chi_{123}, \quad '\chi_{456} = -\chi_{465}. \tag{2·26$'$}$$

For this, the matrix $\Lambda(\beta)$ of (2·24) or (2·24)$'$ is

$$\Lambda(\beta) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},$$

from which, by (2·22) (with β's for α's), we obtain

$$\{\beta_1, \beta_2, \beta_3, \beta_4\} = \pm\frac{1}{\sqrt{2}}\{0, -1, 1, 0\}.$$

Choosing the + sign, we get for ${}_2L(\beta)$ the matrix‡

$$-\{465\}: \qquad \Omega_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}. \tag{2·27}$$

The corresponding matrices for the other odd permutations of χ_{456} are

$$-\{654\}: \qquad \Omega_5 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \tag{2·28}$$

† All three of the χ's have been given minus signs, but alternatively one could be given a minus sign and the other two plus. A reference to this possibility is made at the end of § 2.

‡ The $-\{465\}$ on the left of (2·27) indicates the transformation (2·26), and is inserted for convenience of reference.

$-\{546\}$: $$\Omega_6 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}. \tag{2·29}$$

The even permutation

$$'\chi_{123} = \chi_{123}, \quad '\chi_{456} = +\chi_{564},$$

for which $$\Lambda(\beta) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

gives for ${}_2L(\beta)$ the matrix

$+\{564\}$: $$E_5 = \tfrac{1}{2}\begin{bmatrix} -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}. \tag{2·30}$$

Similarly for the other even permutations, including 456 itself, we have

$+\{645\}$: $$E_6 = \tfrac{1}{2}\begin{bmatrix} -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \tag{2·31}$$

$+\{456\}$: $$E_4 = \text{the unit matrix}. \tag{2·32}$$

The transformations (2·27)–(2·32), or rather the transformations of the χ's which lead to them, will be referred to as *signed permutations*.†

Thus, for example, if we start with a basic ennuple h_i^a and obtain the corresponding χ's, then the transformation (2·26) in S_5 leads to the transformation Ω_4 in S_3, and hence, by (2·11) and (2·27), to the new orthogonal ennuple

$$'h_i^a = \Omega_{4|b}^{\ a}\, h_i^b = \frac{1}{\sqrt{2}}(h_i^2+h_i^3,\ -h_i^1+h_i^4,\ -h_i^1-h_i^4,\ -h_i^2+h_i^3). \tag{2·33}$$

If the coordinates χ_{123} are permuted instead of χ_{456}, a similar set of matrices is obtained, this time of type ${}_1L(\alpha)$. They are obtained, in fact, by changing

† The six matrices behave like the quaternions $\Omega_4 = (j-k)/\sqrt{2}$, $\Omega_5 = (k-i)/\sqrt{2}$, $\Omega_6 = (i-j)/\sqrt{2}$, $E_4 = 1$, $E_5 = -\frac{1}{2}(1-i-j-k)$, $E_6 = \frac{1}{2}(1+i+j+k)$. Cf. Veblen and Young, 1918, pp. 337, 338.

the signs of the elements in the last row and the last column of each of the matrices Ω, E already obtained. For example, the transformation

$$'\chi_{123} = -\chi_{132}, \quad '\chi_{456} = \chi_{456},$$

for which
$$\Lambda(\alpha) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},$$

gives for ${}_1L(\alpha)$ the matrix

$-\{132\}$:
$$\Omega_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \tag{2·34}$$

(cf. (2·27)). In this way we could obtain matrices Ω_1, Ω_2, Ω_3, E_1, E_2, E_3 of which E_1, like E_4, is the unit matrix. Now Ω_1 is obtainable from Ω_4 not only by changing the signs of the last row and column, but also by interchanging rows 2, 3 and changing the sign of row 4. Similarly, Ω_2 is obtainable from Ω_5 by interchanging rows 3, 1 and changing the sign of row 4; and so on. In each case, it will be noted, the distribution of zeros remains unaltered. Likewise E_2 has the same rows as E_6 and E_3 as E_5, though in each case in a different order. Thus for a given initial ennuple in V_4, the matrices $\Omega_1, \ldots, E_3$ produce the same sets of vectors as $\Omega_4, \ldots, E_6$ but differently ordered, and, in the case of the last vector of each set, oppositely directed. For example, Ω_1 gives

$$''h^a_i = \Omega_{1|b}^{\;a}, \quad h^b_i = \frac{1}{\sqrt{2}}(h^2_i + h^3_i,\ -h^1_i - h^4_i,\ -h^1_i + h^4_i,\ h^2_i - h^3_i), \tag{2·35}$$

which may be compared with (2·33). In so far as *orthogonal ennuple* is taken to mean an *ordered* set of four mutually perpendicular unit vectors, this is different from the ennuple (2·33). But in so far as it is taken to mean a set of four mutually perpendicular vectors irrespective of their order and individual orientations, this is the same ennuple as that defined by (2·33). The latter meaning is the one that must be adopted for the purposes of this paper: in S_3 each ennuple represents a tetrahedron self-polar with respect to the fundamental quadric, and it is a matter of indifference in which order their vertices are taken. So far as the underlying V_4 is concerned, the essential point is that, given a basic ennuple at (x^i), then the six matrices $\Omega_4, \ldots, E_6$, or alternatively $\Omega_1, \ldots, E_3$, define a total of 24 directions which fall into 6 orthogonal sets of 4. Ennuples which are composed of the same vectors but in different orders or with different orientations will be called *effectively* the same.

It should be noted that the product of any two or more of the whole set of twelve matrices Ω, E, also defines a rotation, which, applied to the basic ennuple, produces an ennuple effectively the same as one of the six already obtained. Nothing essentially new is therefore to be gained by considering the signed permutations of χ_{123} as well as those of χ_{456}, the matrices $\Omega_4, \ldots, E_6$ being sufficient by themselves to define the 24 directions associated with a given basic ennuple.

If in (2·26) two of the signs had been taken plus and only one minus, and similarly in the case of the other permutations, effectively the same ennuples would again have been obtained.

3. *First consequences of the condition of self-polarity*

The conditions (1·4), (1·6) for the Riemann complex to be self-polar of the first or second kind with respect to the fundamental quadric become, in S_5,

$$R^{\alpha\beta} = \pm\,{}^{\circ}R^{\alpha\beta}. \tag{3·1}$$

Either of these equations states that the g-polar $R^{\alpha\beta}$ of the 4-quadric $R_{\alpha\beta}$ is the same as the ϵ-polar ${}^{\circ}R^{\alpha\beta}$ of the same 4-quadric. This being so, assume the g-characteristic to be [111111], and let Σ be the common self-polar simplex of the 4-quadrics $R_{\alpha\beta}$ and $g_{\alpha\beta}$. Take the polar reciprocal of the whole configuration in S_5 with respect to the g-quadric. Then, as the simplex Σ, the ϵ-quadric and the g-quadric itself are all self-polar with respect to the g-quadric, they reciprocate into themselves, but the quadric $R_{\alpha\beta}$ reciprocates into $R^{\alpha\beta}$, which, by (3·1), is the same as ${}^{\circ}R^{\alpha\beta}$. So Σ is likewise the common self-polar simplex of the g-quadric and the quadric ${}^{\circ}R^{\alpha\beta}$. Now take the polar reciprocal with respect to the ϵ-quadric. Then the g-quadric, the quadric ${}^{\circ}R^{\alpha\beta}$ and their common self-polar simplex Σ reciprocate respectively into the g-quadric (which is self-polar with respect to ϵ), the original quadric $R_{\alpha\beta}$ and the common self-polar simplex of the g- and R-quadrics, i.e. into Σ. Thus Σ reciprocates into itself, and so, *when the Riemann complex in S_3 is self-polar of the first or second kind with respect to the fundamental quadric and is of g-characteristic* [111111], *the common self-polar simplex of the 4-quadrics $R_{\alpha\beta}$, $g_{\alpha\beta}$ in S_5 is self-reciprocal with respect to the 4-quadric $\epsilon_{\alpha\beta}$.*

It is described as self-reciprocal and not as self-polar because, when the complex is self-polar of the *second* kind, the simplex is not self-polar with respect to the ϵ-quadric in the sense of having each vertex the polar of the opposite face, though it does transform into itself. This may be seen as follows. Suppose that the Riemann tensor satisfies the condition (1·7) for a self-polar complex of the second kind. Choose as basis the ennuple of

Ricci vectors at (x^i), that is, the set of four vectors h^i_a, assumed distinct, defined by the Ricci equation

$$(R_{ij}-\rho g_{ij})\,h^j = 0. \tag{3·2}$$

In S_3, $(h^i_1, \ldots, h^i_4)$ are the vertices of the common self-polar tetrahedron of the Ricci quadric $R_{ij}X^iX^j = 0$ and the fundamental quadric, and so, with this tetrahedron as frame of reference, the equations of the two quadrics reduce to sums of squares, namely

$$\sum_{a=1}^{4} \rho_a (X^a)^2 = 0, \quad \sum_{a=1}^{4} (X^a)^2 = 0,$$

ρ_a being the roots of the Ricci characteristic equation. In other words, if we form the ennuplet components of the Ricci and fundamental tensors, we obtain

$$R_{ab} = \operatorname{diag}\{\rho_1, \rho_2, \rho_3, \rho_4\}, \quad g_{ab} = \delta_{ab}.$$

Expressing (1·7) in ennuplet form by writing *abcd* for *ijkl*, we then find that the ennuplet components R_{abcd} of the Riemann tensor are zero except when *ab*, *cd* are the same; in fact

$$R_{\alpha\beta} = -\tfrac{1}{2}\operatorname{diag}\{\rho_2+\rho_3,\ \rho_3+\rho_1,\ \rho_1+\rho_2,\ \rho_1+\rho_4,\ \rho_2+\rho_4,\ \rho_3+\rho_4\},$$

$R_{\alpha\beta}$ being obtained as usual by replacing suffix-pairs by single numerals as in (1·12). Also by (1·8), which may be written $\sum_{a=1}^{4} R_{aa} = 0$, we have $\sum_{a=1}^{4} \rho_a = 0$, whence

$$R_{\alpha\beta} = \operatorname{diag}\{\sigma_1, \sigma_2, \sigma_3, -\sigma_1, -\sigma_2, -\sigma_3\}, \tag{3·3}$$

where $\sigma_1 = -\tfrac{1}{2}(\rho_2+\rho_3)$, etc. Thus the 4-quadric $R_{\alpha\beta}$ has equation

$$R_{\alpha\beta}p^\alpha p^\beta \equiv \sigma_1(p^1)^2+\sigma_2(p^2)^2+\sigma_3(p^3)^2-\sigma_1(p^4)^2-\sigma_2(p^5)^2-\sigma_3(p^6)^2 = 0,$$

while by (1·21) and (1·16), the g- and ϵ-quadrics are

$$\left.\begin{aligned} g_{\alpha\beta}p^\alpha p^\beta &\equiv (p^1)^2+(p^2)^2+\ldots+(p^6)^2 = 0,\\ \epsilon_{\alpha\beta}p^\alpha p^\beta &\equiv 2(p^1p^4+p^2p^5+p^3p^6) = 0. \end{aligned}\right\} \tag{3·4}$$

The equations of the R- and g-quadrics are thus sums of squares, and the selected coordinate-system is therefore the one having their common self-polar simplex as frame of reference. By (3·4) it is evident that each of the vertices of the simplex, for example $p^\alpha \equiv (1,0,0,0,0,0) \equiv \delta^\alpha_1$, lies on the ϵ-quadric, a fact otherwise obvious because the vertices are the points of S_5 corresponding to *lines* of S_3, namely, the edges of the common self-polar tetrahedron of the Ricci and fundamental quadrics. The polar 4-plane of the vertex δ^α_1 has coordinates $\epsilon_{\alpha\beta}\delta^\beta_1$, or $(0,0,0,1,0,0)$. Thus the common

self-polar simplex of $R_{\alpha\beta}$ and $g_{\alpha\beta}$ is inscribed in the ϵ-quadric in such a way that, if its vertices are numbered 1–6 and the opposite faces also numbered 1–6, then the polar reciprocals of the vertices 1, 2, 3, 4, 5, 6 with respect to the ϵ-quadric are respectively the faces 4, 5, 6, 1, 2, 3, and *vice versa*.

If $\mu_1, \ldots, \mu_6$ are the g-roots of the self-polar complex of the second kind, then, by (3·3),

$$\mu_1+\mu_4 = 0 = \mu_2+\mu_5 = \mu_3+\mu_6.$$

The ϵ-roots $\lambda_1, \ldots, \lambda_6$ are respectively i times the g-roots, and hence also

$$\lambda_1+\lambda_4 = 0 = \lambda_2+\lambda_5 = \lambda_3+\lambda_6$$

(cf. **3**, end of § 5). The complex is of the type known as *harmonic* (Jessop, 1903, p. 133). It is beyond the scope of the present paper to consider the special cases that arise when there are equalities among the Ricci roots $\rho_1, \ldots, \rho_4$.

When, however, the Riemann complex is self-polar of the *first* kind, the common self-polar tetrahedron of the 4-quadrics $R_{\alpha\beta}$, $g_{\alpha\beta}$ is also self-polar, in the sense of having each vertex the pole of the opposite face, with respect to the ϵ-quadric; that is, *for a self-polar Riemann complex of the first kind, it is possible to choose in S_5 a coordinate-system such that the equations of the 4-quadrics $R_{\alpha\beta}$, $g_{\alpha\beta}$, $\epsilon_{\alpha\beta}$ all reduce to sums of squares.* To prove this, choose an arbitrary initial ennuple in V_4 as basis, and take the condition of self-polarity in the form (3·1) (with the plus sign). Then

$$R^{\alpha\beta} = {}^{\circ}R^{\alpha\beta}, \tag{3·5}$$

that is, $$g^{\alpha\gamma}R_{\gamma\delta}g^{\delta\beta} = \epsilon^{\alpha\gamma}R_{\gamma\delta}\epsilon^{\delta\beta},$$

or in matrix notation, $$gRg = \epsilon R\epsilon,$$

whence by (1·16) and (1·21), since g is the unit 6×6 matrix,

$$\begin{bmatrix} P & S \\ S' & Q \end{bmatrix} = \begin{bmatrix} O & I \\ I & O \end{bmatrix}\begin{bmatrix} P & S \\ S' & Q \end{bmatrix}\begin{bmatrix} O & I \\ I & O \end{bmatrix}$$
$$= \begin{bmatrix} Q & S' \\ S & P \end{bmatrix},$$

and therefore $P = Q$ and $S = S'$. So S, as well as P and Q, is a symmetric matrix, and we have: *for a self-polar complex of the first kind, the Riemann 4-quadric in S_5, referred to any basic coordinate-system p^α, has the form*

$$R_{\alpha\beta} = \begin{bmatrix} P & S \\ S & P \end{bmatrix}, \tag{3·6}$$

where P, S are symmetric 3×3 *matrices.*

It follows from (2·5) that, if we now transfer to the pseudo-Klein coordinates χ^α defined by (2·1), then $R_{\alpha\beta}$ takes the form

$$R_{\alpha\beta} = \begin{bmatrix} P+S & O \\ O & P-S \end{bmatrix} \tag{3·7}$$

$$= \begin{bmatrix} U & O \\ O & V \end{bmatrix}, \tag{3·8}$$

where $U \equiv P+S$, $V \equiv P-S$ are symmetric 3×3 matrices, while $\epsilon_{\alpha\beta}$ and $g_{\alpha\beta}$ take the forms (2·6). The $R_{\alpha\beta}$ of (3·7) should strictly be distinguished from that of (3·6) by a subscript (χ) as in (2·5), but there will now be no danger of confusion if this tensor (and similarly $g_{\alpha\beta}$, $\epsilon_{\alpha\beta}$) is represented by the same symbol in different coordinate-systems.

If, as we have been assuming, the g-roots of $R_{\alpha\beta}$ are all different, then the latent roots of U and V are all different. Hence, as in the ordinary theory of conics in a plane, it is possible to find a linear transformation $\chi_{123} \rightarrow {}'\chi_{123}$ of the variables χ^1, χ^2, χ^3 which leaves the quadratic form $(\chi^1)^2 + (\chi^2)^2 + (\chi^3)^2$ invariant and simultaneously reduces U to diagonal form. If the transformation is

$${}'\chi_{123} = \Lambda(\alpha)\,\chi_{123} \tag{3·9}$$

(cf. (2·23)$'$), we thus have

$$\Lambda(\alpha)\,.\,U\,.\,\Lambda^t(\alpha) = \operatorname{diag}\{\mu_1, \mu_2, \mu_3\} \equiv M_{123}, \text{ say}, \tag{3·10}$$

where μ_1, μ_2, μ_3 are the latent roots of U. The orthogonal transformation $\Lambda(\alpha)$ is unique, inasmuch as the common self-polar triangle of a pair of unrelated conics is unique, except in so far as the variables χ_{123} may be permuted among themselves in the manner discussed in § 2. Likewise an orthogonal transformation

$${}'\chi_{456} = \Lambda(\beta)\,\chi_{456} \tag{3·11}$$

can be found which reduces V to diagonal form: thus

$$\Lambda(\beta)\,.\,V\,.\,\Lambda^t(\beta) = \operatorname{diag}\{\mu_4, \mu_5, \mu_6\} \equiv M_{456}, \text{ say}, \tag{3·12}$$

μ_4, μ_5, μ_6 being the latent roots of V. This transformation is also unique except for permutations of χ_{456}. Taking (3·9) and (3·11) together, we therefore have a transformation of type (2·25), namely,

$${}'\chi = \begin{bmatrix} \Lambda(\alpha) & O \\ O & \Lambda(\beta) \end{bmatrix} \chi, \tag{3·13}$$

such that, in the ${}'\chi$-system of coordinates in S_5, $R_{\alpha\beta}$ has the diagonal form

$$R_{\alpha\beta} = \begin{bmatrix} M_{123} & O \\ O & M_{456} \end{bmatrix}. \tag{3·14}$$

In V_4, as seen in § 2, there is a corresponding transformation $L \equiv {}_1L(\alpha)\,.\,{}_2L(\beta)$ which rotates the initial ennuple h^a_i into the new basic ennuple ${}'h^a_i$ with

respect to which the $'\chi^\alpha$ are pseudo-Klein coordinates. As the matrix $[g_{\alpha\beta}]$ in any such system of coordinates is the unit 6×6 matrix, it follows that *$\mu_1, \ldots, \mu_6$ are the g-roots of $[R_{\alpha\beta}]$*. The ennuple $'h_i^a$ will be called a *principal ennuple* and the coordinates $'\chi^\alpha$ associated with it a *principal* system of pseudo-Klein coordinates in S_5.

It is convenient here to change the notation and to represent $'h_i^a$, $'\chi^\alpha$ by v_i^a, ϖ^α respectively. We then have the following result, predicted above:

When the Riemann tensor satisfies the Einstein condition and the roots of the equation $\det|R_{\alpha\beta}-\mu g_{\alpha\beta}| = 0$ *are all different, it is possible to choose as basis an orthogonal ennuple v_i^a at (x^i) in such a way that $g_{abcd}\,p^{ab}p^{cd}$, $\epsilon_{abcd}\,p^{ab}p^{cd}$, $R_{abcd}\,p^{ab}p^{cd}$ are each reduced to sums of squares, namely, $\Sigma(\varpi^\alpha)^2$, $\Sigma(\pm)_\alpha(\varpi^\alpha)^2$, $\Sigma\mu_\alpha(\varpi^\alpha)^2$ [$(\pm)_\alpha$ being defined by* (2·7)], *when expressed in terms of the pseudo-Klein coordinates defined by*

$$\varpi^\alpha = K^\alpha_\beta p^\beta \tag{3·15}$$

(*cf.* (2·2)). *Here $\{p^\alpha\}\equiv\{p^{23}, p^{31}, \ldots, p^{34}\}$ are ennuplet coordinates with v_i^a as basis, and are thus given by*

$$p^{ab} = p^{ij}v_i^a v_j^b.$$

The point-equation of the fundamental quadric has the usual ennuplet form $\sum_{a=1}^{4}(X^a)^2 = 0$, from which it follows (Jessop, 1903, p. 100; Hudson, 1905, pp. 32, 41, 47) that *the quadric g_{ij} in S_3 is one of the ten "fundamental quadrics" with respect to which the complex is self-polar*—a not unexpected result. As we shall not be concerned with the other nine quadrics, the term *fundamental* will still refer exclusively to the quadric g_{ij}.

In the principal coordinate-system ϖ^α,

$$\det|R_{\alpha\beta}-\lambda\epsilon_{\alpha\beta}| \equiv \det\begin{vmatrix} M_{123}-\lambda I & O \\ O & M_{456}+\lambda I\end{vmatrix}, \tag{3·16}$$

by (3·14) and (2·6), and hence the ϵ-roots of $[R_{\alpha\beta}]$ are

$$(\lambda_1, \ldots, \lambda_6) \equiv (\mu_1, \mu_2, \mu_3, -\mu_4, -\mu_5, -\mu_6). \tag{3·17}$$

The cyclic condition
$$R_{ijkl}+R_{iklj}+R_{iljk} \equiv 0 \tag{3·18}$$

satisfied by the Riemann tensor may be written

$$\tfrac{1}{4}\epsilon^{ijkl}R_{ijkl} \equiv 0.$$

In S_5 this gives
$$\epsilon^{\alpha\beta}R_{\alpha\beta} = 0, \tag{3·19}$$

which means that the 4-quadrics $\epsilon_{\alpha\beta}$, $R_{\alpha\beta}$ are apolar. Taking the principal coordinates ϖ^α in S_5 and again using (3·14) and (2·6), we get from (3·19)

$$\mu_1+\mu_2+\mu_3-\mu_4-\mu_5-\mu_6 = 0, \tag{3·20}$$

that is, *the sum of the ϵ-roots is zero.*

4. *The six sets of orthogonal principal directions*

It was remarked above that the principal coordinates $'\chi^\alpha$, now denoted by ϖ^α, were unique except for signed permutations of ϖ_{123} and ϖ_{456}. As seen in § 2, such permutations lead to rotations of the basic ennuple in V_4, essentially the same ennuples being obtained from the signed permutations of ϖ_{123} as from those of ϖ_{456}. We may thus confine our attention to the latter. It follows that, if v_i^a are the covariant components of the principal orthogonal ennuple already obtained, five other essentially distinct principal ennuples are obtained by subjecting it to the rotations $\Omega_4, \ldots, E_6$ defined by (2·27), ..., (2·31). With any one of these as basis, the equations of the g-, ϵ- and R-quadrics, in the corresponding pseudo-Klein coordinates in S_5, are sums of squares. In S_3 the contravariant components of these ennuples are respectively the coordinates of the six fundamental tetrahedra which are self-polar with respect to the fundamental quadric (cf. Hudson, 1905, p. 42; Lamson, 1930, p. 713).

Struik (1927–8), following Kretschmann, shows how the Riemann tensor may be used to define 15 sets of four "principal directions" at a point of V_4. His work is discussed below in §§ 6, 8. The directions defined by the principal ennuples obtained above are six of Struik's sets for the case of the self-polar complex.

When the fundamental quadratic form at (x^i) is positive-definite, all of the six principal ennuples define real directions of V_4. For in that case all components of the initial ennuple h_i^a are real, and hence the ennuplet components of the Riemann tensor are real. Therefore P, S in (3·6), and consequently U, V in (3·8), are real symmetric 3×3 matrices, so the orthogonal transformations $\Lambda(\alpha)$, $\Lambda(\beta)$ of (3·10) and (3·11), which reduce them to diagonal form, are also real (see, e.g., Ferrar, 1941, p. 153). It follows almost immediately from (2·22) and (2·19) that the α's and β's are real. Hence the matrices ${}_1L(\alpha)$, ${}_2L(\beta)$ defined by (2·17), (2·18) are real, and therefore their product defines a real rotation in V_4. The product-rotation transforms the initial ennuple h_i^a into the principal ennuple v_i^a, which is therefore real. Finally, since the matrices Ω, E are real, so are all the six principal ennuples.

Now suppose g_{ij} at (x^i) in V_4 to be of signature $(+\ +\ +\ -)$. (This covers all cases of one minus sign, since we are at liberty from the beginning to number the coordinates as we please.) Then in the initial ennuple, h_1^i, h_2^i, h_3^i are real and h_4^i is purely imaginary. Hence the ennuplet Riemann tensor R_{abcd} is real except when one (and only one) of its suffixes is 4, in which case it is purely imaginary. Therefore all the elements of P in (3·6) are real and all those of S are purely imaginary, so U, V in (3·8) are complex conjugates, and the matrix of the transformation (2·14) is therefore $L \equiv {}_1L(\alpha) \,.\, {}_2L(\alpha^*)$,

the asterisk denoting the complex conjugate. From (2·17) and (2·18) it is immediately obvious that L is of the form

$$\begin{bmatrix} \mathrm{r} & \mathrm{r} & \mathrm{r} & \mathrm{i} \\ \mathrm{r} & \mathrm{r} & \mathrm{r} & \mathrm{i} \\ \mathrm{r} & \mathrm{r} & \mathrm{r} & \mathrm{i} \\ \mathrm{i} & \mathrm{i} & \mathrm{i} & \mathrm{r} \end{bmatrix}, \tag{4·1}$$

where "r" denotes "real" and "i" "purely imaginary". Since, with the same symbolism,

$$h_i^a \equiv (h_i^1, \ldots, h_i^4) = (\mathrm{r}, \mathrm{r}, \mathrm{r}, \mathrm{i}),$$

it follows that the rotation (4·1) applied to h_i^a produces a principal ennuple v_i^a which is also of type (r, r, r, i) and which, therefore, defines a set of four real directions in V_4. But by (2·27), ..., (2·31) it is obvious that the rotations Ω_4, Ω_5, Ω_6, E_5, E_6 applied to v_i^a do not produce ennuples of the same type. So the remaining five principal ennuples do not define real directions of V_4, and we are thus able to conclude that, *when the Riemann tensor satisfies the Einstein condition and the signature of* g_{ij} *is* $+2$, *only one of the six principal orthogonal ennuples defines real directions of* V_4 (cf. Lamson, 1930, p. 721). The same conclusion holds if the signature is -2 [case $(- - - +)$]. From the point of view of relativity the existence in empty space-time of a unique set of real principal directions intimately connected with the Riemann tensor is not without interest, especially in view of the non-existence of Ricci principal directions. It suggests that, at every world-point at which the momentum-energy tensor is zero, there exists a coordinate-system which is mathematically, if not physically, to be preferred. It will be shown in a later paper that the principal directions defined for a gravitational field by A. G. D. Watson (1941) are, in fact, the same as those obtained above.

If the signature is $(+ + - -)$, which covers all cases of two minus signs, no definite statement can be made about the reality of the principal directions. For suppose as usual that the initial ennuple h_i^a defines real directions. It is then of type (r, r, i, i), and P, S, U, V are all of type

$$\begin{bmatrix} \mathrm{r} & \mathrm{r} & \mathrm{i} \\ \mathrm{r} & \mathrm{r} & \mathrm{i} \\ \mathrm{i} & \mathrm{i} & \mathrm{r} \end{bmatrix}. \tag{4·2}$$

Suppose, as a particular case, that the imaginary elements of U, V happen to be zero, so that they are both of the type

$$\begin{bmatrix} \mathrm{r} & \mathrm{r} & 0 \\ \mathrm{r} & \mathrm{r} & 0 \\ 0 & 0 & \mathrm{r} \end{bmatrix}. \tag{4·3}$$

Then they are both real and symmetric, and the matrices $\Lambda(\alpha)$, $\Lambda(\beta)$ which reduce them to diagonal form are therefore also real and have zeros distributed as in (4·3). By (2·22), (2·17) and (2·18) it at once follows that ${}_1L(\alpha)$, ${}_2L(\beta)$ have one or other of the forms

$$\begin{bmatrix} \mathrm{r} & \mathrm{r} & 0 & 0 \\ \mathrm{r} & \mathrm{r} & 0 & 0 \\ 0 & 0 & \mathrm{r} & \mathrm{r} \\ 0 & 0 & \mathrm{r} & \mathrm{r} \end{bmatrix}, \tag{4·4}$$

$$\begin{bmatrix} 0 & 0 & \mathrm{r} & \mathrm{r} \\ 0 & 0 & \mathrm{r} & \mathrm{r} \\ \mathrm{r} & \mathrm{r} & 0 & 0 \\ \mathrm{r} & \mathrm{r} & 0 & 0 \end{bmatrix}. \tag{4·5}$$

They may both be of form (4·4) or both of form (4·5), or one may be of form (4·4) and one of form (4·5). In each case their product L also has one of these forms. Applied to the initial ennuple (r, r, i, i), L therefore leads to a principal ennuple either of type (r, r, i, i) or of type (i, i, r, r). Of these the former defines real directions in V_4, but the other principal ennuples obtained from it by applying the rotations Ω, E do not yield real directions. The latter does not itself define real directions, but the rotation Ω_6 (see (2·29)) makes it (r, r, i, i) and so produces an ennuple which does define real directions. So in the case $(+\,+\,-\,-)$ there may be a set of real principal directions.

On the other hand, there may be none. For suppose that there were always such a set, defined by a principal ennuple $v^i_a = (\mathrm{r}, \mathrm{r}, \mathrm{i}, \mathrm{i})$. With this as basis, U and V would both have diagonal form with *real* elements, as follows from (4·2) with all but the diagonal elements zero, and these real elements would be the respective latent roots of the matices U, V. But the latent roots of symmetric (not Hermitian) matrices of type (4·2) are not in general all real,† so there cannot in general be a principal ennuple defining real directions of V_4.

5. *Equal g-roots: example of the Schwarzschild space-time*

It has been assumed throughout that the g-roots $\mu_1, \ldots, \mu_6$ are all different. This restriction may now be removed, but it must still be assumed that the elementary divisors of $[R_{\alpha\beta} - \mu g_{\alpha\beta}]$ are simple.

In § 3 the uniqueness of the transformations $\Lambda(\alpha)$, $\Lambda(\beta)$ (but for signed permutations of the principal coordinates ϖ_{123} or ϖ_{456}) merely required the

† For example, if all the real elements are zero and the purely imaginary ones non-zero, one latent root is zero and the other two are purely imaginary.

assumption that $\mu_{123} \equiv \{\mu_1, \mu_2, \mu_3\}$ and $\mu_{456} \equiv \{\mu_4, \mu_5, \mu_6\}$ were all different. It did not require that the whole set of six numbers $\mu_1, \ldots, \mu_6$ should be all different. Therefore the uniqueness is unaffected if one or more of μ_{123} are respectively equal to one or more of μ_{456}, provided that no two numbers of the same triad are equal. So the transformations remain unique, but for the signed permutations, and the number of sets of principal orthogonal directions is still six, when

either (i) $\mu_1 = \mu_4$ with the μ's otherwise unequal;

or (ii) $\mu_1 = \mu_4, \quad \mu_2 = \mu_5, \quad \mu_3 = \mu_6 \quad (\mu_1, \mu_2, \mu_3$ unequal).

This is exhaustive. For by (3·20) we cannot have $\mu_1 = \mu_4$ and $\mu_2 = \mu_5$ without also having $\mu_3 = \mu_6$, while a possibility such as $\mu_1 = \mu_4$, $\mu_2 = \mu_6$, $\mu_3 = \mu_5$ is taken care of by applying the signed permutation $'\varpi_{456} = -\varpi_{465}$ to case (ii). The g-characteristics in these two cases are respectively [(11) 1111], [(11)(11)(11)]. Since by (3·17) the ϵ-roots are $(\mu_{123}, -\mu_{456})$, the ϵ-characteristic in each case is in general [111111], but has a more specialized form if any of the equal μ's are zero or if any of μ_{123} are equal to any of $-\mu_{456}$.

If any of μ_{123} are equal, there are an infinite number of matrices $\Lambda(\alpha)$ which reduce U in (3·8) to diagonal form. For example, if $\mu_2 = \mu_3 \neq \mu_1$, then, interpreting χ_{123} momentarily as coordinates in a projective plane, the conics $\chi'_{123} U \chi_{123} = 0$, $\chi'_{123} \chi_{123} = 0$ have double contact and therefore have an infinite number of common self-polar triangles. Similarly, if μ_{456} are not all different, there are an infinite number of matrices $\Lambda(\beta)$ which reduce V to diagonal form. An example is provided by the Schwarzschild space-time of relativity, for which

$$ds^2 = -r\rho^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 + \rho r^{-1} dt^2 \quad (\rho \equiv r - 2m, m \text{ const.}).$$

The non-zero distinct components of the Riemann tensor are

$$R_{2323} = -2mr \sin^2\theta, \quad R_{3131} = m\rho^{-1} \sin^2\theta, \quad R_{1212} = m\rho^{-1},$$
$$R_{1414} = 2mr^{-3}, \qquad R_{2424} = -m\rho r^{-2}, \qquad R_{3434} = -m\rho r^{-2} \sin^2\theta.$$

Taking as basic ennuple

$$\left.\begin{array}{llll} h_1^i \equiv i(\rho/r)^{\frac{1}{2}}, & 0, & 0, & 0; \\ h_2^i \equiv \quad 0, & i/r, & 0, & 0; \\ h_3^i \equiv \quad 0, & 0, & i/(r\sin\theta), & 0; \\ h_4^i \equiv \quad 0, & 0, & 0, & (r/\rho)^{\frac{1}{2}}, \end{array}\right\} \tag{5·1}$$

the covariant components of which are obtained by replacing the non-zero elements by their reciprocals, we obtain the following ennuplet components:

$$R_{2323} = -2m/r^3 = R_{1414}, \quad R_{3131} = m/r^3 = R_{1212} = R_{2424} = R_{3434}.$$

Hence the matrix $R_{\alpha\beta}$ of (3·6) is in this case already in diagonal form (cf. Watson, 1941, p. 16) and (5·1) is a principal ennuple. We have in fact

$$R_{\alpha\beta} = mr^{-3}\operatorname{diag}\{-2, 1, 1, -2, 1, 1\}, \tag{5·2}$$

so the S and P of (3·6) have the values $S = 0$ and $P = mr^{-3}\operatorname{diag}\{-2, 1, 1\}$. Hence the U, V of (3·8) are both equal to

$$\frac{m}{r^3}\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5·3}$$

and the g-roots are therefore $\mu_1 = \mu_4 = -2m/r^3$, $\mu_2 = \mu_3 = \mu_5 = \mu_6 = m/r^3$. If

$$\Lambda(\alpha) \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha \\ 0 & -\sin 2\alpha & \cos 2\alpha \end{bmatrix}, \tag{5·4}$$

α being any angle, then $\quad \Lambda(\alpha) . U . \Lambda^t(\alpha) \equiv U,$

that is, U is invariant under the transformation $\Lambda(\alpha)$. Similarly, V is invariant under any transformation $\Lambda(\beta)$, β being any angle. With $\Lambda(\alpha)$ given by (5·4), the α's of (2·22) have the values

$$\alpha_1 = \sin\alpha, \quad \alpha_2 = 0 = \alpha_3, \quad \alpha_4 = \cos\alpha$$

(or minus these). Similarly, for $\Lambda(\beta)$,

$$\beta_1 = \sin\beta, \quad \beta_2 = 0 = \beta_3, \quad \beta_4 = \cos\beta.$$

Substituting in the formulae (2·17) and (2·18) for ${}_1L(\alpha)$, ${}_2L(\beta)$, and multiplying, we obtain in V_4 the rotation

$$L = \begin{bmatrix} \cos(\alpha-\beta) & 0 & 0 & \sin(\alpha-\beta) \\ 0 & \cos(\alpha+\beta) & \sin(\alpha+\beta) & 0 \\ 0 & -\sin(\alpha+\beta) & \cos(\alpha+\beta) & 0 \\ -\sin(\alpha-\beta) & 0 & 0 & \cos(\alpha-\beta) \end{bmatrix}. \tag{5·5}$$

This, applied to the covariant components of the ennuple (5·1) (see (2·11)) produces a new ennuple, which, taken as basis for pseudo-Klein coordinates, reduces $R_{\alpha\beta}$ to diagonal form whatever the angles α, β. The principal ennuple (5·1) is therefore unique except in so far as the symmetry properties of the Schwarzschild V_4 permit this particular type of rotation.

If the elementary divisors of $U - \mu I$ or $V - \mu I$ and hence of $[R_{\alpha\beta} - \mu g_{\alpha\beta}]$, are not simple, then U or V are not reducible to diagonal form by orthogonal transformations, and there are no principal directions in the sense defined above.

6. *The fifteen principal tetrahedra in* S_3

The geometrical background of the preceding analysis is so well known that only a brief account of it need be given here.

A general quadratic complex defines in S_3 a set of 30 lines which are the edges of 15 tetrahedra. These have a total of 60 vertices and 60 faces, the whole forming a Klein's 60_{15} configuration (Hudson, 1905, pp. 41, 42; Jessop, 1903, p. 37). Of the 15 tetrahedra, 9 are inscribed in any one of the 10 fundamental quadrics and the remaining 6 are self-polar (Hudson, *ibid.* p. 42). The 6 that are self-polar with respect to the quadric g_{ij} define the 6 sets of principal orthogonal directions discussed above.

The existence of the 15 principal tetrahedra lies behind the work of Struik (1927–8) and Churchill (1932). Struik, dealing with the general case when the Riemann tensor does not necessarily satisfy the Einstein condition, defines 15 sets of 4 "principal directions", which are, in fact, the directions in V_4 corresponding to the vertices of the 15 tetrahedra in S_3. He makes only passing reference to the geometry. Churchill, dealing with the self-dual part G_{ijkl} of the Riemann tensor, obtains in his § 13 six "elementary six-vector skeletons". Such a skeleton is a set of six simple bivectors, which, in S_3, represent the six edges of one of the six self-polar tetrahedra.

The existence of the 30 lines is obvious from the 5-dimensional representation. For, in S_5, the common self-polar simplex of the 4-quadrics $R_{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ has 6 vertices. The lines joining these are 15 ($= {}_6C_2$) in number, and each of them cuts the ϵ-quadric in two points corresponding to lines of S_3. That the 30 lines form 15 tetrahedra is less obvious, but is easily arguable in terms of the geometry either of S_5 or of S_3. In the latter space they are the pairs of directrices of the linear congruences obtained by taking in pairs the linear complexes corresponding to the vertices of the self-polar simplex in S_5. All this is so well known that no further discussion is needed here, §§ 7 and 8 below being devoted to working out the analytical consequences of the geometry more completely than is done by Struik or by Churchill. The latter's treatment, being 4-dimensional, rather obscures the connexion of his theory with these classical elementary properties of quadratic complexes.

7. *The principal bivectors*

What may be conveniently called a *principal ϵ-bivector* at the point (x^i) of V_4 is any skew tensor q^{ij} that satisfies the equation

$$(R_{ijkl} - \lambda\epsilon_{ijkl})\, q^{kl} = 0 \tag{7·1}$$

for some value of λ. When the ϵ-roots $\lambda_1, \ldots, \lambda_6$ are all different, which we assume from now on, there are six such bivectors, each one being

determinate but for a scalar factor of proportionality. In S_5, (7·1) becomes

$$(R_{\alpha\beta}-\lambda\epsilon_{\alpha\beta})\,q^{\beta}=0, \tag{7·2}$$

or

$$R_{\alpha\beta}q^{\beta}=\lambda\epsilon_{\alpha\beta}q^{\beta},$$

which states that the point q^{α} has the same polar 4-plane with respect to the 4-quadrics $R_{\alpha\beta}$ and $\epsilon_{\alpha\beta}$. The six points q^{α} thus obtained are in fact the vertices of the common self-polar simplex of the 4-quadrics. In S_3 the q^{ij} define the *fundamental linear complexes* of the Riemann complex (Jessop, 1903, p. 26; Hudson, 1905, p. 33).

A *principal g-bivector* r^{ij} is likewise defined, except to the extent of a scalar factor, by the equation

$$(R_{ijkl}-\mu g_{ijkl})\,r^{kl}=0, \tag{7·3}$$

or, in S_5,

$$(R_{\alpha\beta}-\mu g_{\alpha\beta})\,r^{\beta}=0, \tag{7·4}$$

μ being one of the six g-roots, also assumed all different.† The six points r^{α} in S_5 are the vertices of the common self-polar simplex of the 4-quadrics $R_{\alpha\beta}$, $g_{\alpha\beta}$.

It was seen in § 3 that, when the Riemann complex is self-polar of the first kind, the Riemann tensor thereby satisfying the Einstein condition at (x^i), the two simplexes q^{α} and r^{α} are the same, and that, when this common self-polar simplex is taken as frame of reference for coordinates ϖ^{α} (the "principal coordinates") in S_5, then

$$R_{\alpha\beta}=\begin{bmatrix}M_{123} & O\\ O & M_{456}\end{bmatrix},\quad \epsilon_{\alpha\beta}=\begin{bmatrix}I & O\\ O & -I\end{bmatrix},\quad g_{\alpha\beta}=\begin{bmatrix}I & O\\ O & I\end{bmatrix}, \tag{7·5}$$

where $M_{123}=\operatorname{diag}\{\mu_1,\mu_2,\mu_3\}$, $M_{456}=\operatorname{diag}\{\mu_4,\mu_5,\mu_6\}$, the μ's being the g-roots. The ϵ-roots are $(\mu_1,\mu_2,\mu_3,-\mu_4,-\mu_5,-\mu_6)$. So in this case we have *one* set of six points $q^{\alpha}_{(\kappa)}\equiv(q^{\alpha}_{(1)},\ldots,q^{\alpha}_{(6)})$ such that, by (7·2) and (7·4),

$$R_{\alpha\beta}q^{\beta}_{(\kappa)}=(\pm)_{\kappa}\mu_{\kappa}\epsilon_{\alpha\beta}q^{\beta}=\mu_{\kappa}g_{\alpha\beta}q^{\beta} \tag{7·6}$$

(not summed for κ), where, as in (2·9), $(\pm)_{\kappa}=1$ if $\kappa=1,2$ or 3, and $(\pm)_{\kappa}=-1$ if $\kappa=4,5$ or 6. In S_3, and hence V_4, these give the equations

$$\tfrac{1}{2}R_{ijkl}q^{kl}_{(\kappa)}=\tfrac{1}{2}(\pm)_{\kappa}\mu_{\kappa}\epsilon_{ijkl}q^{kl}_{(\kappa)}=\tfrac{1}{2}\mu_{\kappa}g_{ijkl}q^{kl}_{(\kappa)} \tag{7·7}$$

satisfied by the principal linear complexes (bivectors). The last equality may be written

$$(\pm)_{\kappa}\,{}^{\circ}q_{(\kappa)ij}=q_{(\kappa)ij},$$

† The existence in a general Riemannian space of two distinct sets of principal bivectors is a peculiarity of V_4, in which the ϵ-tensors share with the Riemann tensor the property of having four suffixes. A V_n $(n\neq 4)$ has principal g-bivectors but not principal ϵ-bivectors.

which means, in S_3, that the linear complexes $q^{ij}_{(\kappa)}$ are self-polar with respect to the fundamental quadric, and, in V_4, that the three bivectors for which $\kappa = 1, 2, 3$ are self-dual and the other three anti-self-dual.

As the $q^{\alpha}_{(\kappa)}$ in S_5 are the vertices of the simplex of reference of the principal coordinate-system ϖ^{α}, we have, in that system, by a suitable choice of the factor of proportionality,

$$q^{\alpha}_{(1)} = (1, 0, 0, 0, 0, 0), \ldots,$$

or, as one (non-covariantive) equation,

$$q^{\alpha}_{(\kappa)} = \delta^{\alpha}_{\kappa}. \tag{7·8}$$

From this and (7·5) follow at once the covariantive equations

$$R_{\alpha\beta} q^{\alpha}_{(\kappa)} q^{\beta}_{(\sigma)} = \mu_{\kappa} \delta_{\kappa\sigma}, \tag{7·9}$$

$$\epsilon_{\alpha\beta} q^{\alpha}_{(\kappa)} q^{\beta}_{(\sigma)} = (\pm)_{\kappa} \delta_{\kappa\sigma}, \tag{7·10}$$

$$g_{\alpha\beta} q^{\alpha}_{(\kappa)} q^{\beta}_{(\sigma)} = \delta_{\kappa\sigma} \tag{7·11}$$

(no summation for κ). It will be seen that these are in agreement with (7·6). For $\kappa \neq \sigma$, these equations express the conjugacy of the six points $q^{\alpha}_{(\kappa)}$ in S_5 with respect to each of the 4-quadrics $R_{\alpha\beta}$, $\epsilon_{\alpha\beta}$, $g_{\alpha\beta}$. In V_4 they give the relations

$$\tfrac{1}{4} R_{ijkl} q^{ij}_{(\kappa)} q^{kl}_{(\sigma)} = \mu_{\kappa} \delta_{\kappa\sigma}, \tag{7·12}$$

$$\tfrac{1}{4} \epsilon_{ijkl} q^{ij}_{(\kappa)} q^{kl}_{(\sigma)} = (\pm)_{\kappa} \delta_{\kappa\sigma}, \tag{7·13}$$

$$\tfrac{1}{4} g_{ijkl} q^{ij}_{(\kappa)} q^{kl}_{(\sigma)} = \delta_{\kappa\sigma}, \tag{7·14}$$

satisfied by the principal bivectors. For $\kappa \neq \sigma$, (7·13) states that the linear complexes $q^{ij}_{(\kappa)}$ in S_3 are mutually in involution; (7·14) that each is in involution with its polar $q_{(\sigma)ij}$ with respect to the fundamental quadric; and (7·12) that each $q^{ij}_{(\kappa)}$ is in involution with the polar complex $R_{ijkl} q^{kl}_{(\sigma)}$ of any one of the others with respect to the Riemann complex.

If $q_{(\kappa)\alpha} \equiv g_{\alpha\beta} q^{\beta}_{(\kappa)}$, then, by (7·5) and (7·8),

$$R_{\alpha\beta} = \sum_{\kappa=1}^{6} \mu_{\kappa} q_{(\kappa)\alpha} q_{(\kappa)\beta}, \tag{7·15}$$

$$\epsilon_{\alpha\beta} = \sum_{\kappa=1}^{6} (\pm)_{\kappa} q_{(\kappa)\alpha} q_{(\kappa)\beta}, \tag{7·16}$$

$$g_{\alpha\beta} = \sum_{\kappa=1}^{6} q_{(\kappa)\alpha} q_{(\kappa)\beta}, \tag{7·17}$$

from which we obtain: *If $q_{(\kappa)ij}$ ($\kappa = 1, \ldots, 6$) are the covariant components, suitably normalized, of the principal bivectors defined by the Riemann tensor*

at a point of V_4 at which the Einstein condition is satisfied, and if the g-roots μ_κ are all different, then

$$R_{ijkl} = \sum_\kappa \mu_\kappa q_{(\kappa)ij} q_{(\kappa)kl}, \tag{7·18}$$

$$\epsilon_{ijkl} = \sum_\kappa (\pm)_\kappa q_{(\kappa)ij} q_{(\kappa)kl}, \tag{7·19}$$

$$g_{ijkl} = \sum_\kappa q_{(\kappa)ij} q_{(\kappa)kl}. \tag{7·20}$$

These relations may also be established as follows. The formula giving the Riemann tensor in terms of the $R_{\alpha\beta}$ of S_5 is

$$R_{ijkl} = R_{\alpha\beta} \gamma^\alpha_{ij} \gamma^\beta_{kl}, \tag{7·21}$$

γ^α_{ij} being defined as in (1·13), with similar relations for ϵ_{ijkl} and g_{ijkl}. As observed in § 1, α, β may be taken to relate to any coordinate-system in S_5 provided that γ^α_{ij} is treated as a tensor of the type indicated by its suffixes. If in S_5 we take the principal coordinates ϖ^α, then, by (7·5), (7·21) becomes

$$R_{ijkl} = \sum_{\alpha=1}^{6} \mu_\alpha \gamma^\alpha_{ij} \gamma^\alpha_{kl},$$

with similar relations for ϵ_{ijkl} and g_{ijkl}. Comparison with (7·18), (7·19), (7·20) shows that the two sets of equations are the same if this particular γ^α_{ij} is identified, as it must be, with $q_{(\alpha)ij}$.

8. *The 6 + 9 sets of principal directions*

Taking any system of coordinates χ^α in S_5, define $l^\alpha_{(\kappa)}$ by the relations

$$\left.\begin{aligned} l^\alpha_{(1)} &= \frac{1}{\sqrt{2}}(q^\alpha_{(1)} + q^\alpha_{(4)}), \quad l^\alpha_{(2)} = \frac{1}{\sqrt{2}}(q^\alpha_{(2)} + q^\alpha_{(5)}), \quad l^\alpha_{(3)} = \frac{1}{\sqrt{2}}(q^\alpha_{(3)} + q^\alpha_{(6)}), \\ l^\alpha_{(4)} &= \frac{1}{\sqrt{2}}(q^\alpha_{(1)} - q^\alpha_{(4)}), \quad l^\alpha_{(5)} = \frac{1}{\sqrt{2}}(q^\alpha_{(2)} - q^\alpha_{(5)}), \quad l^\alpha_{(6)} = \frac{1}{\sqrt{2}}(q^\alpha_{(3)} - q^\alpha_{(6)}) \end{aligned}\right\} \tag{8·1}$$

[cf. Struik, 1927–8, eqns. (11), (12); Churchill, 1932, eqns. (13·2)], which, by (2·2) and (2·4), may be written

$$l^\alpha_{(\sigma)} = K^\tau_\sigma q^\alpha_{(\tau)}. \tag{8·2}$$

By (7·10) all the l's satisfy the equation

$$\epsilon_{\alpha\beta} l^\alpha l^\beta = 0, \tag{8·3}$$

so they all represent points on the ϵ-quadric in S_5. $l^\alpha_{(1)}$ and $l^\alpha_{(4)}$ are thus the points where the line joining the vertices $q^\alpha_{(1)}$, $q^\alpha_{(4)}$ of the common self-polar simplex of the 4-quadrics meets the ϵ-quadric, and similarly $l^\alpha_{(2)}$, $l^\alpha_{(5)}$ and $l^\alpha_{(3)}$, $l^\alpha_{(6)}$ are points on the edges 25, 36 of the simplex. The relations (8·1) will be symbolized by

$$l = q(1 \pm 4, 2 \pm 5, 3 \pm 6), \tag{8·4}$$

the upper signs referring to the first line and the lower to the second line of (8·1).

By (7·10),
$$\epsilon_{\alpha\beta} l^\alpha_{(\kappa)} l^\beta_{(\sigma)} = 0, \quad |\kappa-\sigma| \neq 3, \tag{8·5}$$
$$= 1, \quad |\kappa-\sigma| = 3. \tag{8·6}$$

These equations mean that each point $l^\alpha_{(\kappa)}$ in S_5 is conjugate with respect to the ϵ-quadric to all the others, with the exception of the point $l^\alpha_{(\kappa\pm 3)}$ on the same edge of the simplex as itself.

In S_3, (8·3) becomes
$$\tfrac{1}{4}\epsilon_{ijkl} l^{ij} l^{kl} = 0, \tag{8·7}$$
expressing the otherwise obvious fact that the $l^{ij}_{(\kappa)}$ are the coordinates of lines in S_3 and not of non-special linear complexes; and (8·5) becomes
$$\tfrac{1}{4}\epsilon_{ijkl} l^{ij}_{(\kappa)} l^{kl}_{(\sigma)} = 0, \quad |\kappa-\sigma| \neq 3,$$
which, for $\kappa = \sigma$, is the same as (8·7), and for $\kappa \neq \sigma$ states that each of the lines $l^{ij}_{(\kappa)}$ meets four of the others but not the fifth (namely, the one for which $|\kappa-\sigma| = 3$). So the six lines are the edges of a tetrahedron in S_3. Moreover, since by (7·11) and (8·1),
$$g_{\alpha\beta} l^\alpha_{(\kappa)} l^\beta_{(\sigma)} = \delta_{\kappa\sigma},$$
which, for $\kappa \neq \sigma$, states that each point $l^\alpha_{(\kappa)}$ in S_5 is conjugate to every other with respect to the g-quadric, we have in S_3
$$\tfrac{1}{4} g_{ijkl} l^{ij}_{(\kappa)} l^{kl}_{(\sigma)} = \delta_{\kappa\sigma},$$
which may be written
$$g_{ik} g_{jl} l^{ij}_{(\kappa)} l^{kl}_{(\sigma)} = 0 \quad (\kappa \neq \sigma), \tag{8·8}$$
$$= 2 \quad (\kappa = \sigma). \tag{8·9}$$

Equations (8·8) mean that each edge $l^{ij}_{(\kappa)}$ of the tetrahedron meets the polar of every other edge with respect to the fundamental quadric, and (8·9) that it does not meet its own polar. Hence the tetrahedron is self-polar with respect to the fundamental quadric. If, then, $v^i_a \equiv (v^i_1, \ldots, v^i_4)$ are the four vertices of the tetrahedron, the polar plane $g_{ij} v^j_b$ of any one of the vertices contains each of the other three vertices, and so
$$g_{ij} v^i_a v^j_b = 0 \quad (a \neq b).$$

If the v's are normalized so as to make the left-hand side equal to unity when $a = b$, which, by (8·9), is consistent with taking the Plücker coordinates of the edges to be exactly
$$\left.\begin{aligned} l^{ij}_{(1)} &= v^i_2 v^j_3 - v^i_3 v^j_2, \\ l^{ij}_{(2)} &= v^i_3 v^j_1 - v^i_1 v^j_3, \\ &\cdots\cdots\cdots\cdots\cdots \\ l^{ij}_{(6)} &= v^i_3 v^j_4 - v^i_4 v^j_3, \end{aligned}\right\} \tag{8·10}$$

and not scalar multiples of these, we have

$$g_{ij} v^i_a v^j_b = \delta_{ab}, \tag{8·11}$$

and the v's therefore constitute an orthogonal ennuple at (x^i) in V_4.

Take v^i_a as basic ennuple in V_4. The ennuplet components of the Riemann tensor are then $R_{ijkl} v^i_a v^j_b v^k_c v^l_d$, which, because of the skewness of R_{ijkl} in i, j and k, l, is identically equal to

$$\tfrac{1}{4} R_{ijkl}(v^i_a v^j_b - v^i_b v^j_a)(v^k_c v^l_d - v^k_d v^l_c).$$

Replacing suffix-pairs by single numerals in the usual way, it is at once evident from (8·10) that the equation of the R-quadric in S_5 in the basic coordinate-system p^α associated with v^i_a is

$$R_{\sigma\tau} p^\sigma p^\tau = 0,$$

where
$$\begin{aligned} R_{\sigma\tau} &= \tfrac{1}{4} R_{ijkl} l^{ij}_{(\sigma)} l^{kl}_{(\tau)} \\ &= K^\rho_\sigma K^\nu_\tau . \tfrac{1}{4} R_{ijkl} q^{ij}_{(\rho)} q^{kl}_{(\nu)} \quad \text{by (8·2)} \\ &= \sum_{\rho=1}^{6} \mu_\rho K^\rho_\sigma K^\rho_\tau \quad \text{by (7·12)}, \end{aligned}$$

whence
$$R_{\sigma\tau} p^\sigma p^\tau = \sum_{\rho=1}^{6} \mu_\rho (\varpi^\rho)^2 \tag{8·12}$$

by (3·15), ϖ^ρ being the pseudo-Klein coordinates associated with v^i_a. The fact that the quadratic form on the right of (8·12) is a sum of squares with coefficients μ_ρ shows that the ϖ^ρ are the principal coordinates for S_5 found in § 3, and therefore that v^i_a, as defined by (8·10), is the principal ennuple associated with them.

Five other principal ennuples were obtained in § 4 by imposing signed permutations upon the ϖ's. These are also obtainable by taking sets of edges of the self-polar simplex in S_5 other than the edges 14, 25, 36 of (8·1) or (8·4). They arise, in fact, if the $q^\alpha_{(4)}, q^\alpha_{(5)}, q^\alpha_{(6)}$ of (8·1) are subjected to exactly the same signed permutations as $\varpi^4, \varpi^5, \varpi^6$. So, for example, the ennuple derived from the signed permutation $-\{465\}$ of (2·26) and (2·27) is obtained if, in (8·1), we replace $q^\alpha_{(4)}, q^\alpha_{(5)}, q^\alpha_{(6)}$ respectively by $-q^\alpha_{(4)}, -q^\alpha_{(6)}, -q^\alpha_{(5)}$. In this way we obtain, in the symbolic notation of (8·4), the following five other ways of defining the l's in terms of the q's:

$$\begin{aligned} l &= q(1 \mp 4, 2 \mp 6, 3 \mp 5), \\ l &= q(1 \mp 6, 2 \mp 5, 3 \mp 4), \\ l &= q(1 \mp 5, 2 \mp 4, 3 \mp 6), \\ l &= q(1 \pm 5, 2 \pm 6, 3 \pm 4), \\ l &= q(1 \pm 6, 2 \pm 4, 3 \pm 5). \end{aligned}$$

These correspond respectively to the transformations (2·27), ..., (2·31), and yield exactly the same principal orthogonal ennuples as before.

The six formulae for the l's in terms of the q's are characterized by having the suffixes 1, 2, 3 of the q's paired in all possible ways with 4, 5, 6, but never with one another. If 1, 2, 3 are allowed to pair with one another as well as with 4, 5, 6, we obtain the complete set of 15 ways of pairing the suffixes of the q's into sets of three, namely the six given above and nine others. Consider, for example, the effect of taking the combination 14, 23, 56. Define the l's by the formulae

$$\left.\begin{aligned} l^{\alpha}_{(1)} &= \frac{1}{\sqrt{2}}(q^{\alpha}_{(1)}+q^{\alpha}_{(4)}), & l^{\alpha}_{(2)} &= \frac{1}{\sqrt{2}}(q^{\alpha}_{(2)}-iq^{\alpha}_{(3)}), & l^{\alpha}_{(3)} &= \frac{1}{\sqrt{2}}(-iq^{\alpha}_{(5)}+q^{\alpha}_{(6)}), \\ l^{\alpha}_{(4)} &= \frac{1}{\sqrt{2}}(q^{\alpha}_{(1)}-q^{\alpha}_{(4)}), & l^{\alpha}_{(5)} &= \frac{1}{\sqrt{2}}(q^{\alpha}_{(2)}+iq^{\alpha}_{(3)}), & l^{\alpha}_{(6)} &= \frac{1}{\sqrt{2}}(-iq^{\alpha}_{(5)}-q^{\alpha}_{(6)}). \end{aligned}\right\} \tag{8·13}$$

The factor $\pm i$ appears before one of the q's whenever two of the suffixes 1, 2, 3 or two of 4, 5, 6 appear together, and the signs are chosen so as to make (8·13) consistent with (8·15), (8·16) below. As before,

$$\left.\begin{aligned} \epsilon_{\alpha\beta} l^{\alpha}_{(\kappa)} l^{\beta}_{(\sigma)} &= 0, \quad |\kappa-\sigma| \neq 3, \\ &= 1, \quad |\kappa-\sigma| = 3, \end{aligned}\right\} \tag{8·14}$$

so the corresponding lines of S_3 again form a tetrahedron. This tetrahedron is self-reciprocal with respect to the fundamental quadric, but not in the sense that each vertex is the pole of the opposite face. For by (7·6) we now have

$$\begin{aligned} g_{\alpha\beta} l^{\beta}_{(\kappa)} &= \epsilon_{\alpha\beta} l^{\beta}_{(\kappa)} \quad \text{if} \quad \kappa = 2 \text{ or } 5, \\ &= -\epsilon_{\alpha\beta} l^{\beta}_{(\kappa)} \quad \text{if} \quad \kappa = 3 \text{ or } 6, \end{aligned}$$

which express the fact that, in S_3, the lines $l^{ij}_{(\kappa)}$ for $\kappa = 2, 3, 5, 6$ are self-polar with respect to the fundamental quadric, that is, that they are generators. Also it follows from (8·14) that generators 2, 5 meet each of 3, 6 but do not meet one another, so 2, 5 belong to one regulus and 3, 6 to the other. Hence the lines $l^{ij}_{(\kappa)}$ for $\kappa = 2, 3, 5, 6$ form a skew quadrilateral on the fundamental quadric, and the remaining lines ($\kappa = 1, 4$) are its diagonals. The vertices u^i_a of this quadrilateral, defined by

$$l^{ij}_{(1)} = u^i_2 u^j_3 - u^i_3 u^j_2, \text{ etc.}, \tag{8·15}$$

thus lie on the fundamental quadric and so define a principal set of *null* vectors of V_4.

The same set of null vectors could be obtained by subjecting the principal coordinates ϖ^{α} to the transformation

$$\left.\begin{aligned} '\varpi^1 &= \varpi^1, & '\varpi^2 &= \varpi^2, & '\varpi^3 &= i\varpi^5, \\ '\varpi^4 &= \varpi^4, & '\varpi^5 &= i\varpi^3, & '\varpi^6 &= \varpi^6 \end{aligned}\right\} \tag{8·16}$$

in the manner of § 2. To this transformation of coordinates in S_5 corresponds in S_3 the transformation

$$'X^a = L^a_b X^b, \tag{8·17}$$

where

$$[L^a_b] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix}. \tag{8·18}$$

The covariant components of the null vectors u^i_a are then connected with those of the principal orthogonal ennuple v^i_a by the formula

$$u^a_i = L^a_b v^b_i.$$

The matrix (8·18) is non-orthogonal. It is easy to show from (8·17) and (8·18) that

$$\sum_{a=1}^{4} (X^a)^2 = 2i('X^2\,'X^3 - 'X^1\,'X^4),$$

so that the form $\Sigma(X^a)^2$ of the fundamental quadric is not preserved when in V_4 we take as basis the set of vectors u^i_a instead of v^i_a; which is geometrically to be expected because the tetrahedron u^i_a in S_3 is not self-polar with respect to, but is inscribed in, the fundamental quadric. It should be noted that the transformation (8·16), unlike those of § 2, does not permute the ϖ_{123} or the ϖ_{456} among themselves, but virtually interchanges ϖ^3 and ϖ^5. It leaves invariant the form $\Sigma(\pm)_\alpha(\varpi^\alpha)^2$ of the ϵ-quadric, but not the form $\Sigma(\varpi^\alpha)^2$ of the g-quadric.

It is beyond the scope of the present paper to study the 9 sets of null directions in detail. It is enough to say that each of the corresponding 9 tetrahedra in S_3 has two pairs of opposite edges as generators forming a skew quadrilateral on the fundamental quadric, and that, if the whole set of 9 are taken, 6 generators of each system are involved. These two sets of 6 generators correspond respectively to the points in S_5 where the edges 23, 31, 12 and 56, 64, 45 of the common self-polar simplex meet the ϵ-quadric. All other edges meet the ϵ-quadric in points corresponding to the diagonals of the 9 skew quadrilaterals, out of which diagonals are also formed the 6 self-polar tetrahedra already considered. The 36 intersections of the $6+6$ generators give 36 points u^i defining null vectors in V_4 (these being separable into 9 sets of 4, as above), the 6 null vectors corresponding to the points on any one generator being mutually perpendicular because the points are all conjugate with respect to the fundamental quadric. As each point is the intersection of 2 generators, one of each system, each of the 36 null vectors in V_4 is perpendicular to itself and to 10 of the others.

In the general case when the Riemann tensor does not satisfy the Einstein condition there are still 15 sets of "principal directions" defined by the

Riemann tensor, but they are in general all non-null and non-orthogonal, the corresponding points in S_3 having then no special relationship to the fundamental quadric. When they are orthogonal they coincide with the Ricci principal directions (Struik, 1927–8, § 4). It is the self-polarity of the Riemann complex with respect to the fundamental tensor that separates the 15 sets into two sets of 6 and 9 as described above.

Some other properties of the self-polar Riemann complex of the first kind are discussed in §§ 4, 5 of an earlier paper (Ruse, 1944*b*). Lamson (1930, p. 722) gives a list of ϵ-characteristics which are algebraically impossible for such a complex. They are twelve in number.

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The University
Leeds

A SUMMATION FORMULA IN THE THEORY OF PRIME NUMBERS

By A. P. GUINAND

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1. *Introduction*

It has long been known that there is a close connexion between the prime numbers and the non-trivial zeros of the Riemann zeta-function,* but the precise nature of this connexion has not been elucidated. However, relations are known† between certain sums involving the prime numbers and other sums involving the zeros of $\zeta(s)$.

In this paper it is shown that, if the Riemann hypothesis is true, then there exists a general summation formula connecting sums of the above types. With appropriate conditions on $f(x)$ the main result is

$$\lim_{T\to\infty}\left\{\sum_{0<m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}}f(m\log p)-\int_0^T f(t)\,e^{\frac{1}{2}t}\,dt\right\}-\frac{1}{2}\int_0^\infty f(t)\left(\frac{1}{t}-\frac{e^{-\frac{1}{2}t}}{\sinh t}\right)dt$$
$$=-(2\pi)^{\frac{1}{2}}\lim_{T\to\infty}\left\{\sum_{0<\gamma<T}g(\gamma)-\frac{1}{2\pi}\int_0^T g(t)\log\frac{t}{2\pi}\,dt\right\}, \quad (1\cdot1)$$

where‡
$$g(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\to\infty} f(t)\cos xt\,dt,$$

p runs through the prime numbers, m through the positive integers, and $\frac{1}{2}+i\gamma$ through the non-trivial zeros§ of $\zeta(s)$.

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* E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, 1 (Leipzig, 1909), 367–368.

† E. Landau, *loc. cit.* 348–368, and A. E. Ingham, *The Distribution of Prime Numbers* (Cambridge, 1932), chapter IV.

‡ We use the notation $\int^{\to\infty}$ for $\lim\limits_{X\to\infty}\int^X$.

§ If multiple zeros of $\zeta(s)$ exist then the corresponding terms $g(\gamma)$ in (1·1) are to be multiplied by the orders of the zeros. This convention will be used throughout the paper.

This result resembles Poisson's summation formula, and the method of proof used here was originally applied to the latter formula.*

An expression is derived for the remainder in the Riemann-von Mangoldt formula† for $N(T)$, the number of zeros of $\zeta(s)$ in $0 < I(s) < T$.

Some examples and applications of the summation formula (1·1) are given in the last section.

The truth of the Riemann hypothesis is assumed throughout the paper.

2. *A pair of Hankel transforms*

Consider the integral

$$\int_0^{\to\infty} (xt)^{\frac{1}{2}} J_{\frac{3}{2}}(xt) \left\{N(t) - \left(\frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi}\right)\right\} \frac{dt}{t}. \tag{2·1}$$

Now
$$J_{\frac{3}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left(\frac{\sin z}{z} - \cos z\right),$$

and hence
$$\left(\frac{x}{t}\right)^{\frac{1}{2}} J_{\frac{3}{2}}(xt) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dt}\left(\frac{\sin xt}{xt}\right).$$

Also, by definition,
$$N(t) = \sum_{0<\gamma<t} 1,$$

using the convention explained earlier if multiple zeros of $\zeta(s)$ occur. Hence the integral (2·1) may be written

$$-\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\to\infty} \left\{\sum_{0<\gamma<t} 1 - \left(\frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi}\right)\right\} \frac{d}{dt}\left(\frac{\sin xt}{xt}\right) dt.$$

If we now write γ_r to distinguish the rth value of γ, this becomes

$$\begin{aligned}
-\left(\frac{2}{\pi}\right)^{\frac{1}{2}} &\lim_{N\to\infty} \int_0^{\gamma_{N+1}} \left\{\sum_{0<\gamma_r<t} 1 - \left(\frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi}\right)\right\} \frac{d}{dt}\left(\frac{\sin xt}{xt}\right) dt \\
&= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lim_{N\to\infty} \left\{-\sum_{r=1}^{N} r \int_{\gamma_r}^{\gamma_{r+1}} \frac{d}{dt}\left(\frac{\sin xt}{xt}\right) dt + \left[\frac{\sin xt}{xt}\left(\frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi}\right)\right]_0^{\gamma_{N+1}} \right. \\
&\qquad\qquad \left. - \frac{1}{2\pi}\int_0^{\gamma_{N+1}} \frac{\sin xt}{xt} \log\frac{t}{2\pi}\, dt\right\} \\
&= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lim_{N\to\infty} \left\{\frac{1}{x}\sum_{r=1}^{N} r\left(\frac{\sin x\gamma_r}{\gamma_r} - \frac{\sin x\gamma_{r+1}}{\gamma_{r+1}}\right)\right. \\
&\qquad\qquad \left. + \frac{\sin x\gamma_{N+1}}{x\gamma_{N+1}}\left(\frac{\gamma_{N+1}}{2\pi}\log\frac{\gamma_{N+1}}{2\pi} - \frac{\gamma_{N+1}}{2\pi}\right)\right\} \\
&\qquad\qquad - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{1}{2\pi}\right) \int_0^{\to\infty} \frac{\sin xt}{xt} \log\frac{t}{2\pi}\, dt.
\end{aligned} \tag{2·2}$$

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* A. P. Guinand, *Annals of Math.* 42 (1941), 591–603.

† E. Landau, *loc. cit.* chapter xx.

Now $$\int_0^{\to\infty} \sin u \frac{du}{u} = \tfrac{1}{2}\pi, \quad \int_0^{\to\infty} \sin u \log u \frac{du}{u} = -\tfrac{1}{2}\pi C,$$
where C is Euler's constant. Hence the integral term in (2·2) is equal to
$$-\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{1}{2\pi x}\right)\int_0^{\to\infty} \sin u(\log u - \log 2\pi x)\frac{du}{u} = \frac{1}{2x(2\pi)^{\frac{1}{2}}}(C + \log 2\pi x),$$
and (2·2) becomes
$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \lim_{N\to\infty}\left\{\frac{1}{x}\sum_{r=1}^{N}\frac{\sin x\gamma_r}{\gamma_r} - \frac{\sin x\gamma_{N+1}}{x\gamma_{N+1}}\left(N - \frac{\gamma_{N+1}}{2\pi}\log\frac{\gamma_{N+1}}{2\pi} + \frac{\gamma_{N+1}}{2\pi}\right)\right\}$$
$$+\frac{1}{2x(2\pi)^{\frac{1}{2}}}(C + \log 2\pi x). \tag{2·3}$$

Now,* as $t \to \infty$ $$N(t) - \left(\frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi}\right) = O(\log t). \tag{2·4}$$
Hence, as $N \to \infty$
$$\frac{\sin x\gamma_{N+1}}{\gamma_{N+1}}\left(N - \frac{\gamma_{N+1}}{2\pi}\log\frac{\gamma_{N+1}}{2\pi} + \frac{\gamma_{N+1}}{2\pi}\right) = O\left(\frac{\log \gamma_{N+1}}{\gamma_{N+1}}\right)$$
and (2·3) becomes
$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{1}{x}\right)\left\{\sum_{r=1}^{\infty}\frac{\sin x\gamma_r}{\gamma_r} + \tfrac{1}{4}(C + \log 2\pi x)\right\}, \tag{2·5}$$
if this series converges. Now it is known that†
$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\leqslant x}{}' \Lambda(n)\, n^{-s} - \frac{x^{1-s}}{1-s} - \sum_{q=1}^{\infty}\frac{x^{-2q-s}}{2q+s} + \sum_{\rho}\frac{x^{\rho-s}}{\rho - s} \quad (x>1, s\neq 1, \rho, -2n), \tag{2·6}$$
where the dash indicates that the term $n = x$, if it occurs, is to be halved, and ρ runs through the non-trivial zeros of $\zeta(s)$. The series $\sum_{\rho}$ is to be interpreted as
$$\lim_{T\to\infty}\left\{\sum_{-T<I(\rho)<T}\right\}.$$
Putting $s = \frac{1}{2}$, $x = e^z$, and $\rho = \frac{1}{2} + i\gamma$, this becomes
$$-\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})} = \sum_{n\leqslant e^z}{}' \Lambda(n)\, n^{-\frac{1}{2}} - 2e^{\frac{1}{2}z} - \sum_{q=1}^{\infty}\frac{e^{-(2q+\frac{1}{2})z}}{2q+\frac{1}{2}} + \sum_{\gamma}\frac{e^{i\gamma z}}{i\gamma}$$
$$= \sum_{\log n\leqslant z}{}' \Lambda(n)\, n^{-\frac{1}{2}} - 2e^{\frac{1}{2}z} + 2e^{-\frac{1}{2}z} - \tfrac{1}{2}\log\coth\tfrac{1}{4}z$$
$$-\arctan e^{-\frac{1}{2}z} + 2\sum_{\gamma>0}\frac{\sin\gamma z}{\gamma}.$$

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* E. C. Titchmarsh, *The Zeta-function of Riemann* (Cambridge, 1930), 4. Referred to in the sequel as *Tract*.

† E. C. Titchmarsh, *Tract*, 81.

Hence the series in (2·5) does converge, and it follows that the integral (2·1) exists. Using the definition* of $\Lambda(n)$ we obtain

$$\sum_{\gamma>0}\frac{\sin\gamma z}{\gamma}=-\frac{1}{2}\sum_{m\log p\leqslant z}{}'\frac{\log p}{p^{\frac{1}{2}m}}+2\sinh\tfrac{1}{2}z$$
$$+\tfrac{1}{4}\log\coth\tfrac{1}{4}z+\tfrac{1}{2}\arctan e^{-\frac{1}{2}z}-\tfrac{1}{4}C-\tfrac{1}{8}\pi-\tfrac{1}{4}\log 8\pi, \qquad (2\cdot 7)$$

since†
$$\frac{\zeta'(\frac{1}{2})}{\zeta(\frac{1}{2})}=\tfrac{1}{2}C+\tfrac{1}{4}\pi+\tfrac{1}{2}\log 8\pi.$$

Substituting (2·7) in (2·5) we obtain

THEOREM 1. *If the Riemann hypothesis is true and $x>0$, then*

$$\int_0^{\to\infty}(xt)^{\frac{1}{2}}J_{\frac{3}{2}}(xt)\left\{N(t)-\left(\frac{t}{2\pi}\log\frac{t}{2\pi}-\frac{t}{2\pi}\right)\right\}\frac{dt}{t}$$
$$=-\frac{1}{x(2\pi)^{\frac{1}{2}}}\left\{\sum_{0<m\log p\leqslant x}{}'\frac{\log p}{p^{\frac{1}{2}m}}-4\sinh\tfrac{1}{2}x\right\}$$
$$+\frac{1}{x(2\pi)^{\frac{1}{2}}}\{\tfrac{1}{2}\log(\tfrac{1}{4}x\coth\tfrac{1}{4}x)+\arctan e^{-\frac{1}{2}x}-\tfrac{1}{4}\pi\}.$$

If we put
$$F(x)=\frac{1}{x}\left\{N(x)-\left(\frac{x}{2\pi}\log\frac{x}{2\pi}-\frac{x}{2\pi}\right)\right\}, \qquad (2\cdot 8)$$

and
$$G(x)=-\frac{1}{x(2\pi)^{\frac{1}{2}}}\left\{\sum_{0<m\log p\leqslant x}{}'\frac{\log p}{p^{\frac{1}{2}m}}-4\sinh\tfrac{1}{2}x\right\}$$
$$+\frac{1}{x(2\pi)^{\frac{1}{2}}}\{\tfrac{1}{2}\log(\tfrac{1}{4}x\coth\tfrac{1}{4}x)+\arctan e^{-\frac{1}{2}x}-\tfrac{1}{4}\pi\}, \qquad (2\cdot 9)$$

then theorem 1 implies that $G(x)$ is the Hankel transform‡ of order $\frac{3}{2}$ of $F(x)$.

We can deduce the inverse transformation from theorem 1 by using the following lemmas:

LEMMA§ α. *If $F(x)$ belongs to $L^p(0,\infty)$ for some p in $1<p\leqslant 2$, then there exists a function $G(x)$ belonging to $L^{p'}(0,\infty)$ for $p'=p/(p-1)$, defined almost everywhere by*

$$G(x)=\underset{T\to\infty}{\text{l.i.m.}}\int_0^T(xt)^{\frac{1}{2}}J_{\frac{3}{2}}(xt)\,F(t)\,dt. \qquad (2\cdot 10)$$

Further
$$F(x)=\underset{T\to\infty}{\text{l.i.m.}}\int_0^T(xt)^{\frac{1}{2}}J_{\frac{3}{2}}(xt)\,G(t)\,dt,$$

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* $\Lambda(n)=\log p$ when $n=p^m$ and vanishes elsewhere.

† This follows on putting $s=\frac{1}{2}$ in the functional equation for $\zeta'(s)/\zeta(s)$. See A. E. Ingham, *loc. cit.* 73.

‡ E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937), 214–215. Referred to in the sequel as *F.I.*

§ H. Kober, *Quart. J. of Math.* 8 (1937), 186–199, and I. W. Busbridge, *ibid.* 9 (1938), 148–160.

and if $f(x)$, $g(x)$ is another pair of such transforms for the same value of p, then

$$\int_0^\infty f(x)\,G(x)\,dx = \int_0^\infty F(x)\,g(x)\,dx. \tag{2·11}$$

LEMMA* β. *If $F(x)$ belongs to $L^2(0,\infty)$ and is of bounded variation in some neighbourhood of $x = y$, and $G(x)$ is defined by* (2·10), *then*

$$\tfrac{1}{2}\{F(y+0)+F(y-0)\} = \int_0^{\to\infty} (yt)^{\frac{1}{2}} J_{\frac{3}{2}}(yt)\,G(t)\,dt.$$

Now if $F(x)$ is defined by (2·8), then it follows from (2·4) that $F(x)$ belongs to $L^p(0,\infty)$ for any $p>1$, and the conditions of lemmas α and β are fulfilled for any $y>0$. Hence

$$\frac{1}{y}\left[\tfrac{1}{2}\{N(y+0)+N(y-0)\}-\left(\frac{y}{2\pi}\log\frac{y}{2\pi}-\frac{y}{2\pi}\right)\right]$$
$$= \int_0^{\to\infty}(yt)^{\frac{1}{2}} J_{\frac{3}{2}}(yt)\left[-\frac{1}{t(2\pi)^{\frac{1}{2}}}\left\{\sum_{0<m\log p\leqslant t}{}' \frac{\log p}{p^{\frac{1}{2}m}}-4\sinh\tfrac{1}{2}t\right\}\right.$$
$$\left.+\frac{1}{t(2\pi)^{\frac{1}{2}}}\{\tfrac{1}{2}\log(\tfrac{1}{4}t\coth\tfrac{1}{4}t)+\arctan e^{-\frac{1}{2}t}-\tfrac{1}{4}\pi\}\right]dt.$$

We can deal with the first part of this integral as we dealt with the integral (2·1). The second part of the integral can be evaluated by partial integration and the use of known integrals. We obtain:

THEOREM 2. *If the Riemann hypothesis is true and $T\geqslant 0$, then*

$$\tfrac{1}{2}\{N(T+0)+N(T-0)\}-\left(\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}\right)$$
$$= -\frac{1}{\pi}\lim_{N\to\infty}\left[\sum_{n=1}^{N}\Lambda(n)\frac{\sin(T\log n)}{n^{\frac{1}{2}}\log n}-\int_1^N\frac{\sin(T\log t)}{t^{\frac{1}{2}}\log t}dt\right.$$
$$\left.-\frac{\sin(T\log N)}{\log N}\left\{\sum_{n=1}^{N}\Lambda(n)\,n^{-\frac{1}{2}}-2N^{\frac{1}{2}}\right\}\right]+\frac{1}{2\pi}\{\operatorname{am}\Gamma(\tfrac{1}{2}+iT)-T\log T+T\}$$
$$+\frac{1}{\pi}\arctan 2T-\frac{1}{4\pi}\arctan(\sinh\pi T), \tag{2·12}$$

where am $\Gamma(\tfrac{1}{2}+iT)$ *is defined by making* am $\Gamma(\tfrac{1}{2}) = 0$ *and continuing analytically along any path not meeting the real axis.*

Other writers have remarked† previously that a connexion exists between $N(T)$ and the first series in (2·12), but as far as I am aware no exact formula of this type for $N(T)$ has been given before.

* E. C. Titchmarsh, *F.I.* 83 and 266.

† E. C. Titchmarsh, *Tract*, 63; A. Wintner, *Duke Math. J.* 10 (1943), 99–105; and A. Selberg, *Avhandlinger utgitt av det Norske Videnskaps-Akademi i Oslo*, no. 1 (1944). 1–27.

Now it has been shown* that, if the Riemann hypothesis is true, then as $N \to \infty$

$$\sum_{n=1}^{N} \Lambda(n) - N = O(N^{\frac{1}{2}} \log^{\alpha} N) \tag{2·13}$$

holds for $\alpha = 2$, and also that the expression on the left is

$$\Omega \pm (N^{\frac{1}{2}} \log\log\log N).$$

These results are the best of their kind yet proved, so it is still possible that (2·13) may hold for some α in $0 < \alpha < 2$.

If we put $s = 0$ and $s = \frac{1}{2}$ successively in (2·6), then we can show that

$$\begin{aligned}\left\{\sum_{n=1}^{N} \Lambda(n)\, n^{-\frac{1}{2}} - 2N^{\frac{1}{2}}\right\} - N^{-\frac{1}{2}}\left\{\sum_{n=1}^{N} \Lambda(n) - N\right\} \\ &= -\sum_{\rho} N^{\rho-\frac{1}{2}}\left\{\frac{1}{\rho-\frac{1}{2}} - \frac{1}{\rho}\right\} + O(1) \\ &= -\frac{1}{2}\sum_{\gamma} \frac{N^{i\gamma}}{i\gamma(\frac{1}{2}+i\gamma)} + O(1) \\ &= O(1),\end{aligned}$$

since $\sum_{\gamma} \gamma^{-2}$ is absolutely convergent.

Hence
$$\sum_{n=1}^{N} \Lambda(n)\, n^{-\frac{1}{2}} - 2N^{\frac{1}{2}} = O(\log^{\alpha} N). \tag{2·14}$$

If this holds for some $\alpha < 1$, then in theorem 2

$$\lim_{T\to\infty} \frac{\sin(T\log N)}{\log N}\left\{\sum_{n=1}^{N} \Lambda(n)\, n^{-\frac{1}{2}} - 2N^{\frac{1}{2}}\right\} = 0,$$

so this term could be omitted from (2·11).

3. *The summation formula*

We now need the following lemmas:†

LEMMA γ. *If $f(x)$ is an integral, tends to zero as x tends to infinity, and $xf'(x)$ belongs to $L^p(0,\infty)$ for some $p > 1$, then $f(x)$ belongs to $L^p(0,\infty)$ and $x^{1/p}f(x)$ tends to zero as x tends to $+0$ or to infinity.*

LEMMA δ. *If $f(x)$ satisfies the conditions of lemma γ for some p in $1 < p \leqslant 2$, then it has a Fourier cosine transform $g(x)$ given for all positive x by*

$$g(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\to\infty} f(t)\cos xt\,dt.$$

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* A. E. Ingham, *loc. cit.* theorems 30 and 34.

† A. P. Guinand, *loc. cit.* lemmas 2 and 4.

Further $$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\to\infty} g(t) \cos xt\, dt,$$

$g(x)$ is an integral, both $g(x)$ and $xg'(x)$ belong to $L^{p'}(0,\infty)$, $x^{1/p'}g(x)$ tends to zero as x tends to $+0$ or to infinity, and $xf'(x)$, $xg'(x)$ is a pair of Hankel transforms of order $\frac{3}{2}$ in the sense of lemma α.

Now if $f(x)$ and $g(x)$ satisfy the conditions of lemma δ and $F(x)$ and $G(x)$ are defined by (2·7) and (2·8), then by (2·10)

$$\int_0^\infty xf'(x)\, G(x)\, dx = \int_0^\infty xg'(x)\, F(x)\, dx. \tag{3·1}$$

The left-hand side of (3·1) is

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty f'(x) \left\{ - \sum_{0<m\log p\leqslant x}{}' \frac{\log p}{p^{\frac{1}{2}m}} + 4 \sinh \tfrac{1}{2}x + \tfrac{1}{2} \log(\tfrac{1}{4}x \coth \tfrac{1}{4}x) + \arctan e^{-\frac{1}{2}x} - \tfrac{1}{4}\pi \right\} dx. \tag{3·2}$$

Now $$\frac{1}{x}\{\tfrac{1}{2}\log(\tfrac{1}{4}x \coth \tfrac{1}{4}x) + \arctan e^{-\frac{1}{2}x} - \tfrac{1}{4}\pi\}$$

belongs to $L^p(0,\infty)$ for any $p > 1$. Hence the integral

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty f'(x) \{\tfrac{1}{2}\log(\tfrac{1}{4}x \coth \tfrac{1}{4}x) + \arctan e^{-\frac{1}{2}x} - \tfrac{1}{4}\pi\}\, dx \tag{3·3}$$

converges absolutely. Integrating by parts, the integrated terms vanish by lemma γ, and (3·3) becomes

$$-\frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty f(x) \left\{ \frac{1}{2x} - \frac{\operatorname{cosech}^2 \frac{1}{4}x}{8 \coth \frac{1}{4}x} - \frac{\frac{1}{2}e^{-\frac{1}{2}x}}{1+e^{-x}} \right\} dx$$
$$= \frac{1}{2(2\pi)^{\frac{1}{2}}} \int_0^\infty f(x) \left(\frac{e^{\frac{1}{2}x}}{\sinh x} - \frac{1}{x} \right) dx. \tag{3·4}$$

The other terms in (3·2) are

$$-\frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty f'(x) \left\{ \sum_{0<m\log p\leqslant x}{}' \frac{\log p}{p^{\frac{1}{2}m}} - 4\sinh \tfrac{1}{2}x \right\} dx$$
$$= -\frac{1}{(2\pi)^{\frac{1}{2}}} \lim_{T\to\infty} \left\{ \left[f(x) \left(\sum_{0<m\log p\leqslant x}{}' \frac{\log p}{p^{\frac{1}{2}m}} - 4\sinh\tfrac{1}{2}x \right) \right]_0^T - \int_0^T f(x)\, d\left(\sum_{0<m\log p\leqslant x}{}' \frac{\log p}{p^{\frac{1}{2}m}} \right) + 2\int_0^T f(x) \cosh \tfrac{1}{2}x\, dx \right\}$$
$$= -\frac{1}{(2\pi)^{\frac{1}{2}}} \lim_{T\to\infty} \left\{ f(T) \left(\sum_{0<m\log p\leqslant T}{}' \frac{\log p}{p^{\frac{1}{2}m}} - 2e^{\frac{1}{2}T} \right) - \sum_{0<m\log p<T} \frac{\log p}{p^{\frac{1}{2}m}} f(m\log p) + \int_0^T f(x)\, e^{\frac{1}{2}x}\, dx \right\}$$
$$- \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty f(x)\, e^{-\frac{1}{2}x}\, dx, \tag{3·5}$$

since the last integral converges absolutely. Adding (3·4) and (3·5) we find that (3·2) is equal to

$$\frac{1}{(2\pi)^{\frac{1}{2}}}\lim_{T\to\infty}\left\{\sum_{0<m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}}f(m\log p)-\int_0^T f(x)\,e^{\frac{1}{2}x}\,dx\right.$$
$$\left.-f(T)\left(\sum_{0<m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}}-2e^{\frac{1}{2}T}\right)\right\}$$
$$-\frac{1}{(2\pi)^{\frac{1}{2}}}\int_0^\infty f(x)\,e^{-\frac{1}{2}x}\,dx+\frac{1}{2(2\pi)^{\frac{1}{2}}}\int_0^\infty f(x)\left(\frac{e^{\frac{1}{2}x}}{\sinh x}-\frac{1}{x}\right)dx.$$

The sum of the last two integrals is equal to

$$\frac{1}{2(2\pi)^{\frac{1}{2}}}\int_0^\infty f(x)\left(\frac{e^{-\frac{3}{2}x}}{\sinh x}-\frac{1}{x}\right)dx.$$

Now consider the right-hand side of (3·1). It is equal to

$$\int_0^\infty g'(x)\left\{N(x)-\left(\frac{x}{2\pi}\log\frac{x}{2\pi}-\frac{x}{2\pi}\right)\right\}dx$$
$$=\lim_{T\to\infty}\left\{\left[g(x)\left(N(x)-\frac{x}{2\pi}\log\frac{x}{2\pi}+\frac{x}{2\pi}\right)\right]_0^T-\int_0^T g(x)\,dN(x)\right.$$
$$\left.+\frac{1}{2\pi}\int_0^T g(x)\log\frac{x}{2\pi}\,dx\right\}$$
$$=-\lim_{T\to\infty}\left\{\sum_{0<\gamma<T}g(\gamma)-\frac{1}{2\pi}\int_0^T g(x)\log\frac{x}{2\pi}\,dx\right\},$$

since the integrated terms vanish by (2·4) and lemma δ. Hence we have, altogether:

THEOREM 3. *If* (i) *the Riemann hypothesis is true,*

(ii) $f(x)$ *is an integral, tends to zero as* x *tends to infinity, and* $xf'(x)$ *belongs to* $L^p(0,\infty)$ *for some* p *in* $1<p\leqslant 2$, *and*

(iii) $$g(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\to\infty}f(t)\cos xt\,dt,$$

then $$\lim_{T\to\infty}\left\{\sum_{0<m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}}f(m\log p)-\int_0^T f(x)\,e^{\frac{1}{2}x}\,dx\right.$$
$$\left.-f(T)\left(\sum_{0<m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}}-2e^{\frac{1}{2}T}\right)\right\}-\frac{1}{2}\int_0^\infty f(x)\left(\frac{1}{x}-\frac{e^{-\frac{3}{2}x}}{\sinh x}\right)dx$$
$$=-(2\pi)^{\frac{1}{2}}\lim_{T\to\infty}\left\{\sum_{0<\gamma<T}g(\gamma)-\frac{1}{2\pi}\int_0^T g(x)\log\frac{x}{2\pi}\,dx\right\}. \quad (3·6)$$

We can simplify this result slightly by making further assumptions. By lemma γ and (2·14)

$$f(T)\left(\sum_{0<m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}}-2e^{\frac{1}{2}T}\right)=O\{T^{\alpha}f(T)\}$$
$$=o(T^{\alpha-(1/p)}). \qquad (3·7)$$

Hence:

THEOREM 3′. *If, in addition to the assumptions of theorem* 3, *either* (2·13) *holds for* $\alpha = 1/p$ *or* $T^2f(T)$ *tends to zero as* T *tends to infinity, then the term* (3·7) *can be omitted from* (3·6).

Further, if it is desired, the left-hand side of (3·6) can be expressed in terms of $\Lambda(n)$. Putting $x = \log t$, $U = e^T$ we have

$$\lim_{T\to\infty}\left\{\sum_{0<m\log p<T}\frac{\log p}{p^{\frac{1}{2}m}}f(m\log p)-\int_0^T f(x)\,e^{\frac{1}{2}x}\,dx\right\}$$
$$=\lim_{U\to\infty}\left\{\sum_{n<U}\frac{\Lambda(n)}{n^{\frac{1}{2}}}f(\log n)-\int_1^U f(\log t)\frac{dt}{t^{\frac{1}{2}}}\right\}.$$

4. *Examples and applications*

Numerous examples of functions to which the formula (3·6) is applicable can easily be found, corresponding to the many known applications of Poisson's summation formula.* Two examples chosen for their special interest are given here.

(A) Suppose† $\frac{1}{2}<R(s)<1$, and

$$f_1(x)=\begin{cases}x^{-s} & (x\leqslant 1),\\ e^{1-x} & (x\geqslant 1),\end{cases}$$

$$f_2(x)=\begin{cases}0 & (x\leqslant 1),\\ x^{-s}-e^{1-x} & (x\geqslant 1).\end{cases}$$

Then $f_1(x)$ satisfies the conditions of theorem 3 for $1<p<1/R(s)$, and $f_2(x)$ satisfies them for $p = 2$. Hence the result (3·6) holds for both functions, and by addition it also holds for the function

$$f(x)=f_1(x)+f_2(x)=x^{-s}.$$

This gives
$$g(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\to\infty}t^{-s}\cos xt\,dt$$
$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\{\Gamma(1-s)\sin\tfrac{1}{2}s\pi\}\,x^{1-s}.$$

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* E. C. Titchmarsh, *F.I.* 64–65.

† Cf. A. P. Guinand, *loc. cit.* 597.

Also if we write for the infinite integral in (3·6)

$$I(s) = \frac{1}{2}\int_0^\infty x^{-s}\left(\frac{1}{x} - \frac{e^{-\frac{1}{2}x}}{\sinh x}\right) dx,$$

then this defines an analytic function of s for $0 < R(s) < 1$, and in this strip

$$\begin{aligned} I(s) &= \frac{1}{2}\int_0^1 x^{-s}\left(\frac{1}{x} - \frac{e^{-\frac{1}{2}x}}{\sinh x}\right) dx + \frac{1}{2}\int_1^\infty x^{-1-s}\,dx - \frac{1}{2}\int_1^\infty \frac{x^{-s}e^{-\frac{1}{2}x}}{\sinh x}\,dx \\ &= \frac{1}{2}\int_0^1 x^{-s}\left(\frac{1}{x} - \frac{e^{-\frac{1}{2}x}}{\sinh x}\right) dx + \frac{1}{2s} - \frac{1}{2}\int_1^\infty \frac{x^{-s}e^{-\frac{1}{2}x}}{\sinh x}\,dx. \end{aligned}$$

This defines the analytic continuation of $I(s)$ for $R(s) < 0$, where it is

$$\begin{aligned} I(s) &= \frac{1}{2}\int_0^1 x^{-s-1}\,dx + \frac{1}{2s} - \frac{1}{2}\int_0^\infty \frac{x^{-s}e^{-\frac{1}{2}x}}{\sinh x}\,dx \\ &= -\int_0^\infty \frac{x^{-s}e^{-\frac{1}{2}x}}{e^x - e^{-x}}\,dx \\ &= -\sum_{n=0}^\infty \int_0^\infty x^{-s}e^{-(2n+\frac{3}{2})x}\,dx \\ &= -\Gamma(1-s)\sum_{n=0}^\infty (2n+\tfrac{5}{2})^{s-1} \\ &= -2^{1-s}\Gamma(1-s)\sum_{n=1}^\infty (4n+1)^{s-1}. \end{aligned}$$

Now, if we write

$$\begin{aligned} \eta(s) &= 1^{-s} - 3^{-s} + 5^{-s} - \ldots \quad \{R(s) > 1\} \\ &= \sum_{n=0}^\infty (4n+1)^{-s} - \sum_{n=0}^\infty (4n+3)^{-s}, \end{aligned}$$

then

$$\begin{aligned} \sum_{n=1}^\infty (4n+1)^{-s} &= \tfrac{1}{2}\eta(s) + \frac{1}{2}\sum_{n=0}^\infty (4n+1)^{-s} + \frac{1}{2}\sum_{n=0}^\infty (4n+3)^{-s} - 1 \\ &= \tfrac{1}{2}\eta(s) + \frac{1}{2}\sum_{n=0}^\infty (2n+1)^{-s} - 1 \\ &= \tfrac{1}{2}\eta(s) + \tfrac{1}{2}(1-2^{-s})\zeta(s) - 1. \end{aligned}$$

Hence

$$I(s) = \Gamma(1-s)\{2^{1-s} - 2^{-s}\eta(1-s) - (2^{-s} - \tfrac{1}{2})\zeta(1-s)\}.$$

If (2·13) holds for some $\alpha < R(s)$, then (3·6) gives

$$\begin{aligned} &\lim_{T\to\infty}\left\{\sum_{0<m\log p<T} \frac{\log p}{p^{\frac{1}{2}m}}(m\log p)^{-s} - \int_0^T x^{-s}e^{\frac{1}{2}x}\,dx\right\} \\ &\qquad - \Gamma(1-s)\{2^{1-s} - 2^{-s}\eta(1-s) - (2^{-s} - \tfrac{1}{2})\zeta(1-s)\} \\ &= -2\Gamma(1-s)\sin\tfrac{1}{2}s\pi \lim_{T\to\infty}\left\{\sum_{0<\gamma<T}\gamma^{s-1} - \frac{T^s}{2\pi s}\log\frac{T}{2\pi} + \frac{T^s}{2\pi s^2}\right\}. \end{aligned}$$

(B) Put $y > 0$ and

$$f(x) = \begin{cases} \cos yx - \cos yX & (0 \leqslant x \leqslant X), \\ 0 & (x \geqslant X). \end{cases}$$

The conditions of theorem 3′ are satisfied, and

$$g(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{y}{x(y^2-x^2)} (x \sin yX \cos xX - y \cos yX \sin xX)$$

if $x \neq y$. If $x = y$ then this becomes

$$g(y) = \frac{1}{(2\pi)^{\frac{1}{2}}} \left(X - \frac{\sin 2yX}{2y}\right). \tag{4·1}$$

Substituting this in (3·6) we obtain

$$\begin{aligned} \sum_{0<m\log p<X} \frac{\log p}{p^{\frac{1}{2}m}} \{\cos(ym \log p) - \cos yX\} - \int_0^X (\cos yx - \cos yX) e^{\frac{1}{2}x}\, dx \\ - \frac{1}{2}\int_0^X (\cos yx - \cos yX)\left(\frac{1}{x} - \frac{e^{-\frac{1}{2}x}}{\sinh x}\right) dx \\ = -2\sum_{\gamma>0} \frac{y}{\gamma(y^2-\gamma^2)} (\gamma \sin yX \cos \gamma X - y \cos yX \sin \gamma X) \\ + \frac{1}{\pi}\int_0^\infty \frac{y}{x(y^2-x^2)} (x \sin yX \cos xX - y \cos yX \sin xX) \log \frac{x}{2\pi}\, dx, \end{aligned} \tag{4·2}$$

where the term $\gamma = y$, if it occurs, is to be interpreted as in (4·1).

Now let X tend to infinity. For fixed y

$$\begin{aligned} \frac{1}{2}\int_0^X (\cos yx - \cos yX)\left(\frac{1}{x} - \frac{e^{-\frac{1}{2}x}}{\sinh x}\right) dx &= O(1) - \tfrac{1}{2}\cos yX \int_1^X \frac{dx}{x} \\ &= O(1) - \tfrac{1}{2}\cos yX \log X. \end{aligned}$$

Also

$$\begin{aligned} &\left| \sum_{\gamma>0,\, \gamma\neq y} \frac{y}{\gamma(y^2-\gamma^2)} (\gamma \sin yX \cos \gamma X - y \cos yX \sin \gamma X) \right| \\ &\leqslant \sum_{\gamma>0,\, \gamma\neq y} \frac{y(\gamma+y)}{\gamma\,|y^2-\gamma^2|} = y \sum_{\gamma>0,\, \gamma\neq y} \frac{1}{\gamma\,|y-\gamma|} \\ &= O(1), \end{aligned}$$

since the latter series converges.* The term $\gamma = y$, if it occurs, gives a term

$$-X + \frac{\sin 2yX}{2y} = -X + O(1).$$

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* A. E. Ingham, *loc. cit.* theorem 25*b*.

Further

$$\frac{1}{\pi}\int_0^\infty \frac{y}{x(y^2-x^2)}(x\sin yX\,\cos xX - y\cos yX\,\sin xX)\log\frac{x}{2\pi}\,dx$$

$$= \frac{1}{\pi}\int_0^\infty \frac{y}{x(y^2-x^2)}\{(x-y)\cos yX\,\sin xX - x\sin(x-y)X\}\log\frac{x}{2\pi}\,dx$$

$$= -\frac{1}{\pi}y\cos yX\int_0^\infty \frac{\sin xX}{x(y+x)}\log\frac{x}{2\pi}\,dx + \frac{1}{\pi}y\int_0^\infty \frac{\sin(x-y)X}{(x^2-y^2)}\log\frac{x}{2\pi}\,dx$$

$$= \frac{1}{\pi}\cos yX\int_0^\infty \frac{\sin xX}{y+x}\log\frac{x}{2\pi}\,dx - \frac{1}{\pi}\cos yX\int_0^\infty \frac{\sin xX}{x}\log\frac{x}{2\pi}\,dx$$

$$+\frac{1}{\pi}y\int_0^\infty \frac{\sin(x-y)X}{x^2-y^2}\log\frac{x}{2\pi}\,dx. \qquad (4\cdot3)$$

By Fourier's single integral theorem* the first of these integrals tends to zero as X tends to infinity, and the last tends to

$$\tfrac{1}{2}\log(y/2\pi) = O(1).$$

There remains in (4·3)

$$-\frac{1}{\pi}\cos yX\int_0^\infty \frac{\sin xX}{x}\log\frac{x}{2\pi}\,dx = -\frac{1}{\pi}\cos yX\int_0^\infty \frac{\sin t}{t}\log\left(\frac{t}{2\pi X}\right)dt$$

$$= -\tfrac{1}{2}\cos yX(C+\log 2\pi X)$$

$$= O(1)+\tfrac{1}{2}\cos yX\,\log X.$$

Hence (4·2) becomes

$$\sum_{0<m\log p<X}\frac{\log p}{p^{\frac{1}{2}m}}\{\cos(ym\log p)-\cos yX\} - \int_0^X(\cos yx-\cos yX)\,e^{\frac{1}{2}x}\,dx$$

$$= \begin{cases} -X+O(1) & (y=\gamma), \\ O(1) & (y\neq\gamma). \end{cases}$$

Hence we have:

THEOREM 4. *If the Riemann hypothesis is true, then as X tends to infinity*

$$\sum_{0<m\log p<X}\frac{\log p}{p^{\frac{1}{2}m}}\cos(ym\log p) - \frac{e^{\frac{1}{2}X}}{\frac{1}{4}+y^2}(\tfrac{1}{2}\cos yX + y\sin yX)$$

$$-\cos yX\left\{\sum_{0<m\log p<X}\frac{\log p}{p^{\frac{1}{2}m}} - 2e^{\frac{1}{2}X}\right\}$$

$$= \begin{cases} -X+O(1) & (y=\gamma), \\ O(1) & (y\neq\gamma). \end{cases}$$

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* E. C. Titchmarsh, *F.I.* theorem 12.

Inversely, if we interchange $f(x)$ and $g(x)$ we can prove:

THEOREM 5. *If the Riemann hypothesis is true, then as X tends to infinity*

$$\sum_{0<\gamma<X} \cos y\gamma = \begin{cases} -\dfrac{X}{2\pi}\dfrac{\log p}{p^{\frac{1}{2}m}} + O(\log X) & (y = m\log p), \\ O(\log X) & (y \neq m\log p). \end{cases}$$

However, this result is a simple consequence of the known result* for $x > 1$

$$\sum_{0<I(\rho)<T} x^{\rho} = \begin{cases} -\dfrac{T}{2\pi}\log p + O(\log T) & (x = p^m), \\ O(\log T) & (x \neq p^m). \end{cases}$$

New College
Oxford

* E. C. Titchmarsh, *Tract*, theorem 42.

ON ANALYTIC FUNCTIONS POSSESSING CERTAIN PROPERTIES OF UNIVALENCY

By ZEEV NEHARI

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1. In the present paper, various classes of analytic functions possessing certain properties of univalency are investigated. Throughout the paper, ample use is made of the principle of subordination (principle of the majorant function)* which appears to be particularly suited to the treatment of the problems under discussion. In all cases, "best possible" results are obtained —a characteristic feature of the majorant method when appropriately used.

The functions discussed will usually be of the form

$$w = f(z) = z + a_2 z^2 + \dots, \tag{a}$$

and will be assumed to be regular in the unit circle $|z| < 1$. In order to submit the properties of such functions to a closer study, the very great generality still allowed by the class of functions (a) is usually qualified by various additional assumptions which lead to more specialized classes of functions $f(z)$. One of the most fruitful of these additional assumptions is the condition that the domain D on which $w = f(z)$ maps the unit circle should be "schlicht", i.e. should not overlap itself. Functions $f(z)$ satisfying this condition form the class S of univalent or "schlicht" functions which has been thoroughly studied and for which a great many important results have been obtained. Most noteworthy among these results is the classical distortion theorem according to which any point $w = d$ outside D satisfies the inequality $|d| \geqslant \frac{1}{4}$.

Now this condition of D being "schlicht" can be generalized in many ways. Some of these generalizations will be investigated in the following, and it will be shown that in spite of considerably weaker assumptions results of the type of the distortion theorem still hold. As may be expected,

* For a detailed account of the principle of subordination and its many applications see Littlewood, *Lectures on the Theory of Functions* (Oxford, 1944).

the constant $\frac{1}{4}$ will have to be replaced in these cases by suitable smaller constants.

2. In the first theorem to be proved, the condition that D be "schlicht" will be replaced by the much weaker assumption that there exists a point d on the boundary of D such that $w = d$ may be connected with $w = 0$ by a "schlicht" curve, i.e. a curve covered by D exactly once. Stating the latter condition in a different way, we may say that for all points w' on this curve the equation $f(z) = w'$ shall have exactly one solution satisfying $|z| < 1$.

THEOREM I. *Let*

$$w = f(z) = z + a_2 z^2 + \dots$$

be regular in the unit circle and let D denote the domain on which $|z| < 1$ is mapped by $w = f(z)$; let, further, $w = d$ be a point on the boundary of D which may be connected with $w = 0$ by a "schlicht" curve. Then

$$|d| \geqslant \frac{1}{\pi^2}.$$

This inequality is the best possible.

Before we begin the proof of this theorem, we discuss the special case in which the curve in question is the stretch $0 \leqslant w < d$ (d may be assumed to be positive without loss of generality). If we consider the function

$$w = g(z) = \frac{f(z)}{d} = \frac{z}{d} + \dots,$$

we see without difficulty that $w = g(z)$ is subordinate to the function $w = \mathfrak{F}(z) = \alpha z + \dots$ which maps $|z| < 1$ on a Riemann surface $\bar{D}$ of the following description: $\bar{D}$ covers the whole finite w-plane an infinity of times, with the exception of the stretch $0 \leqslant w < 1$ which is covered only once and the points $w = 1$ and $w = \infty$ which are outside $\bar{D}$ (i.e. $\bar{D}$ has no boundary points not satisfying either $0 \leqslant w \leqslant 1$ or $w = \infty$); furthermore, $\bar{D}$ is everywhere "locally schlicht", i.e. has no inner branch-points. Here we should show that $\bar{D}$ is simply-connected, but this may be dispensed with, as we shall presently construct the function $w = \mathfrak{F}(z)$ which maps $|z| < 1$ on $\bar{D}$.

Since both $w = g(z)$ and $w = \mathfrak{F}(z)$ vanish for $z = 0$ and the conformal representations of the unit circle by both functions cover the stretch $0 \leqslant w < 1$ in a "schlicht" manner, the function

$$\omega(z) = \mathfrak{F}^{-1}[g(z)]$$

is regular—and $|\omega(z)| \leqslant 1$—for those values z which correspond to $0 \leqslant g(z) < 1$. Since no other value taken by $w = g(z)$ (for $|z| < 1$) coincides with a boundary point of $\bar{D}$ (i.e. $0 \leqslant w \leqslant 1$, $w = \infty$) or a branch-point (of which $\bar{D}$ has none),

we may conclude that $w(z)$ is uniform and $|\omega(z)| \leqslant |z|$ for $|z| < 1$ or, in other words, that $g(z)$ is subordinate to $\mathfrak{F}(z)$. The developments of $g(z)$ and $\mathfrak{F}(z)$ being $g(z) = z/d + \ldots$ and $\mathfrak{F}(z) = \alpha z + \ldots$ respectively, this entails

$$\frac{1}{d} \leqslant |\alpha|, \quad d \geqslant \frac{1}{|\alpha|}. \tag{1}$$

We now proceed to construct the function $w = \mathfrak{F}(z)$ which maps $|z| < 1$ on this surface $\bar{D}$. To this end we consider the conformal mapping properties of the function $w = \sin x$. This function maps the half-strip

$$\mathfrak{I}\{x\} > 0, \quad -\tfrac{1}{2}\pi < \mathfrak{R}\{x\} < \tfrac{1}{2}\pi$$

on the half-plane $\mathfrak{I}\{w\} > 0$, the points $x = \frac{1}{2}\pi, -\frac{1}{2}\pi, \infty$ and $w = 1, -1, \infty$ corresponding respectively. As both the half-strip and the half-plane are bounded by straight lines, the function $w = \sin x$ can be analytically continued with the help of Schwarz's principle of symmetry. By inverting the half-strip continually along the rays $\mathfrak{R}\{x\} = -\frac{1}{2}\pi$, $0 < \mathfrak{I}\{x\} < \infty$ and $\mathfrak{R}\{x\} = \frac{1}{2}\pi$, $0 < \mathfrak{I}\{x\} < \infty$ and the rays corresponding to them in the half-strips so obtained, the whole half-plane $\mathfrak{I}\{x\} > 0$ will eventually be covered (in a "schlicht" manner). In the w-plane, the corresponding inversions will be made with regard to the rays $-\infty < w < -1$ and $1 < w < \infty$. The function $w = \sin x$ will therefore map the half-plane $\mathfrak{I}\{x\} > 0$ on the whole finite w-plane, covered an infinity of times, with the exception of the stretch $-1 \leqslant w \leqslant 1$ which is not covered at all. All boundary points of this surface satisfy either $-1 \leqslant w \leqslant 1$ or $w = \infty$. We now use the symmetry principle once more, this time with regard to the stretch $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$. In the x-plane we thus obtain the entire plane slit along the rays $-\infty < x < -\frac{1}{2}\pi$ and $\frac{1}{2}\pi < x < \infty$; in the w-plane the result is a surface D' very nearly related to $\bar{D}$ (D' leaves out the points $w = \pm 1, \infty$ and covers simply the stretch $-1 < w < 1$). Since the conformal representation of $|z| < 1$ on an x-plane, cut in the way just described, is effected by the function

$$x = \frac{\pi z}{1 + z^2},$$

it follows that $w = \mathfrak{G}(z) = \sin\left[\dfrac{\pi z}{1 + z^2}\right]$ maps the unit circle on D'. Now the relation between $\mathfrak{G}(z)$ and the function $\mathfrak{F}(z)$ mapping $|z| < 1$ on $\bar{D}$ is $\mathfrak{G}(z) = \sqrt{[\mathfrak{F}(z^2)]}$, this transformation obviously linking $\bar{D}$ and D'. Hence

$$\mathfrak{F}(z) = \mathfrak{G}^2(\sqrt{z}) = \sin^2\left[\frac{\pi\sqrt{z}}{1 + z}\right] = \pi^2 z + \ldots.$$

Combined with (1), this finally yields

$$d \geqslant \frac{1}{\pi^2}.$$

So far the simply covered curve has been the stretch $0 \leqslant w < d$. If we now turn to the general case, we start by considering the function

$$w = g(z) = \sqrt{\left(\frac{f(z^2)}{d}\right)} = \frac{z}{\sqrt{d}} + \dots$$

which is also regular for $|z| < 1, f(z)$ not vanishing there except for $z = 0$. $w = g(z)$ does not take the values $w = \pm 1$ and both $w = 1$ and $w = -1$ can be connected with $w = 0$ by curves, say c_1 and c_2, which are simply covered by the surface on which $w = g(z)$ maps $|z| < 1$. $w = g(z)$ being an odd function of z, c_1 and c_2 are symmetrical with regard to $w = 0$. Let now E be the surface arising from the surface D' considered earlier by deforming the stretch $-1 < w < 1$ so as to take the shape of the combined curves c_1 and c_2, i.e. E is locally "schlicht", leaves out the points $w = \pm 1, \infty$, covers the curves c_1 and c_2 only once, and all finite boundary points of E lie on c_1 and c_2. Since E is obtained from D' by a continuous deformation of the w plane, E is also simply-connected and can therefore be obtained as a conformal representation of the unit circle by a function $w = \mathfrak{G}(z) = \beta z + \dots$. Considerations similar to those applied earlier to the domain D' show that $w = g(z) = \dfrac{z}{\sqrt{d}} + \dots$ is subordinate to $w = \mathfrak{G}(z) = \beta z + \dots$, whence we obtain

$$\frac{1}{\sqrt{d}} \leqslant |\beta|, \quad d \geqslant \frac{1}{|\beta|^2}. \tag{2}$$

In order to find an upper bound for $|\beta|$, we consider the function

$$u = h(z) = \frac{2}{\pi} \arcsin \mathfrak{G}(z) = \frac{2\beta}{\pi} z + \dots,$$

which is regular and uniform for $|z| < 1$, since the critical points of $u = \arcsin w$, viz. $w = \pm 1, \infty$, are left out by $w = \mathfrak{G}(z)$. The simply-covered curves c_1 and c_2 are transformed by the main branch of $h(z)$ into two other simply covered curves c_1' and c_2' connecting $u = 1$ and $u = -1$ with $u = 0$. Since, for values w' on the curves c_1 and c_2, the equation $w' = f(z)$ had only one solution z satisfying $|z| < 1$, the analytical continuation of $u = h(z)$ for all values $|z| < 1$ leaves out all points $u' + 2n$ $(n = \pm 1, \pm 2, \dots)$, where u' is any point of c_1' or c_2'. We may therefore conclude that the points $u = \pm 1$ belong to the "outer boundary" of the domain F on which $u = h(z)$ maps the unit circle, i.e. $u = \pm 1$ may be connected with $u = \infty$ by curves p_1 and p_2 respectively, not belonging to F. $u = h(z)$ being an odd function of z, p_1 and p_2 may be chosen symmetrical with regard to $u = 0$. $u = h(z)$ is therefore subordinate to the odd "schlicht" function $u = k(z) = \gamma z + \dots$ mapping $|z| < 1$ on the entire z-plane slit along p_1 and p_2. Now, by the

distortion theorem, odd "schlicht" functions $u = k(z) = \gamma z + \ldots$ not taking (for $|z| < 1$) the values $u = \pm 1$ satisfy $|\gamma| \leqslant 2$. The function

$$u = h(z) = 2\beta z/\pi + \ldots$$

being subordinate to $u = k(z)$, this entails

$$\frac{2|\beta|}{\pi} \leqslant 2.$$

Combined with (2), this finally yields

$$d \geqslant \frac{1}{|\beta|^2} \geqslant \frac{1}{\pi^2}.$$

It is clear that the case $d = 1/\pi^2$ can only occur for one of the functions

$$\frac{1}{K\pi^2} \sin^2 \left[\frac{\pi \sqrt{(Kz)}}{1+Kz} \right],$$

where $|K| = 1$. This completes the proof of theorem I.

3. The hypothesis of the next theorem will be half-way between that of theorem I and the assumption that the function considered is "schlicht".

THEOREM II. *Let $w = f(z) = z + a_2 z^2 + \ldots$ be regular for $|z| < 1$, and let D denote the domain on which $w = f(z)$ maps the unit circle; let further $w = d$ be a point of the boundary of D. If the interior of the circle $|w| = |d|$ is covered by D in a "schlicht" manner, i.e. if for $|w'| < |d|$ the equation $w' = f(z)$ has exactly one solution z satisfying $|z| < 1$, we have*

$$|d| \geqslant \frac{1}{e^2}.$$

This inequality is the best possible.

The proof of this theorem is a straightforward majorant affair. If (without loss of generality) d is assumed positive, it is clear that the function

$$w = \frac{f(z)}{d} = g(z) = \frac{z}{d} + \ldots$$

is subordinate to the function $w = \mathfrak{F}(z) = dz + \ldots$ mapping the unit circle on a surface $\bar{D}$ defined as follows: $\bar{D}$ is locally "schlicht" and covers the finite w-plane an infinity of times, with the exception of the circle $|w| < 1$ which is covered only once and the points $w = 1$ and $w = \infty$ which are not covered at all. All boundary points of $\bar{D}$ satisfy either $|w| = 1$ or $w = \infty$. $\bar{D}$ is simply-connected, as will be shown by actually finding the function $w = \mathfrak{F}(z) = \alpha z + \ldots$ mapping $|z| < 1$ on $\bar{D}$.

$w = g(z) = z/d + \ldots$ being subordinate to $w = \mathfrak{F}(z) = \alpha z + \ldots$, we have

$$\frac{1}{d} \leqslant |\alpha|, \quad d \geqslant \frac{1}{|\alpha|}. \tag{3}$$

Now it is easily seen that $\mathfrak{F}(z)$ is of the form

$$\mathfrak{F}(z) = z \exp\left(2 \cdot \frac{1-z}{1+z}\right).$$

Indeed, for $|z| = 1$ (and $z \neq -1$) we have $|\mathfrak{F}(z)| = 1$, and $\mathfrak{F}(z)/z$ does not vanish for $|z| < 1$. Obviously $w = \mathfrak{F}(z)$ takes all values $|w| < 1$ only once for $|z| < 1$. The derivative of $\mathfrak{F}(z)$ is

$$\mathfrak{F}'(z) = \exp\left(2 \cdot \frac{1-z}{1+z}\right)\left(\frac{1-z}{1+z}\right)^2. \tag{4}$$

As this expression does not vanish except for $z = 1$, it follows that $z \exp\left(2 \cdot \frac{1-z}{1+z}\right)$ maps $|z| < 1$ on a locally "schlicht" surface, say $\bar{D}$. Furthermore, (4) shows that $\mathfrak{F}'(z)$ has a double zero at $z = 1$. At this point the curve into which $w = \mathfrak{F}(z)$ transforms $|z| = 1$ turns by an angle of 2π. Because of $\mathfrak{F}(1) = 1$ this means that the point $w = 1$ is not covered by the particular sheet of the surface $\bar{D}$ in which lie the points $w = \mathfrak{F}(\theta)$, where θ is positive and slightly smaller than 1. If $w = 1$ were covered by another sheet of $\bar{D}$, the same would be true of the points $w = \mathfrak{F}(\theta)$. However, all points $|w| < 1$ are taken by $w = \mathfrak{F}(z)$ only once and therefore we have $F(z) \neq 1$ for $|z| < 1$.

The function $\quad w = \mathfrak{F}(z) = z \exp\left(2 \cdot \frac{1-z}{1+z}\right) = e^2 z + \ldots$

mapping $|z| < 1$ on the surface $\bar{D}$ described above, we have, by (3),

$$d \geqslant \frac{1}{e^2}.$$

Here, equality is only possible for functions of the form $\frac{1}{e^2 K}\mathfrak{F}(Kz)$ with $|K| = 1$.

This completes the proof.

4. Theorems I and II both occupy intermediate positions between the classical distortion theorem on the one hand, and Hurwitz's theorem on functions "schlicht in one point" on the other. If written in the order of increasing generality, these four theorems may be enunciated as follows:

Let $w = f(z)$ be an analytic function, regular for $|z| < 1$, and let $w = d$ be a point on the boundary of the domain D on which the unit circle is mapped by $w = f(z)$. Then:

(*a*) *if the whole domain D is "schlicht", we have*

$$|d| \geqslant \frac{1}{2^2};$$

(*b*) *if the circle* $|w| < |d|$ *is covered by* D *in a "schlicht" manner, we have*

$$|d| \geqslant \frac{1}{e^2};$$

(*c*) *if a curve connecting* $w = d$ *with* $w = 0$ *is covered by* D *in a "schlicht" manner, we have*

$$|d| \geqslant \frac{1}{\pi^2};$$

(*d*) *if the point* d *is covered by* D *in a "schlicht" manner, we have*

$$|d| \geqslant \frac{1}{4^2}.$$

In the last case it has to be further stipulated that $f(z) \neq d$ *for* $|z| < 1$.

5. The object of this section will be a generalization of theorem I. In addition to the hypotheses of theorem I we shall further assume that the function $w = f(z) = z + \ldots$ is bounded by a positive number M (> 1), i.e.

$$|f(z)| \leqslant M, \quad |z| < 1.$$

It is clear that this additional condition will result in an increased value of the lower bound for the distance of the boundary point $w = d$ from the origin.

Following the usual procedure in the case of bounded functions, we shall divide the function $f(z)$ by M and, in the sequel, deal with functions $w = f(z) = \alpha z + \ldots$ satisfying $|f(z)| \leqslant 1$ for $|z| < 1$. The theorem to be proved may then be enunciated as follows:

THEOREM III. *Let* $w = f(z) = \alpha z + \ldots$ *be an analytic function, regular for* $|z| < 1$ *and satisfying there* $|f(z)| \leqslant 1$; *let further* d $(0 < |d| < 1)$ *be a point of the boundary of the domain* D *on which* $|z| < 1$ *is mapped by* $w = f(z)$, *such that* $w = d$ *may be connected with* $w = 0$ *by a "schlicht" curve. Then we have*

$$|\alpha| \leqslant \tau^2 \vartheta_2^4(0;\tau) \tanh^2 \frac{\pi}{2\tau}, \tag{5}$$

where τ $(\tau > 0)$ *is given by the equation*

$$|d| = \frac{\vartheta_2^2(0;2\tau)}{\vartheta_3^2(0;2\tau)}, \tag{6}$$

ϑ_2 *and* ϑ_3 *being the Jacobian* ϑ-*functions. This bound for* $|\alpha|$ *is the best possible.*

This result can also be interpreted in the opposite way, namely, if τ is determined by assuming equality in (5), then the right-hand side of (6) gives a lower bound for $|d|$.

Before we enter upon the proof of theorem III, we remark that for $1/\alpha \uparrow \infty$ theorem I is obtained as a limiting case of theorem III. In fact, if we denote

by (ϵ) quantities that tend to 0 for $\tau \uparrow \infty$, the asymptotic expressions for the ϑ-functions are

$$\frac{\vartheta_2^2(0;\,2\tau)}{\vartheta_3^2(0;\,2\tau)} = 4e^{-4\tau}[1+(\epsilon)] \quad \text{and} \quad \vartheta_2^4(0;\,\tau) = 16e^{-4\tau}[1+(\epsilon)].$$

Assuming equality in (5), the case $1/\alpha \uparrow \infty$ corresponds therefore to $\tau \uparrow \infty$, and in view of

$$\tau^2 \tanh^2 \frac{\pi}{2\tau} = \frac{\pi^2}{4} + (\epsilon),$$

we obtain
$$|\, d' \,| = \frac{|\, d \,|}{|\, \alpha \,|} \geqslant \frac{4e^{-4\tau}}{16e^{-4\tau}\,\frac{1}{4}\pi^2} + (\epsilon) = \frac{1}{\pi^2} + (\epsilon),$$

which is equivalent to theorem I.

The proof of theorem III resembles very closely the proof of theorem I. We shall first assume the "schlicht" curve to be the stretch $0 \leqslant w < d$, where d is again assumed positive without loss of generality. The role of the surface $\bar{D}$ will here be played by a surface E obtained from $\bar{D}$ by contracting it in the ratio $1:d$ and then cutting away the outer parts of all its sheets along the circle $|\, w \,| = 1$. The (locally "schlicht") surface E may consequently be defined as follows: E covers the circle $|\, w \,| < 1$ an infinity of times, apart from the stretch $0 \leqslant w < d$ which is covered by E exactly once and the point $w = d$ which is not covered at all. E is simply-connected as is clear from the manner by which it was obtained from the simply-connected surface $\bar{D}$. Obviously, the function $w = f(z) = \alpha z + \ldots$ is subordinate to the function $w = \mathfrak{F}(z) = \beta z + \ldots$ mapping $|\, z \,| < 1$ on E, whence we obtain $|\, \alpha \,| \leqslant |\, \beta \,|$.

Our next aim is, therefore, to find an analytical expression for the function $w = \mathfrak{F}(z)$ or, at least, for its first coefficient β. For this purpose we consider the conformal mapping properties of the Jacobian elliptic function $w = \operatorname{sn} x = x + \ldots$ satisfying the differential equation $w'^2 = (1-w^2)(1-k^2w^2)$. If $4K$ and $2iK'$ (K, K' real) denote the elementary periods of $w = \operatorname{sn} x$, this function maps the rectangle $[-K;\, K;\, K+iK';\, -K+iK']$ in the x-plane on the half-plane $\mathfrak{F}\{w\} > 0$, the points

x	$-K$	K	$K+iK'$	$-K+iK'$	0	iK'
w	-1	1	k^{-1}	$-k^{-1}$	0	∞

corresponding. The expression of k in terms of the period ratio $\tau = K'/K$ is of the form

$$k = \frac{\vartheta_2^2(0;\,\tau)}{\vartheta_3^2(0;\,\tau)},$$

where ϑ_2 and ϑ_3 are the Jacobian ϑ-functions.

In order to continue $w = \operatorname{sn} x$ analytically for the whole strip $0 < \mathfrak{I}\{x\} < K'$, we again use Schwarz's symmetry principle. Since to the inversions of the mentioned rectangle with respect to the stretches $\mathfrak{R}\{x\} = K$, $0 < \mathfrak{I}\{x\} < K'$ and $\mathfrak{R}\{x\} = -K$, $0 < \mathfrak{I}\{x\} < K'$ correspond inversions of the upper half-plane $\mathfrak{I}\{w\} > 0$ with respect to the stretches $1 < w < k^{-1}$ and $-k^{-1} < w < -1$, it follows that $w = \operatorname{sn} x$ maps the strip $0 < \mathfrak{I}\{x\} < K'$ on a locally "schlicht" surface covering the finite w-plane an infinity of times, with the exception of the stretch $-1 \leqslant w \leqslant 1$ and the rays $-\infty \leqslant w \leqslant -k^{-1}$ and $k^{-1} \leqslant w \leqslant \infty$ which are not covered at all. Another application of Schwarz's symmetry principle, this time with respect to the stretch $-K < x < K$, shows that $w = \operatorname{sn} x$ maps the strip $-K' < \mathfrak{I}\{x\} < K'$, cut along the rays $-\infty \leqslant x \leqslant -K$ and $K \leqslant x \leqslant \infty$ on the locally "schlicht" surface E' defined as follows: All boundary points of E' satisfy one of the inequalities

$$-1 \leqslant w \leqslant 1, \quad -\infty \leqslant w \leqslant -k^{-1}, \quad k^{-1} \leqslant w \leqslant \infty;$$

the rays $-\infty \leqslant w \leqslant -k^{-1}$ and $k^{-1} \leqslant w \leqslant \infty$ and the points $w = \pm 1$ are outside E, and the stretch $-1 < w < 1$ is covered by one sheet of E' only.

Now this surface E' has many traits in common with the surface we attempt to construct. In fact, since the function $w = \dfrac{2u}{k(1+u^2)}$ maps $|u| < 1$ on the entire w-plane cut along the rays $-\infty \leqslant w \leqslant -k^{-1}$ and $k^{-1} \leqslant w \leqslant \infty$, we may infer that the function $u = p(x) = \frac{1}{2}kx + \ldots$ defined by

$$\frac{2p(x)}{k[1+p^2(x)]} = \operatorname{sn} x \tag{7}$$

maps the strip $-K' < \mathfrak{I}\{x\} < K'$, cut along the rays $-\infty \leqslant x \leqslant -K$ and $K \leqslant x \leqslant \infty$, on a surface E'' connected with E by a root transformation, i.e. if $u = \mathfrak{G}(z)$ maps $|z| < 1$ on E'' and $u = \mathfrak{F}(z)$ maps $|z| < 1$ on E, the relation

$$\mathfrak{G}(z) = \sqrt{[\mathfrak{F}(z^2)]} \tag{8}$$

holds. Indeed, in view of (7) the surface E'' on which $u = p(x)$ maps the strip in question covers $|u| < 1$ an infinity of times, apart from a stretch $-\delta < u < \delta$ (δ to be defined presently) which is covered by E'' exactly once and the points $u = \pm\delta$ which are not covered at all. A glance at (8) shows that $\delta = \sqrt{d}$, where $w = d$ was the point left out by $w = \mathfrak{F}(z)$ ($|z| < 1$). In view of (7), the modulus k of the elliptic function is therefore determined by

$$\frac{2\sqrt{d}}{k(1+d)} = 1 \quad [0 < d < 1],$$

i.e.

$$\frac{2\sqrt{d}}{1+d} = k = \frac{\vartheta_2^2(0;\tau)}{\vartheta_3^2(0;\tau)}.$$

Since the modulus of τ is connected with the modulus belonging to 2τ by the relation

$$\frac{2\sqrt{[k(2\tau)]}}{1+k(2\tau)} = k(\tau),$$

this may also be written $\sqrt{d} = \sqrt{[k(2\tau)]}$,

i.e. $$d = k(2\tau) = \frac{\vartheta_2^2(0;\, 2\tau)}{\vartheta_3^2(0;\, 2\tau)}.$$

In order to complete the construction of the function $\mathfrak{F}(z)$, we have now to find a function, say $x = r(z)$, mapping $|z| < 1$ on the strip $-K' < \mathfrak{J}\{x\} < K'$, cut along the rays $-\infty \leqslant x \leqslant -K$ and $K \leqslant x \leqslant \infty$; $\mathfrak{F}(z)$ will then be of the form $\mathfrak{F}(z) = p^2[r(\sqrt{z})]$. But this function can be built up by elementary conformal representations. The function

$$x = \frac{2K'}{\pi}\log\left(\frac{1+\xi}{1-\xi}\right) = \frac{4K'}{\pi}\xi + \ldots$$

maps $|\xi| < 1$ on the strip $-K' < \mathfrak{J}\{x\} < K'$. If we cut this strip along the rays $-\infty \leqslant x \leqslant -K$ and $K \leqslant x \leqslant \infty$, the corresponding cuts in the ξ-plane will be $-1 \leqslant \xi \leqslant -\theta$ and $\theta \leqslant \xi \leqslant 1$, θ being defined by

$$\frac{2K'}{\pi}\log\frac{1+\theta}{1-\theta} = K,$$

i.e. $$\theta = \tanh\frac{\pi}{4\tau} \quad [\tau = K'/K].$$

The last step will now be to find the function $\xi = h(z)$ mapping $|z| < 1$ on the circle $|\xi| < 1$ cut along the stretches $-1 \leqslant \xi \leqslant -\theta$ and $\theta \leqslant \xi \leqslant 1$. But it is easily seen that this conformal representation is effected by the function

$$\xi = h(z) = g^{-1}\{\lambda g(z)\},$$

where $$g(z) = \frac{z}{1+z^2} \quad \text{and} \quad \lambda = \frac{2\theta}{1+\theta^2}.$$

The first coefficient of the power series of $h(z)$—which is all the information we need for our purpose—is

$$\lambda = \frac{2\theta}{1+\theta^2} = \frac{2\tanh\frac{1}{4}\pi/\tau}{1+\tanh^2\frac{1}{4}\pi/\tau} = \tanh\tfrac{1}{2}\pi/\tau.$$

The development of the function $r(z)$ will therefore start with

$$r(z) = \frac{2K'}{\pi}\log\frac{1+h(z)}{1-h(z)} = \left\{\frac{4K'}{\pi}\tanh\tfrac{1}{2}\pi/\tau\right\}z + \ldots.$$

$\mathfrak{F}(z)$ being of the form $p^2\{r(\sqrt{z})\}$, we obtain finally

$$\mathfrak{F}(z) = p^2(r(\sqrt{z})) = \left\{\frac{4K'k^2}{\pi^2}\tanh^2\tfrac{1}{2}\pi/r\right\}z + \ldots.$$

In view of $$k = \frac{\vartheta_2^2(0;\tau)}{\vartheta_3^2(0;\tau)} \quad \text{and} \quad K' = \frac{\pi\tau}{2}\vartheta_3^2(0;\tau),$$

the first coefficient of $\mathfrak{F}(z)$ may also be written

$$\beta = \tau^2\vartheta_2^4(0;\tau)\tanh^2 \tfrac{1}{2}\pi/\tau.$$

This completes the proof in the case in which the "schlicht" curve connecting $w = 0$ with $w = d$ was the stretch $0 \leqslant w < d$. The extension of this result to the case of a general "schlicht" curve connecting the points $w = 0$ and $w = d$ can be achieved in very much the same way as in the proof of theorem I. The place of the function $2/\pi \sin^{-1}(w)$ used there will be taken here by the inverse of the elliptic function we utilized for the proof of the particular case of theorem III. Instead of the usual distortion theorem which was there used to clinch the argument, recourse has to be made here to a well-known generalization of the distortion theorem which asserts that the first coefficient α of an odd "schlicht" function $w = \alpha z + \ldots$, bounded by unity and leaving out the points $w = \pm l$ $(l > 0)$, attains its largest modulus for the particular function mapping $|z| < 1$ on the full circle $|w| < 1$ cut radially from $w = l$ to $w = 1$ and from $w = -l$ to $w = -1$.

5. In this section we shall make use of the conformal representation effected by the function $w = \sin\left\{\frac{\pi z}{1+z^2}\right\}$ discussed earlier, in order to prove a theorem on doubly-connected domains. The main argument used here will in many respects be similar to the principle of subordination used in the case of simply-connected domains.

A function $w = f(z)$ mapping an annulus $1 < |z| < m$ on a doubly-connected domain D may be said to be subordinate to a function $w = \mathfrak{F}(z)$ mapping an annulus $1 < |z| < M$ on a doubly-connected domain $\bar{D}$, if the relation

$$f(z) = \mathfrak{F}[\omega(z)]$$

holds, where $\omega(z)$ is regular for $1 < |z| < m$, satisfies

$$1 \leqslant |\omega(z)| \leqslant M \quad (1 < |z| < m),$$

and the argument of $\omega(z)$ does not return to its initial value if z describes any circle $|z| = \zeta$ $(1 < \rho < m)$.

In case $\bar{D}$ does not overlap itself, i.e. is a "schlicht" domain, this definition of subordination is equivalent to saying that, for $1 < |z| < m$, $f(z)$ takes only values inside $\bar{D}$ (and that, inside $\bar{D}$, the inner or outer boundary of D cannot shrink continuously into evanescence).

The applicability of this definition of subordination lies in the well-known fact that in the case of both domains D and $\bar{D}$ being "schlicht",

the modulus m of D cannot be greater than the modulus M of $\bar{D}$, the modulus of a doubly-connected "schlicht" domain being a steadily increasing function of the domain.

However, this property of the modulus steadily increasing with the domain breaks down if the domains D and $\bar{D}$ are not both "schlicht" or, which is somewhat less, if $\omega(z)$ is not a "schlicht" function of z for $1<|z|<m$.

Nevertheless, the inequality $m \leqslant M$ holds also in the general case, as is shown by the following lemma due to M. Schiffer:

LEMMA.* Let D be a doubly connected domain contained in the annulus $1<|w|<M$. If the argument of a point w does not return to its initial value when this point describes once the inner boundary of D (or the outer boundary, for that matter), the modulus m of D satisfies the inequality

$$m \leqslant M.$$

Here the sign of equality can only hold if D is identical with the annulus $1<|w|<M$.

In order to prove the lemma, we denote by $w=f(z)$ the function mapping $1<|z|<m$ on D and consider the function

$$g(z)=z^{-n}f(z), \tag{9}$$

where $2\pi n$ is the total increase of the argument of $f(z)$ when z once describes any circle $|z|=r$, $1<r<m$. Here the integer n has been assumed to be positive, as otherwise we could have considered the function $M/f(z)$ instead. Obviously, n must be finite since otherwise D would be simply-connected.

From (9) it follows that the argument of $g(z)$ returns to its initial value when z once describes any circle $|z|=r$, $1<r<m$. Now, D being contained in the annulus $1<|w|<M$, we have, for $|z|=1$,

$$|g(z)|=|f(z)| \geqslant 1,$$

and for $|z|=m$, $$|g(z)|=m^{-n}|f(z)| \leqslant Mm^{-n}.$$

(Here $f(z)$ is supposed to be regular on both boundaries of the annulus; if this is not the case, an annulus $1+\epsilon<|z|<m-\epsilon$ with sufficiently small ϵ has to be considered instead.) We now suppose for a moment that $M<m$. This would require that, for $|z|=1$, we should have $|g(z)| \geqslant 1$ and for $|z|=m$ we should have $|g(z)|<1$, i.e. the two boundaries of the domain D' on which $w=g(z)$ maps $1<|z|<m$ would be separated from each other by a circle $|w|=1-\delta$ (δ sufficiently small) which is fully covered by D' and on which lie no boundary points of D'. But this contradicts the fact that the argument of $g(z)$ returns to its initial value after one full circuit

* I owe this lemma and its proof to a conversation which I had with Dr M. Schiffer.

of z; indeed, $g^{-1}(w)$ would be defined for all points of the circle $|w| = 1-\delta$, and after w describing this circle a sufficient number of times, say n' times, $g^{-1}(w)$ would return to its initial value, i.e. there would exist a closed curve inside the annulus $1 < |z| < m$, such that after z once describing this curve the argument of $g(z)$ would increase by $2\pi n'$, which is impossible.

The case $m = M$ can only occur for $f(z) \equiv Kz, |K| = 1$. In fact, for $|z| = m$ we had $|g(z)| \leqslant Mm^{-n}$; since $|g(z)| < 1$ led to a contradiction, we must have $n = 1$. Moreover, it is clear that the only case for which $|g(z)| \leqslant 1$ will not lead to a contradiction is that in which the domain D' on which $w = g(z)$ maps $1 < |z| < m$ reduces to a single point, i.e. $w = g(z)$ is a constant. This completes the proof of the lemma.

As an application of the lemma we shall prove here the following theorem:

THEOREM IV. *Let D be a doubly-connected—not necessarily "schlicht"—domain of the modulus m; let P be a point of the outer boundary of D and Q a point of its inner boundary, such that P may be connected with Q by a "schlicht" curve; let further R be another point of the inner boundary of D such that P, Q, R lie on a straight line and that Q and R may be connected by a curve not covered by D; if we denote the distances $\overline{PQ}$ and $\overline{PR}$ by d and α respectively, we have*

$$d \geqslant \alpha \operatorname{ctg}^2 \{\tfrac{1}{2}\pi \sqrt{[\mathfrak{T}(m^{-1})]}\},$$

where $\mathfrak{T}(x)$ is the elliptic modular function. This inequality is the best possible.

Remark. The distances α and d being generally more easily accessible than the modulus m, it would perhaps be more appropriate to read this result in reversed order, so as to obtain a bound for the modulus m with the help of α and d which are presumed as known. The function

$$\operatorname{ctg}^2 \{\tfrac{1}{2}\pi \sqrt{[\mathfrak{T}(m^{-1})]}\}$$

increases steadily with m; we have accordingly

$$m \leqslant q\left(\frac{d}{\alpha}\right),$$

where the function $q(x)$ is determined by

$$x = \operatorname{ctg}^2 \{\tfrac{1}{2}\pi \sqrt{[\mathfrak{T}(q^{-1})]}\}.$$

As to the proof, we shall—as in the cases of theorems I and III—proceed by stages, first proving the theorem for a special case and then showing that the result obtained holds also in the general case. This special case will be that in which the curve connecting P and Q is the stretch $\overline{PQ}$, and the stretch $\overline{QR}$ is outside D. Without loss of generality we may assume R to coincide with $w = 0$ and both P and Q to lie on the positive axis. We may further assume that R coincides with $w = 1$ (this involving a trivial trans-

formation). This domain D leaves out the stretch $0 \leqslant w \leqslant c$ ($c = \alpha/(\alpha+d)$ in our previous notation) and covers simply the stretch $c < w < 1$. $w = c$ belongs to the inner boundary of D and $w = 1$ to its outer boundary.

We are now going to construct a surface $\bar{D}$ majorant (in the extended sense defined above) to D. To this end we recall the properties of the surface, used in the proof of theorem I, on which the function $w = \sin^2\left\{\frac{\pi\sqrt{z}}{1+z}\right\}$ maps the circle $|z| < 1$. This surface covers the w-plane an infinity of times, apart from the stretch $0 \leqslant w < 1$ which is covered exactly once and the points $w = 1$ and $w = \infty$ which are not covered at all. If we now cut this surface along the stretch $0 < w < c$, we obtain a doubly-connected surface, say $\bar{D}$, which will have the required majorant properties.

For, let $w = f(z)$ now be the analytic function mapping the annulus $1 < |z| < m$ on D and $w = \mathfrak{F}(z)$ the function mapping $1 < |z| < M$ on $\bar{D}$. If we then consider the function $\omega(z)$ defined by

$$\omega(z) = \mathfrak{F}^{-1}[f(z)],$$

it is clear that $\omega(z)$ is regular and uniform for all values $1 < |z| < m$, $w = f(z)$ taking the values $c < w < 1$ only once and leaving out the values $0 \leqslant w \leqslant c$, $w = 1$ and $w = \infty$. Moreover, the stretch $0 \leqslant w \leqslant c$ connects two points of the inner boundary of D and is not itself covered by D; it follows that the total change of argument of $f(z) - \eta$, where η is a point of $0 < \eta < c$ and z completes a full circuit of the inner boundary of D, cannot reduce to zero. We have therefore, in virtue of the lemma,

$$m \leqslant M,$$

m being the modulus of D and M that of $\bar{D}$.

We now find an analytical expression for $\mathfrak{F}(z)$, or rather for the function $\mathfrak{F}_1(z) = \mathfrak{F}(mz)$ which is more convenient to handle. As in similar previous cases, it will be simpler to deal with the symmetrical function $\mathfrak{G}(z)$ defined by

$$w = \mathfrak{G}(z) = \sqrt{[\mathfrak{F}_1(z^2)]}.$$

This function maps the annulus $M^{-\frac{1}{2}} < |z| < 1$ on a surface covering the w-plane an infinity of times, apart from the stretches $-1 < w < -\sqrt{c}$ and $\sqrt{c} < w < 1$ which are simply covered and the stretch $-\sqrt{c} < w < \sqrt{c}$ and the points $w = \pm 1, \infty$ which are not covered at all. Now this surface is obtained from the surface on which

$$w = \sin\left\{\frac{\pi z}{1+z^2}\right\}$$

maps $|z| < 1$ by applying a cut along the stretch $-\sqrt{c} < w < \sqrt{c}$. $w = \sin\left\{\frac{\pi z}{1+z^2}\right\}$ being real for real z, this corresponds to a cut in the z-plane from $z = -l$ to

$z = l$, where l is defined by $\sqrt{c} = \sin\left\{\frac{\pi l}{1+l^2}\right\}$. The function $w = \mathfrak{G}(z)$ will therefore be of the form

$$\mathfrak{G}(z) = \sin\left\{\frac{\pi p(z)}{1+p^2(z)}\right\},$$

where $u = p(z)$ is the function mapping the annulus $\rho < |z| < 1$ ($\rho = M^{-\frac{1}{4}}$) on the interior of the circle $|u| < 1$, cut along the stretch $-l < u < l$. Our task is therefore reduced to finding an expression for the function $u = p(z)$. Although $u = p(z)$ can be readily expressed in terms of Jacobian elliptic functions by making use of their known conformal mapping properties, it is perhaps more in the spirit of the theory of conformal representation to construct the analytical expression of $u = p(z)$ direct from its mapping properties.

$u = p(z)$ maps the annulus $\rho < |z| < 1$ on the "schlicht" circle $|u| < 1$ cut along the stretch $-l < u < l$. Both domains being symmetrical with regard to the real axis and the origin, we may assume $p(z)$ to be real for real z and have then $p(\rho) = l$, $p(-\rho) = -l$, $p(1) = 1$, $p(-1) = -1$. Since $u = p(z)$ transforms $|z| = 1$ into $|u| = 1$, we have, by Schwarz's symmetry principle, $p(\bar{z}^{-1})\overline{p(z)} = 1$ or (in view of $p(\bar{z}) = \overline{p(z)}$)

$$p(z)\,p(z^{-1}) = 1. \tag{10}$$

As $u = p(z)$ further transforms the circle $|z| = \rho$ into parts of the real axis we have, again by the symmetry principle, $p(\rho^2\bar{z}^{-1}) = \overline{p(z)}$, which, with $p(\bar{z}) = \overline{p(z)}$, gives

$$p(z) = p(\rho^2 z^{-1}). \tag{11}$$

By suitably combining (10) and (11) we obtain the functional equation

$$p(z) = p(\rho^4 z), \tag{12}$$

which makes it clear that $p(e^z)$ is an elliptic function possessing the fundamental periods $2\pi i$ and $4\log\rho$.

From the symmetry of the function $u = p(z)$ it follows that $p(z)$ vanishes for $z = \pm i\rho$, and consequently, in view of (12), also at all points $z = \pm i\rho^{4n+1}$ ($n = \pm 1, \pm 2, \ldots$). The mapping being conformal at these points, all these zeros are simple. As is easily seen, there are no other zeros of $u = p(z)$. In virtue of (10), all poles of $u = p(z)$ are situated at the points $z = \pm i\rho^{4n-1}$ ($n = 0, \pm 1, \pm 2, \ldots$), all of them being simple. We then consider the infinite product

$$\phi(z) = z\frac{\prod_0^\infty (1+\rho^{8n+2}z^{-2})\prod_1^\infty (1+\rho^{8n-2}z^2)}{\prod_1^\infty (1+\rho^{8n-2}z^{-2})\prod_0^\infty (1+\rho^{8n+2}z^2)} = \operatorname{sn}\left\{i\log\frac{z}{\rho};\rho^4\right\}, \tag{13}$$

which converges for all values of z other than $0, \infty$ and the obvious poles. The zeros and poles of $\phi(z)$ are identical with those of $p(z)$. Moreover, we have obviously

$$\phi(\rho^4 z) = \phi(z),$$

so that $\phi(z)$(or rather $\phi(e^z)$) is an elliptic function possessing the same zeros and poles as $p(z)$. By Liouville's theorem, $\phi(z)$ and $p(z)$ are therefore identical, apart from a multiplicative constant which, however, as shown by (10), can only have the values ± 1. Hence we have

$$u = p(z) = \phi(z) = \operatorname{sn}\{i \log z/\rho; \rho^4\},$$

sn being the Jacobian elliptic function. What we are mainly interested in is the length of the stretch $-l < u < l$. Remembering that $p(\rho) = l$, we obtain, in view of (13),

$$l = l(\rho) = p(\rho) = 2\rho \prod_1^\infty \left(\frac{1+\rho^{8n}}{1+\rho^{8n-4}}\right)^2$$
$$= \sqrt[4]{[\mathfrak{T}(\rho^4)]},$$

where $\mathfrak{T}(x)$ denotes the elliptic modular function.

Now l was connected with the quantity c ($c = \alpha/(\alpha+d)$) by the equation

$$\sqrt{c} = \sin\left\{\frac{\pi l}{1+l^2}\right\}, \quad l = l(\rho).$$

$\mathfrak{T}(x)$ satisfying a functional equation which—in terms of l—may be written

$$l^2(\rho) = \frac{2l(\rho^2)}{1+l^2(\rho^2)},$$

we are led to $$c = \sin^2\{\tfrac{1}{2}\pi l^2(\sqrt{\rho})\} = \sin^2\{\tfrac{1}{2}\pi\sqrt{[\mathfrak{T}(\rho^2)]}\}.$$

Remembering that $c = \alpha/(\alpha+d)$ and $\rho = M^{-\frac{1}{4}}$, we obtain finally

$$\frac{\alpha}{\alpha+d} = \sin^2\{\tfrac{1}{2}\pi\sqrt{[\mathfrak{T}(M^{-1}]}\},$$

so that $$d = \alpha \operatorname{ctg}^2\{\tfrac{1}{2}\pi\sqrt{[\mathfrak{T}(M^{-1}])}\}. \tag{14}$$

Since the right-hand side of (14) increases steadily with M, we have—in view of $m \leqslant M$—

$$d \geqslant \alpha \operatorname{ctg}^2\{\tfrac{1}{2}\pi\sqrt{[\mathfrak{T}(m^{-1})]}\}.$$

This completes the proof for the case in which both the simply-covered curve connecting P with Q and the non-covered curve connecting Q with R were linear segments.

As for the case of general curves, this can again be reduced to the special case discussed by considering the function

$$x = q(z) = \frac{2}{\pi}\arcsin\sqrt{[f(z^2)]}.$$

Exactly as in the proof of theorem I it can be shown that the surface H on which $x = q(z)$ maps the annulus $1 < |z| < m$ leaves out a curve connecting $x = 0$ with both points $x = \pm\sin^{-1}\sqrt{c}$ (this curve being inside the inner boundary of H) and two curves connecting both $x = 1$ and $x = -1$ with $x = \infty$ (these curves being outside the outer boundary of H). $x = q(z)$ being an odd function of z, the two latter curves, on the one hand, and the two parts of the first curve connecting $x = 0$ with the points $x = \pm\sin^{-1}\sqrt{c}$ on the other, may be assumed symmetrical with regard to $x = 0$. By the lemma the modulus of H cannot be greater than that of the "schlicht" domain H', symmetrical with regard to $x = 0$, which covers the entire x-plane apart from these curves. By a theorem due to Groetzsch,* the modulus of H' is smaller or equal to that of the special "schlicht" domain which consists of the whole x-plane cut along the stretch

$$-\sin^{-1}\sqrt{c} \leqslant x \leqslant \sin^{-1}\sqrt{c}$$

and the rays $-\infty \leqslant x \leqslant -1$ and $1 \leqslant x \leqslant \infty$. But this special domain leads us exactly to our function which solved the extremum problem in the special case discussed. Consequently, $\mathfrak{F}(z)$ gives also the solution in the general case.

The Hebrew University
Jerusalem

* Groetzsch, *Ber. sächs. Ges. (Akad.) Wiss.* 28 (1928), 497.

ON THE FOUR-COLOUR CONJECTURE

By W. T. TUTTE

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1. *Introduction*

The maps discussed in this paper are dissections of surfaces into simple polygons, called *regions*. In each map it is supposed that the regions are finite in number and that each vertex of a region is common to just three regions. Sides and vertices of the regions will be called *edges* and *vertices* of the map respectively.

A *colouring* of a map M is defined as a set of four mutually exclusive subclasses, called *colour-classes*, of the regions of M such that each region belongs to some colour-class and no two regions of the same colour-class have any edge in common. If Z is a colouring whose colour-classes are C_1, C_2, C_3 and C_4, we write

$$Z = (C_1, C_2, C_3, C_4).$$

The union $A \cup B$ of two colour-classes A and B of a colouring Z we call a *colour-dyad*. The regions of a colour-dyad may or may not form a connected set. In any case we call the disjoint internally connected components *Kempe chains* and denote their number by $c_0(A \cup B)$.

If U is a Kempe chain of $A \cup B$ and V is the remainder of $A \cup B$, then it is clear that the four sets

$$((A \cap U) \cup (B \cap V)), \quad ((B \cap U) \cup (A \cap V)), \quad C, \quad D,$$

where $Z = (A, B, C, D)$ and $A \cap U$, for example, denotes the intersection of A and U, constitute a colouring Z_1 of the map concerned and that Z_1 differs from Z if and only if

$$c_0(A \cup B) > 1.$$

We say that Z_1 is derived from Z by an *exchange operation* on the Kempe chain U. The set of all colourings of the map that can be derived from Z by a finite sequence of exchange operations we call the *colour-system* containing Z and denote it by $\Pi(Z)$. Clearly, if Y is any colouring in $\Pi(Z)$, then

$$\Pi(Y) = \Pi(Z).$$

The problem with which this paper is concerned is as follows.

PROBLEM. Let M be a map on the sphere, and let M contain a pentagon P. Let any colouring Z of that part M_P of M which is exterior to P be said

to be of type I if it has a colour-class A which contains no region of M_P adjacent to P, and of type II otherwise.

The problem is to find M_P and Z such that all the colourings of $\Pi(Z)$ are of type II.

A demonstration that this problem is insoluble would complete the verification of the four-colour conjecture by enabling us to deduce from the existence of a colouring of M_P the existence of a colouring of type I of M_P and thence (by assigning P to A) the existence of a colouring of M.†

The contribution of this paper to the problem is the deduction of some new limitations on the structure of $\Pi(Z)$ which must be satisfied in any solution that may exist.

Use is made of an elementary general theorem on the colourings of spherical maps, to which I have not seen any reference in the literature. This is the "Parity Theorem" proved below.

Reference may be made to a paper by Kittel‡ on the above problem. In comparing this paper with his it should be borne in mind that he counts as distinct colourings which differ only by a redistribution of "colours" among the colour-classes—a distinction which has no meaning with the definitions used here.

2. *The parity theorem*

Let $Z = (A, B, C, D)$ be a colouring of the spherical map M of α_2 regions. Let $\alpha_2(X)$ and $\alpha_2(X \cup Y)$ denote the number of regions in the colour-class X and the colour-dyad $(X \cup Y)$ respectively, and let $\beta_1(X \cup Y)$ be the number of edges in which regions of X meet regions of Y. Then, if $c_1(X \cup Y)$ is the connectivity of the set of regions $X \cup Y$, we have by elementary topology§

$$c_0(A \cup B) - c_1(A \cup B) = \alpha_2(A \cup B) - \beta_1(A \cup B) \tag{1}$$

and

$$c_1(A \cup B) = c_0(C \cup D) - 1. \tag{2}$$

From these equations and the corresponding ones for the colour-dyads $A \cup C$ and $A \cup D$ it follows that

$$\begin{aligned} &c_0(A \cup B) + c_0(A \cup C) + c_0(A \cup D) - c_0(C \cup D) - c_0(B \cup D) - c_0(B \cup C) \\ &\quad = 3\alpha_2(A) + \alpha_2(B) + \alpha_2(C) + \alpha_2(D) - \beta_1(A \cup B) - \beta_1(A \cup C) - \beta_1(A \cup D) - 3 \\ &\quad = 2\alpha_2(A) - \sum_{R \in A} f(R) + \alpha_2 - 3, \end{aligned} \tag{3}$$

where $f(R)$ denotes the number of sides of the polygon R.

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† Heawood, *Quart. J. of Math.* 24 (1890), 332–338.

‡ Kittel, *Bull. American Math. Soc.* 41 (1935), 407–413.

§ See, for example, Newman, *Topology of Plane Sets* (C.U.P. 1939), 194–199. The regions must be dissected into triangles before his results are applied, but this introduces no real difficulty.

Now the right-hand side of (3) depends only on the colour-class A. Hence we have

THEOREM I. *The quantity on the left of equation* (3) *has the same value for all colourings of M for which A is a colour-class.*

We say that two colourings Z_1 and Z_2 of a map are *connected* if there exists a finite sequence of colourings of the map beginning with Z_1 and ending with Z_2 and such that each pair of consecutive members have a colour-class in common. Clearly any two colourings which belong to the same colour-system are connected. (But the converse does not follow.)

For any colouring Z, let $J(Z)$ denote the sum of the quantities $c_0(X \cup Y)$ over the six colour-dyads. We call the parity of $J(Z)$ the *parity of Z*.

THEOREM II (the parity theorem). *If Z_1 and Z_2 are connected colourings of a spherical map M, then*

$$J(Z_1) \equiv J(Z_2), \quad (\text{mod } 2). \tag{4}$$

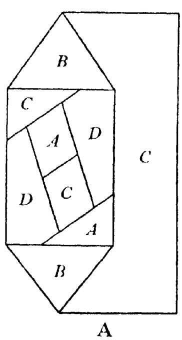
For by (3), for any colouring of M

$$J(Z) \equiv \sum_{R \in A} f(R) + \alpha_2 + 1, \quad (\text{mod } 2),$$

where A is any colour-class of Z.

Hence (4) is true whenever Z_1 and Z_2 have a colour-class in common, and therefore it is true whenever Z_1 and Z_2 are connected.

It may be worth mentioning that the colourings of a particular spherical map need not be restricted to one parity. For example the two colourings shown in Fig. 1 have opposite parities.



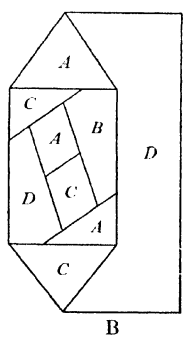


Fig. 1

3. *The pentagon problem*

Let us now return to the problem of the partially coloured map stated in the introduction.

Let F denote the set of regions of M which meet P. We denote these regions by F_1, F_2, F_3, F_4, F_5 in the cyclic order of incidence with the edges of P. We may suppose that they are taken in clockwise order as seen from the centre of the sphere. We have not yet assumed that they are all distinct.

In a colouring of type II each colour-class contains a region of F. Three colour-classes therefore each meet P in one edge only, and the fourth meets it in just two edges, E_1 and E_2 say. E_1 and E_2 do not meet, for if they did the third edge incident with their common vertex would be common to two regions of the fourth colour-class. There is therefore an edge E_3 of the pentagon adjacent to both E_1 and E_2. We call the set of regions of F which have E_1 or E_2 as an edge the *norm* of the colouring, and the region of F having E_3 as an edge the *apex* of the colouring.

The following well-known result will be needed.†

THEOREM III. *If Z is a colouring of M_P such that all the members of $\Pi(Z)$ are of type II, then if any Kempe chain contains the norm of Z it contains also the apex of Z.*

For let $Z = (A, B, C, D)$ where A contains the norm and B the apex of Z. Then if the theorem is false for Z one of the remaining two regions of F, $\epsilon\, C$ say, must be separated in M_P from the apex by a Kempe chain of $A \cup D$ containing the norm. Hence, by an exchange operation on that Kempe chain of $B \cup C$ which contains the apex, a type I colouring of $\Pi(Z)$ can be obtained, contrary to hypothesis.

It follows from theorem III that if all the colourings of $\Pi(Z)$ are of type II, then the five regions F_i are all distinct. We therefore assume their distinctness in what follows.

If F_i is the apex of Z, then the norm of Z is the pair of regions F_{i+1}, F_{i+4}. (The addition in the suffices is modulo 5.) We then denote the Kempe chain containing F_{i+1} and F_{i+2} by $g(Z)$ and the colour-dyad to which it belongs by $G(Z)$. We denote the Kempe chain of $G(Z)$ which contains the other member F_{i+4} of the norm by $h(Z)$. It follows from theorem III that $g(Z)$ and $h(Z)$ are distinct.

We define a λ-*operation* on Z as the application of the exchange operation to each member of a subset Λ of the Kempe chains of $G(Z)$, where Λ contains $g(Z)$ but not $h(Z)$.

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† Heawood, loc. cit.

(*Note.* A similar operation on $G(Z)$ affecting $h(Z)$ but not $g(Z)$ would be equivalent to the λ-operation affecting just those components of $G(Z)$ which the former operation left unchanged.)

We define a λ-*circuit* in $\Pi(Z)$ as a cyclic sequence of colourings belonging to $\Pi(Z)$ such that each member of the sequence can be derived from its immediate predecessor by a single λ-operation. λ-operations and λ-circuits can of course be defined in the same way when $\Pi(Z)$ contains type I colourings but they are of particular interest in the other case, for then it has been shown that every colouring in $\Pi(Z)$ is a member of a λ-circuit.†

If a λ-operation is applied to a colouring Z of apex F_i the norm of the new colouring will be the pair F_{i+2}, F_{i+4}, and its apex will therefore be F_{i+3}. The colour-classes of the old and new apices must be common to both colourings, since neither belongs to $G(Z)$. Each λ-operation thus advances the apex three places in the cyclic sequence of the F_i. Hence if the number of members of any λ-circuit is n, then

$$n \equiv 0, \quad (\text{mod } 5). \tag{5}$$

The object of this paper is to improve upon this result by establishing the following two theorems:

THEOREM IV. $\quad n \equiv 0, \quad (\text{mod } 10).$

THEOREM V. $\quad n \neq 10.$

4. *Proof of theorem IV*

Let λZ be the colouring obtained from Z by application of the λ-operation λ. The intersections of the colour-classes with F will be as shown in the diagrams (i) and (ii) of Fig. 2 for Z and λZ respectively. The colour-classes B and D are common to both colourings.

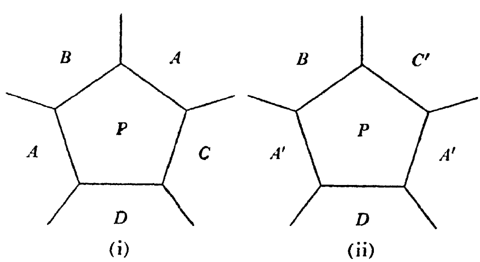

Fig. 2

† Errera, "Du Coloriage des Cartes et de Quelques Questions d'Analysis Situs", Thesis (Gauthier-Villars, 1921). See also Kittel, loc. cit.

Since the regions of F are distinct, a new spherical map N can be formed from M by incorporating P in that region of F which belongs to B in the colourings Z and λZ. If the new region is assigned to B, the colourings Z and λZ will be transformed into colourings Z_N and $(\lambda Z)_N$ respectively of N. These two colourings of N are connected, for they have the colour-class D in common. Hence by the parity theorem

$$J(Z_N) \equiv J((\lambda Z)_N), \quad (\text{mod } 2). \tag{6}$$

Now in the change from Z to Z_N the only two of the six quantities $c_0(X \cup Y)$ not obviously unaltered are $c_0(B \cup C)$ and $c_0(B \cup D)$. But for one of these, let us say $c_0(B \cup C)$, to be altered it is necessary for the two regions in which the corresponding colour-dyad of Z meets F to be in different Kempe chains of that colour-dyad. This colour-dyad $(B \cup C)$ of Z would then contain the apex and would not separate the members of the norm in M_P. The members of the norm would therefore belong to the same component of $(A \cup D)$ in Z, which contradicts theorem III, since the apex of Z is in B and therefore not in $(A \cup D)$. Hence

$$J(Z_N) = J(Z). \tag{7}$$

Again in the change from λZ to $(\lambda Z)_N$ the only quantities $c_0(X \cup Y)$ not obviously unaffected are $c_0(B \cup D)$ and $c_0(B \cup A')$. But the first of these is unchanged, by the last paragraph, since $B \cup D$ is the same in Z as in λZ and in Z_N as in $(\lambda Z)_N$. However, by theorem III the two members of the norm in λZ are in different Kempe chains of $(B \cup A')$ (which does not contain the apex). $c_0(B \cup A')$ therefore decreases by 1 and so

$$J((\lambda Z)_N) = J(\lambda Z) - 1. \tag{8}$$

From (6), (7) and (8) we have

$$J(\lambda Z) \equiv J(Z) + 1, \quad (\text{mod } 2). \tag{9}$$

It follows that the number of members of any λ-circuit must be even. Hence by (5),

$$n \equiv 0, \quad (\text{mod } 10). \tag{10}$$

5. *Proof of theorem V*

Assume that a λ-circuit of 10 members exists. Let its members be in order $Z_0, Z_1, Z_2, \ldots, Z_9$.

Let A_i denote that colour-class of Z_i which contains the apex. By the paragraph immediately preceding equation (5), A_{i-1}, A_i and A_{i+1} are all colour-classes of Z_i. Moreover they are distinct, for the apices of Z_{i-1}, Z_i and Z_{i+1} are distinct regions F_j, F_{j+3} and F_{j+1}. (Addition in the suffices of the A_i and the Z_i is mod 10.)

The ten colourings Z_i are therefore completely determined when the ten colour-classes A_i are given, for when three colour-classes of a colouring are given the fourth is uniquely determined.

If R is any region of M_P we write

$$v_i = v_i(R) = 1 \quad \text{if} \quad R \text{ is in } A_i,$$
$$= 0 \quad \text{if} \quad R \text{ is not in } A_i. \tag{11}$$

Then the set of 10-vectors

$$V(R) = (v_0(R), v_1(R), \ldots, v_9(R))$$

given for all R completely determines the A_i and therefore the Z_i.

As an example we note that F_j is in A_i if and only if it is the apex of Z_i and that this happens just once in any five consecutive members of the λ-circuit. So we may write

$$V(F_1) = (1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0), \tag{12}$$

a vector which we denote hereafter by e.

We then have, by the properties of λ-operations,

$$V(F_1) = Qe, \quad V(F_2) = Q^2 e,$$

and in general,
$$V(F_{1+r}) = Q^{2r} e, \tag{13}$$

where Q is a cyclic permutation defined by

$$Q(v_0, v_1, \ldots, v_9) = (v_9, v_0, v_1, \ldots, v_8). \tag{14}$$

We denote the group of cyclic permutations Q^i by $\mathfrak{Q}$, and say that two 10-vectors are *equivalent* when they can be transformed into one another by operations of $\mathfrak{Q}$.

A 10-vector whose components are restricted to the values 0 and 1 we call a *V-vector*.

We say that a V-vector V is *admissible* if it satisfies the following conditions: (i) no three consecutive signs of V (regarded as a cyclic sequence) include more than one 1; and (ii) there exist three other V-vectors satisfying (i) such that their sum with V is $I = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$.

The other three V-vectors of (ii) are clearly admissible. We call a set of four admissible V-vectors whose sum is I a *tetrad*.

We have at once

LEMMA I. *If V_1 and V_2 are equivalent V-vectors and V_1 is admissible, then V_2 is admissible.*

For the properties (i) and (ii) are invariant under the transformation Q.

LEMMA II. *If R is any region of M_P, then $V(R)$ is an admissible V-vector.*

For first, by (11), $V(R)$ is a V-vector.

Secondly, suppose two of $v_{i-1}(R)$, $v_i(R)$, and $v_{i+1}(R)$ have the value 1. Then, by (11), R belongs to two of the colour-classes A_{i-1}, A_i, A_{i+1} which is impossible since these are distinct colour-classes of the same colouring Z_i. Hence $V(R)$ satisfies (i).

Thirdly, any region R_0 of M_P has some vertex not a vertex of P. For if this were false for F_j then F_{j-1} and F_{j+1} would not be distinct. At this vertex R_0 meets two other regions, R_1 and R_2 say, of M_P. Consider the matrix whose three rows are the vectors $V(R_0)$, $V(R_1)$, $V(R_2)$.

No column of this matrix contains more than one 1, since no two of R_0, R_1 and R_2 belong to the same colour-class in any colouring of M_P. Hence $V' = I - V(R_0) - V(R_1) - V(R_2)$ is a V-vector.

Consider the $(i-1)$th, ith and $(i+1)$th columns (addition mod 10). If two of them consist entirely of 0's, then Z_i has a colour-dyad containing none of the regions R_0, R_1 and R_2 by (11). But this is impossible for it requires that two of these mutually contiguous regions shall belong to the same colour-class of Z_i. It follows that the V-vector V' satisfies (i) and so by the previous result that $V(R)$ is a V-vector satisfying (i), and by the definition of V', it follows that $V(R_0)$ satisfies (ii). This proves the lemma.

COROLLARY. *If three regions R_0, R_1 and R_2 of M_P meet at a vertex, then the four vectors $V(R_0)$, $V(R_1)$, $V(R_2)$, $I - V(R_0) - V(R_1) - V(R_2)$ are admissible V-vectors and constitute a tetrad.*

If V is a V-vector, we denote by $\sigma(V)$ the number of its 1's. By considering in turn the cases $\sigma(V) = 0$, $\sigma(V) = 1$, etc., we find that every V-vector satisfying (i) is equivalent to a member of the following set.

$$\left.\begin{aligned} a &= (0\,0\,0\,0\,0\,0\,0\,0\,0\,0), \\ b &= (1\,0\,0\,0\,0\,0\,0\,0\,0\,0), \\ c &= (1\,0\,0\,1\,0\,0\,0\,0\,0\,0), \\ d &= (1\,0\,0\,0\,1\,0\,0\,0\,0\,0), \\ e &= (1\,0\,0\,0\,0\,1\,0\,0\,0\,0), \\ f &= (1\,0\,0\,1\,0\,0\,1\,0\,0\,0). \end{aligned}\right\} \tag{15}$$

Clearly no two members of this set are equivalent.

If x is one of the vectors satisfying (i) we define $m(x)$ as the smallest integer m not 0 such that $Q^m x = x$. We then have

$$\begin{aligned} m(x) &= 1 \text{ if } x \text{ is equivalent to } a, \\ &= 5 \text{ if } x \text{ is equivalent to } e, \\ &= 10 \text{ otherwise.} \end{aligned} \tag{16}$$

The next step in the proof is the determination of all the tetrads. We note that if the vectors V_i $(i = 1, 2, 3, 4)$ constitute a tetrad, then

$$\sum_i \sigma(V_i) = 10 \tag{17}$$

by the definition of a tetrad; and for each $\sigma(V_i)$ (by (15))

$$0 \leqslant \sigma(V_i) \leqslant 3. \tag{18}$$

The only sets of four integers satisfying (17) and (18) are

$$(3, 3, 3, 1) \quad \text{and} \quad (3, 3, 2, 2). \tag{19}$$

The V-vector a therefore is not admissible.

Now the only vector x of (15) for which $\sigma(x) = 1$ is b and the only one for which $\sigma(x) = 3$ is f. Hence by (19) any tetrad involving b is of the form

$$T = (b, Q^i f, Q^j f, Q^k f).$$

Now f contains just one block of three consecutive 0's. Consider the matrix whose rows are the vectors of T. The three columns which contain the corresponding block in $Q^i f$ must contain three 1's altogether (definition of a tetrad) and this is possible only if they contain the non-zero component of b (by (i)) which implies that $i = 1$, 2, or 3. The same argument applies with i replaced by j or k. Hence if b is contained in a tetrad, that tetrad is

$$(b, Qf, Q^2 f, Q^3 f),$$

and it is easily verified that these four vectors do indeed constitute a tetrad.

If we apply an operation Q^i to each of the four V-vectors of any tetrad T we shall clearly obtain a new tetrad. We denote this by $Q^i T$ and say that it is *equivalent* to T. In listing the tetrads it will suffice to give one member of each set of equivalent tetrads. We may therefore proceed to those tetrads which contain no vector equivalent to b. By (15) and (19) such a tetrad involves just two vectors equivalent to f, and is therefore equivalent to a tetrad involving a pair f, $Q^j f$. We can further suppose j less than 6, for the operation Q^{10-j} transforms the above pair into the pair f, $Q^{10-j} f$ (by (16)).

By comparing the first of the six V-vectors f, Qf, $Q^2 f$, $Q^3 f$, $Q^4 f$, $Q^5 f$ with the other five, we find that the cases $j = 3$ and $j = 4$ are impossible since for each of these there is an s such that

$$v_s(f) = v_s(Q^j f) = 1,$$

contrary to the definition of a tetrad, but that the other cases cannot be ruled out in this way. We may suppose therefore that $j = 1$, 2 or 5.

Now
$$\begin{aligned} I - f - Qf &= (0\,0\,1\,0\,0\,1\,0\,0\,1\,1) \\ &= Q^5 d + Q^8 d \\ &= Q^5 c + Q^9 c, \end{aligned}$$

and these are the only two ways in which the vector (0 0 1 0 0 1 0 0 1 1) can be expressed as the sum of two vectors equivalent to members of the set (15) and satisfying $\sigma(x) = 2$.

Again $$I-f-Q^2f = (0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1),$$

which can be expressed subject to the above conditions only as

$$Q^7d + Q^4e.$$

Finally $$I-f-Q^5f = (0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1),$$

which must be expressed either as

$$Q^2e + Q^4e \quad \text{or as} \quad Q^4c + Q^9c.$$

Every tetrad therefore is equivalent to a member of the following set:

$$\left.\begin{aligned}
T_1 &= (b,\ Qf,\ Q^2f,\ Q^3f),\\
T_2 &= (f,\ Qf,\ Q^5d,\ Q^8d),\\
T_3 &= (f,\ Qf,\ Q^5c,\ Q^9c),\\
T_4 &= (f,\ Q^2f,\ Q^7d,\ Q^4e),\\
T_5 &= (f,\ Q^5f,\ Q^2e,\ Q^4e),\\
T_6 &= (f,\ Q^5f,\ Q^4c,\ Q^9c).
\end{aligned}\right\} \tag{20}$$

The significance of the set W of all admissible V-vectors can best be understood in terms of the dual map M^* of M. The regions of M^* are all triangles and so M^* is a simplicial dissection of the sphere. If R is any region of M, we denote the corresponding vertex of M^* by R^*. The dual map M_P^* of M_P may be defined as the set of all simplexes of M^* which do not have P^* as a vertex. It is therefore a simplicial dissection of a part of the sphere bounded by a simple closed curve F^* which consists of the vertices $F_1^*, F_2^*, \ldots, F_5^*$ and the edges $F_1^* F_2^*, F_2^* F_3^*, \ldots, F_5^* F_1^*$ dual to the five distinct edges of M_P which meet the pentagon P. (If there were not five such distinct edges of M_P, some two vertices of the pentagon would be joined by an edge E in M_P and then at least one of the two regions of M_P incident with E would be incident with two of the edges of P. This would contradict the requirement that the F_i must be distinct.)

It follows that the formal sum

$$\sum_{i=1}^{5} (F_i^*, F_{i+1}^*) = K \text{ say,} \tag{21}$$

where (F_i^*, F_{i+1}^*) is a 1-simplex of F^* with an orientation given by the order of the terms F_i^* and F_{i+1}^*, is a bounding 1-cycle on M_P^*.

We can treat W as a simplicial 3-complex of which the V-vectors are the 0-simplexes and the tetrads the 3-simplexes (and in which each i-simplex is incident with an $(i+1)$-simplex for $i<3$). The correspondence $R^* \to V(R)$ defines a mapping of M_P^* into W which maps vertices on to vertices, and, by the corollary to lemma II, 2-simplexes on to 2-simplexes.

It follows at once that if the correspondence maps the 2-chain U of M_P^* on to the 2-chain U_W of W, where U and U_W have ordinary integers as coefficients, then it maps the boundary of U on to the boundary of U_W. Consequently bounding cycles on M_P^* are mapped on to bounding cycles on W.

Now the bounding 1-cycle K on M_P^* maps on to a 1-cycle

$$K_W = \sum_{i=1}^{5} (V(F_i),\, V(F_{i+1})) \quad \text{by (21)}$$

$$= ((e, Q^2e) + (Q^2e, Q^4e) + (Q^4e, Qe) + (Qe, Q^3e) + (Q^3e, e)) \quad \text{by (13) and (16).}$$

But it can be shown that this is a non-bounding cycle of W. By proving this we shall show that the hypothesis of the existence of a λ-circuit of 10 members leads to a contradiction and so establish theorem V. To do this we first define a function $\Delta(V_1, V_2)$ for each 1-simplex (V_1, V_2) of W by means of the equations

$$\Delta(V_1, V_2) = -\Delta(V_2, V_1), \tag{22}$$

$$\Delta(Q^iV_1, Q^iV_2) = \Delta(V_1, V_2) \tag{23}$$

and the following table.

TABLE I

Reference number	V_1	V_2	$\Delta(V_1, V_2)$	Reference number	V_1	V_2	$\Delta(V_1, V_2)$
1	b	Qf	-1	12	d	Q^2f	-2
2	b	Q^2f	0	13	d	Q^3f	-1
3	b	Q^3f	1	14	d	Q^5f	1
4	c	Q^4c	-1	15	d	Q^6f	2
5	c	Q^5c	0	16	e	Q^2e	2
6	c	Qf	0	17	e	Qf	-1
7	c	Q^2f	1	18	e	Q^3f	1
8	c	Q^5f	-1	19	f	Qf	1
9	c	Q^6f	0	20	f	Q^2f	2
10	d	Q^3d	3	21	f	Q^5f	0
11	d	Q^2e	0				

It may readily be verified (with the help of (16)) that no two of the pairs of this table, even when regarded as unordered, are equivalent under the operations of $\mathfrak{Q}$. The definition of $\Delta(V_1, V_2)$ is therefore consistent.

Other assertions verifiable from the table with the help of equations (16), (22) and (23) are: (i) $\Delta(V_1, V_2)$ is defined for every 1-simplex (V_1, V_2), that is for every ordered pair of V-vectors both contained in the same tetrad; and (ii) if V_1, V_2 and V_3 are distinct members of the same tetrad, then

$$\Delta(V_1, V_2) + \Delta(V_2, V_3) + \Delta(V_3, V_1) = 0. \tag{24}$$

It is sufficient to verify these two assertions for each of the tetrads of (20) by (23). This verification is set out in tabular form in Table II. The numbers in the last column of this table are the references, in order, to the rows of Table I.

Equation (24) states that the sum of the function $\Delta(V_1, V_2)$ over the boundary of any 2-simplex of W is 0. It follows from this, and (22) that the sum of the function over any bounding 1-cycle of W is 0. But its sum over the 1-cycle K_W is, by (23) and the preceding expression for K_W,

$$5\Delta(e, Q^2 e) = 10 \quad \text{(by Table I)}.$$

Thus K_W is shown to be non-bounding and the proof of theorem V is complete.

TABLE II

Tetrad	V_1	V_2	V_3	$\Delta(V_1, V_2)$	$\Delta(V_2, V_3)$	$\Delta(V_3, V_1)$	Sum	References
T_1	b	Qf	Q^2f	-1	1	0	0	1, 19, 2
	b	Q^2f	Q^3f	0	1	-1	0	2, 19, 3
	b	Qf	Q^3f	-1	2	-1	0	1, 20, 3
	Qf	Q^2f	Q^3f	1	1	-2	0	19, 19, 20
T_2	f	Qf	Q^5d	1	-2	1	0	19, 15, 14
	f	Qf	Q^8d	1	1	-2	0	19, 13, 12
	f	Q^5d	Q^8d	-1	3	-2	0	14, 10, 12
	Qf	Q^5d	Q^8d	-2	3	-1	0	15, 10, 13
T_3	f	Qf	Q^5c	1	0	-1	0	19, 9, 8
	f	Qf	Q^9c	1	-1	0	0	19, 7, 6
	f	Q^5c	Q^9c	1	-1	0	0	8, 4, 6
	Qf	Q^5c	Q^9c	0	-1	1	0	9, 4, 7
T_4	f	Q^2f	Q^7d	2	-1	-1	0	20, 14, 13
	f	Q^2f	Q^4e	2	-1	-1	0	20, 18, 17
	f	Q^7d	Q^4e	1	0	-1	0	13, 11, 17
	Q^2f	Q^7d	Q^4e	-1	0	1	0	14, 11, 18
T_5	f	Q^5f	Q^2e	0	-1	1	0	21, 18, 18
	f	Q^5f	Q^4e	0	1	-1	0	21, 17, 17
	f	Q^2e	Q^4e	-1	2	-1	0	18, 16, 17
	Q^5f	Q^2e	Q^4e	-1	2	-1	0	18, 16, 17
T_6	f	Q^5f	Q^4c	0	0	0	0	21, 6, 9
	f	Q^5f	Q^9c	0	0	0	0	21, 9, 6
	f	Q^4c	Q^9c	0	0	0	0	9, 5, 6
	Q^5f	Q^4c	Q^9c	0	0	0	0	6, 5, 9

The simple method used in the proof of theorem V will not suffice to prove the analogous theorem for $n = 20$. For Errera† has given a map M_P and a colouring Z such that a certain sequence of 20 λ-operations transforms Z into itself. Perhaps it is significant that this map contains a region (the central one in Kittel's diagram) whose 20-vector V satisfies $Q^4(V) = V$, where Q and V are defined by equations analogous to (14) and (11) respectively. Even in this case $\Pi(Z)$ contains colourings of type I.

Trinity College
Cambridge

† Errera, loc. cit. and Kittel, loc. cit.

COMBINANT FORMS ASSOCIATED WITH A PENCIL OF CONICS

By J. A. TODD

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Introduction

The purpose of this paper is to determine the complete irreducible system of concomitants of the double form $A_x^2 a_y$, in which x is a ternary variable and y a binary variable. If (x_0, x_1, x_2) are interpreted as coordinates in a plane, then the equation $A_x^2 a_y = 0$ represents a pencil of conics, and the forms of the complete system of $A_x^2 a_y$ represent combinants of the pencil and form a complete system of such combinants.†

While many examples of combinants associated with a pencil of conics are well known, I am not aware of any previous systematic discussion of them. It will appear that the complete system to be considered consists of 65 forms. The details of the reduction processes involved are somewhat lengthy, and in order to keep the paper within reasonable limits of size many of these have been left for verification to the reader. On account of the difficulty of manipulating complicated symbolic expressions, the reductions have been accomplished by making use of a simple canonical form; and their verification is a matter of elementary algebra.

Our method of procedure is as follows. We first construct an auxiliary system $\{M\}$ of concomitants of the form $A_x^2 a_y$ which has the property that any concomitant of $A_x^2 a_y$ is a form belonging to the simultaneous system of the forms of $\{M\}$, regarded as binary forms in y. It is proved that such a system exists with eleven members, one being cubic in y, two being quadratic, three being linear, and the remaining five being independent of y. The second step is the construction of the complete system of the forms of $\{M\}$, regarded as *independent* binary forms in y. A system of 221 forms is thus obtained. Next, the forms of this system are reduced to a set of 65 irreducible forms, the reductions being effected, in part, by making use of syzygies derived from the canonical form. Finally, we add some remarks

† The term combinant is commonly applied only to such forms as do not involve the parametric variables y. But the forms involving y are also of interest, and many of them have simple geometrical interpretations.

of a geometrical character, interpreting some of the simpler members of the system.

I. *Determination of the auxiliary system* $\{M\}$

1. Let
$$S \equiv S_y \equiv A_x^2 a_y \equiv B_x^2 b_y \equiv \ldots \equiv y_0 S_0 + y_1 S_1 \tag{1}$$
be the form under consideration. We write
$$S_0 \equiv A_x'^2 \equiv B_x'^2 \equiv \ldots, \quad S_1 \equiv A_x''^2 \equiv B_x''^2 \equiv \ldots, \tag{2}$$
and represent by α', α'' bracket factors of the form $(B'C')$, $(B''C'')$. Then the complete simultaneous system of the two ternary quadratics S_0, S_1 consists† of 21 irreducible forms, which we may take to be

$$\left.\begin{array}{l}
A_{\alpha'}'^2,\ A_{\alpha'}''^2,\ A_{\alpha''}'^2,\ A_{\alpha''}''^2;\quad A_x'^2,\ A_x''^2,\ (\alpha'\alpha''x)^2,\ (\alpha'\alpha''x)\,A'_{\alpha''}A''_{\alpha'}A'_xA''_x;\\
u_{\alpha'}^2,\ u_{\alpha''}^2,\ (A'A''u)^2,\quad (A'A''u)\,A'_{\alpha''}A''_{\alpha'}u_{\alpha'}u_{\alpha''};\quad u_x,\ A''_{\alpha'}A''_xu_{\alpha'},\ A'_{\alpha''}A'_xu_{\alpha''};\\
(A'A''u)\,A'_xA''_x,\ (\alpha'\alpha''x)\,A'_{\alpha''}A'_xu_{\alpha'},\ (\alpha'\alpha''x)\,A''_{\alpha'}A''_xu_{\alpha''};\\
(\alpha'\alpha''x)\,u_{\alpha'}u_{\alpha''},\ (A'A''u)\,A''_{\alpha'}A'_xu_{\alpha'},\ (A'A''u)\,A'_{\alpha''}A''_xu_{\alpha''}.
\end{array}\right\} \tag{3}$$

2. We construct a system of forms belonging to the complete system of $A_x^2a_y$ with the property that these forms (when independent of y) or the coefficients of the various power products of y_0 and y_1 in these forms, are equivalent to the set (3) in the sense that either system is expressible rationally and integrally in terms of the other. We shall then prove that this system is the system $\{M\}$ desired.

We write down the set of forms to be considered, and then calculate their coefficients and show that these form a set equivalent to (3). The numerical factors introduced are purely a matter of convenience, and are designed with a view to avoiding large numerical coefficients in subsequent work.

Consider the forms (eleven in number)

$$\left.\begin{array}{l}
u \equiv u_x,\\
S \equiv A_x^2a_y \equiv y_0S_0 + y_1S_1,\\
\Sigma \equiv (ABu)^2a_yb_y \equiv y_0^2\Sigma_0 + 2y_0y_1\Phi + y_1^2\Sigma_1,\\
\gamma \equiv \tfrac{1}{2}(ABu)\,(ab)\,A_xB_x,\\
\Delta \equiv \tfrac{1}{6}(ABC)^2a_yb_yc_y \equiv y_0^3\Delta_0 + 3y_0^2y_1\Theta_0 + 3y_0y_1^2\Theta_1 + y_1^3\Delta_1,\\
Q \equiv \tfrac{1}{3}(ABC)\,(ABu)\,(ac)\,C_xb_y \equiv y_0Q_0 + y_1Q_1,\\
\Lambda \equiv (ABC)\,(ABu)\,(CDu)\,(cd)\,D_xa_yb_y \equiv y_0^2\Lambda_0 + 2y_0y_1\Psi + y_1^2\Lambda_1,\\
w \equiv -\tfrac{1}{6}(ABC)\,(ABD)\,(ac)\,(bd)\,C_xD_x,\\
L \equiv \tfrac{1}{6}(ABC)\,(ABD)\,(CEu)\,(ac)\,(bd)\,D_xE_xe_y \equiv y_0L_0 + y_1L_1,\\
j \equiv \tfrac{1}{36}(ABC)\,(ABD)\,(CEF)\,(ac)\,(bd)\,(ef)\,D_xE_xF_x,\\
\zeta \equiv \tfrac{1}{36}(ABC)\,(ABD)\,(CEu)\,(DFu)\,(EFu)\,(ac)\,(bd)\,(ef).
\end{array}\right\} \tag{4}$$

† Grace and Young, *Algebra of Invariants* (Cambridge, 1903), 286.

We have, immediately,

$$\left.\begin{array}{l} S_0 = A_x'^2, \quad S_1 = A_x''^2, \quad \Sigma_0 = (A'B'u)^2 = u_{\alpha'}^2, \quad \Sigma_1 = u_{\alpha''}^2, \quad \Phi = (A'A''u)^2, \\ \Delta_0 = \tfrac{1}{6}(A'B'C')^2 = \tfrac{1}{6}A_{\alpha'}'^2, \quad \Delta_1 = \tfrac{1}{6}A_{\alpha''}''^2, \quad \Theta_0 = \tfrac{1}{6}(A'B'C'')^2 = \tfrac{1}{6}A_{\alpha'}''^2, \quad \Theta_1 = \tfrac{1}{6}A_{\alpha''}'^2. \end{array}\right\} \tag{5}$$

Next,
$$\gamma = \tfrac{1}{2}(ABu)(ab)A_x B_x$$
$$= \tfrac{1}{2}[(A'A''u)A_x'A_x'' - (A''A'u)A_x'A_x''] = (A'A''u)A_x'A_x''. \tag{6}$$

And
$$3Q_0 = (A'B'A'')(A'B'u)A_x'' - (A''B'A')(A''B'u)A_x'$$
$$= A_{\alpha'}''u_{\alpha'}A_x'' - \tfrac{1}{2}(A''B'A')[(A''B'u)A_x' - (A''A'u)B_x']$$
$$= A_{\alpha'}''u_{\alpha'}A_x'' - \tfrac{1}{2}(A''B'A')[(A''B'A')u_x - (B'A'u)A_x'']$$
$$= \tfrac{3}{2}A_{\alpha'}''u_{\alpha'}A_x'' - \tfrac{1}{2}A_{\alpha'}''^2 u_x.$$

Proceeding in the same way for Q_1 we see that

$$Q_0 = \tfrac{1}{2}A_{\alpha'}''u_{\alpha'}A_x'' - \tfrac{1}{6}A_{\alpha'}''^2 u_x, \quad Q_1 = -\tfrac{1}{2}A_{\alpha''}'u_{\alpha''}A_x' + \tfrac{1}{6}A_{\alpha''}'^2 u_x, \tag{7}$$

so that Q_0 and Q_1 may replace the forms $A_{\alpha'}''u_{\alpha'}A_x''$, $A_{\alpha''}'u_{\alpha''}A_x'$ in (3).

Again,

$$\Lambda_0 = (A'B'u)[(A'B'C')(C'D''u)D_x'' - (A'B'C'')(C''D'u)D_x'].$$

The first term vanishes, on permuting A', B', C' cyclically and adding, and the second term is $(D'C''u)C_{\alpha'}''u_{\alpha'}D_x' = (A'A''u)A_{\alpha'}''u_{\alpha'}A_x'$. So, treating Λ_1 similarly, we find

$$\Lambda_0 = (A'A''u)A_{\alpha'}''u_{\alpha'}A_x', \quad \Lambda_1 = -(A'A''u)A_{\alpha''}'u_{\alpha''}A_x''. \tag{8}$$

And $\quad \Psi = (A'B''u)[(A'B''C')(C'D''u)D_x'' - (A'B''C'')(C''D'u)D_x'].$

The first term of this expression, on interchanging the equivalent pairs of symbols $A'C'$, $B''D''$, is

$$\tfrac{1}{2}(A'B''u)(C'D''u)[(A'B''C')D_x'' - (A'D''C')B_x'']$$
$$= -\tfrac{1}{2}(A'D''u)(C'B''u)[(A'B''C')D_x'' - (A'D''C')B_x'']$$
$$= \tfrac{1}{4}[(A'B''u)(C'D''u) - (A'D''u)(C'B''u)][(A'B''C')D_x'' - (A'D''C')B_x'']$$
$$= \tfrac{1}{4}(A'C'u)(B''D''u)[(A'B''C')D_x'' - (A'D''C')B_x'']$$
$$= \tfrac{1}{4}u_{\alpha'}(B''D''u)[-B_{\alpha'}''D_x'' + D_{\alpha'}''B_x''] = \tfrac{1}{4}(\alpha'\alpha''x)u_{\alpha'}u_{\alpha''},$$

and, in like manner, the second term is equal to the same expression. Hence

$$\Psi = \tfrac{1}{2}(\alpha'\alpha''x)u_{\alpha'}u_{\alpha''}. \tag{9}$$

Next,

$$-6w = (ABC)(ABD)(ac)(bd)C_x D_x$$
$$= (A'B'C'')(A'B'D'')C_x''D_x'' - 2(A''B'C')(A''B'D'')C_x'D_x''$$
$$+ (A''B''C')(A''B''D')C_x'D_x'$$
$$= T_1 - 2T_2 + T_3 \text{ (say)}$$

(the second term being the sum of two terms which are equivalent).

Here

$$T_1 = C''_{\alpha'} D''_{\alpha'} C''_x D''_x = \tfrac{1}{2}[C''^2_{\alpha'} D''^2_x + C''^2_x D''^2_{\alpha'} - (C''_{\alpha'} D''_x - C''_x D''_{\alpha'})^2]$$
$$= \tfrac{1}{2}[2C''^2_{\alpha'} D''^2_x - (\alpha'\alpha''x)^2],$$

and similarly $$T_3 = \tfrac{1}{2}[2A'^2_{\alpha''} B'^2_x - (\alpha'\alpha''x)^2],$$

while, writing α' for $(B'C')$ and α'' for $(D''A'')$,

$$T_2 = A''_{\alpha'} D''_x \,.\, B'_{\alpha''} C'_x = \tfrac{1}{4}(A''_{\alpha'} D''_x - D''_{\alpha'} A''_x)(B'_{\alpha''} C'_x - C'_{\alpha''} B'_x)$$
$$= \tfrac{1}{4}(\overline{A''D''}\,\alpha' x)(\overline{B'C'}\,\alpha'' x) = \tfrac{1}{4}(\alpha'\alpha''x)^2,$$

so that $$-6w = A'^2_{\alpha''} S_0 + A''^2_{\alpha'} S_1 - \tfrac{3}{2}(\alpha'\alpha''x)^2, \tag{10}$$

and $(\alpha'\alpha''x)^2$ may be replaced by w as an equivalent form in (3).

Next, $-L$ is the first transvectant of w and S with respect to x. Hence, denoting such a transvectant by $(w, S)^1_x$, we have

$$6L_0 = (A'^2_{\alpha''} S_0 + A''^2_{\alpha'} S_1 - \tfrac{3}{2}(\alpha'\alpha''x)^2, S_0)^1_x$$
$$= -A''^2_{\alpha'}\gamma - \tfrac{3}{2}(\overline{\alpha'\alpha''}\, B'u)(\alpha'\alpha''x) B'_x$$
$$= -A''^2_{\alpha'}\gamma - \tfrac{3}{2}(B'_{\alpha'} u_{\alpha''} - B'_{\alpha''} u_{\alpha'})(\alpha'\alpha''x) B'_x.$$

The second term is

$$-\tfrac{3}{2}(B'C'D')(\overline{C'D'}\,\alpha''x) B'_x u_{\alpha''} = -\tfrac{3}{2}(B'C'D')(C'_{\alpha''} D'_x - C'_x D'_{\alpha''}) B'_x u_{\alpha''}$$

and vanishes, as we see by permuting B', C', D' cyclically and adding. Thus, treating L_1 in the same way, we have

$$6L_0 = \tfrac{3}{2}(\alpha'\alpha''x) A'_{\alpha''} A'_x u_{\alpha'} - A''^2_{\alpha'}\gamma, \quad 6L_1 = -\tfrac{3}{2}(\alpha'\alpha''x) A''_{\alpha'} A''_x u_{\alpha''} + A'^2_{\alpha''}\gamma, \tag{11}$$

so that L_0 and L_1 may be taken as two of the irreducible members of the system (3) in place of $(\alpha\alpha'x) A'_{\alpha''} A'_x u_{\alpha'}$ and the similar form.

Lastly, by expressing j and ζ as transvectants of L and S (with respect to y as well as other variables) it is easy to verify that

$$j = \tfrac{1}{12}(\alpha'\alpha''x) A'_{\alpha''} A''_{\alpha'} A'_x A''_x, \quad \zeta = \tfrac{1}{12}(A'A''u) A'_{\alpha''} A''_{\alpha'} u_{\alpha'} u_{\alpha''}. \tag{12}$$

3. It follows from what has just been shown, that instead of the set (3) we may take the irreducible system of the two ternary quadratics S_0 and S_1 to be the 21 forms

$$\left.\begin{matrix} \Delta_0,\ \Theta_0,\ \Theta_1,\ \Delta_1,\ S_0,\ S_1,\ w,\ j,\ \Sigma_0,\ \Sigma_1,\ \Phi, \\ \zeta,\ u,\ Q_0,\ Q_1,\ \gamma,\ L_0,\ L_1,\ \Lambda_0,\ \Lambda_1,\ \Psi, \end{matrix}\right\} \tag{13}$$

which are defined in (4), and that these forms are the coefficients of power products of y_0, y_1 in the eleven forms belonging to the complete system of S defined above.

Suppose, now, that f is any form belonging to the complete system of S. Since f is covariant with respect to linear transformation of the variables x, the coefficients of the various power products of y_0 and y_1 in f (or the form f itself if these variables are absent) must be covariants—in the extended sense which includes concomitants of any type—of the ternary quadratics S_0 and S_1 and hence, being rational integral functions of the coefficients in S_0 and S_1, must be expressible rationally and integrally in terms of the set of forms (13). Since, further, f is covariant with respect to linear transformation of the variables y, its expression in terms of y_0, y_1 and the forms (13) is unaltered (save for multiplication by a power of the determinant of the transformation) when y_0, y_1 are replaced by independent linear functions y_0', y_1' of them and $S_0, S_1, \ldots$ are replaced by the corresponding $S_0', S_1', \ldots$. Thus, by definition of a simultaneous covariant, f is a simultaneous covariant of the system $u, S, \Sigma, \gamma, \Delta, Q, \Lambda, w, L, j, \zeta$ regarded as a set of binary forms in y. If then we construct the complete system of this set $\{M\}$ of binary forms we shall obtain the complete system of S by picking out from the simultaneous system those forms which are irreducible when the syzygies between the forms (13) are taken into account.

II. *Solution of the binary form problem*

4. Regarded as binary forms in y the forms of $\{M\}$ fall into four groups.

(i) Δ, a cubic in y,
(ii) Σ, Λ, quadratics in y,
(iii) S, Q, L, linear in y,
(iv) u, γ, w, j, ζ which do not involve y.

In forming the complete simultaneous system of $\{M\}$ with respect to y we may ignore the last five forms, provided they are included finally in the complete system. We have thus to consider the complete simultaneous system of one cubic, two quadratic, and three linear forms, and in the present section we shall ignore any special relations between these forms which hold in virtue of syzygies connecting the forms (13).

While the complete system we obtain contains a large number of forms, its actual determination is quite simple, particularly as we have no need to establish its actual irreducibility, since the whole system has in any case to be reconsidered in view of the special relations connecting the forms.

It will be convenient to denote by q_r a product of degree r each of whose factors is one of the quadratics Σ, Λ, and similarly by l_r a product of degree r whose factors are S, Q, L.

5. We first form the simultaneous system of Δ, Σ, Λ. The complete system $\{A_1\}$ of Δ consists of the four forms

$$\Delta, \quad H = (\Delta, \Delta)^2, \quad T = (H, \Delta)^1, \quad D = (H, H)^2, \tag{14}$$

and the simultaneous system $\{B_1\}$ of Σ and Λ of six forms

$$\Sigma, \quad \Lambda, \quad \Gamma = (\Sigma, \Lambda)^1, \quad (\Sigma, \Sigma)^2, \quad (\Sigma, \Lambda)^2, \quad (\Lambda, \Lambda)^2. \tag{15}$$

The last three forms, and the form D of $\{A_1\}$, do not involve y.

The complete system of Δ, Σ, Λ is that derived by transvection from $\{A_1\}$ and $\{B_1\}$. Considerations of the orders of the forms involved and the fact that T and Γ are jacobians show† that the only transvectants which need be considered are

$$\left.\begin{array}{l}(H, q_1)^1, \quad (H, q_1)^2, \quad (H, \Gamma)^2, \\ (\Delta, q_1)^1, \quad (\Delta, q_1)^2, \quad (\Delta, q_2)^3, \quad (\Delta, \Gamma)^2, \quad (\Delta, q_1\Gamma)^3, \quad (\Delta^2, q_3)^6, \quad (\Delta^2, q_2\Gamma)^6, \\ (T, q_1)^2, \quad (T, q_2)^3, \quad (\Delta T, q_3)^6, \\ (T, \Gamma)^2, \quad (T, q_1\Gamma)^3, \quad (\Delta T, q_2\Gamma)^6.\end{array}\right\} \tag{16}$$

Of this set the ones in the last line, involving both T and Γ, are all reducible. For, representing terms in a transvectant by their bracket factors,

$$(T, \Gamma)^2 = ((H\Delta)\, H_y \Delta_y^2, (\Sigma\Lambda)\, \Sigma_y \Lambda_y)^2$$

contains the term

$$\begin{aligned}(H\Delta)(H\Sigma)(\Delta\Lambda)(\Sigma\Lambda) &= (H\Delta)(\Sigma\Lambda).(H\Sigma)(\Delta\Lambda) \\ &= \tfrac{1}{2}[(H\Delta)^2(\Sigma\Lambda)^2 + (H\Sigma)^2(\Delta\Lambda)^2 - (H\Lambda)^2(\Delta\Sigma)^2],\end{aligned}$$

and so is reducible. Moreover, being a linear form, its expression in terms of other forms of the system is a sum of terms each having an invariant factor.‡ Hence any symbolic product containing the factor $(T\Gamma)^2$ is reducible. Since $(T, q_1\Gamma)^3$ and $(\Delta T, q_2\Gamma)^6$ contain such terms it follows that these transvectants are also reducible.

Thus the system $\{A_2\}$ derived by transvection from $\{A_1\}$ and $\{B_1\}$ consists of 41 forms, as follows (in which q_r gives rise to $r+1$ forms):

$$\left.\begin{array}{ll}\text{4 cubics:} & \Delta, \ T, \ (\Delta, q_1)^1; \\ \text{6 quadratics:} & H, \quad \Sigma, \quad \Lambda, \quad \Gamma, \quad (H, q_1)^1; \\ \text{13 linear forms:} & (\Delta, q_1)^2, \quad (T, q_1)^2, \quad (\Delta, \Gamma)^2, \quad (\Delta, q_2)^3, \quad (T, q_2)^3, \\ & (\Delta, q_1\Gamma)^3; \\ \text{18 invariants:} & D, \ (\Sigma, \Sigma)^2, \ (\Sigma, \Lambda)^2, \ (\Lambda, \Lambda)^2, \ (H, q_1)^2, \ (H, \Gamma)^2, \\ & (\Delta^2, q_3)^6, \ (\Delta T, q_3)^6, \ (\Delta^2, q_2\Gamma)^6.\end{array}\right\} \tag{17}$$

† Cf. Grace and Young, *loc. cit. ante*, 164–165 for a discussion of the very similar problem of a cubic and one quadratic.

‡ I.e. a factor not involving y.

6. The complete system $\{B_2\}$ of the three linear forms S, Q, L consists of the six forms

$$S, \quad Q, \quad L, \quad (S,Q)^1, \quad (S,L)^1, \quad (Q,L)^1, \tag{18}$$

the last three being invariants. The complete system of $\Delta, \Sigma, \Lambda, S, Q, L$ is that derived by transvection from $\{A_2\}$ and $\{B_2\}$. Since all the forms of $\{B_2\}$ involving y are linear, the complete system consists of the forms of $\{A_2\}$ and $\{B_2\}$ together with transvectants $(f, l_r)^r$, where f is one of the 23 forms of $\{A_2\}$ of positive order, and r does not exceed this order. From each linear form in $\{A_2\}$ we thus obtain three invariants, from each quadratic three linear forms and six invariants, and from each cubic three quadratics, six linear forms and ten invariants. The complete system derived from $\{A_2\}$ and $\{B_2\}$ thus contains 4 cubics, $6+4.3 = 18$ quadratics, $(13+3)+4.6+6.3 = 58$ linear forms and $(18+3)+4.10+6.6+13.3 = 136$ invariants. Adding to these the five forms u, γ, w, j, ζ we have in all 221 forms to consider; and the complete system of S consists of such of these forms as are irreducible.

III. *The complete irreducible system of S*

7. We now approach the most complicated stage of our problem. The system of 221 forms which we have just obtained includes all the irreducible concomitants of S, but owing to the existence of syzygies between the coefficients in the forms of the set $\{M\}$ many of these 221 forms prove to be reducible.

The first step towards a systematic examination of these forms is to arrange them in ascending order of degree in the coefficients of S; the process of testing for reducibility proceeding from one degree to the next. Unfortunately the number of forms involved is so large as to make this list inconvenient to print. We shall therefore arrange the work in the following manner. First, we give a list of the 65 forms which are finally shown to constitute the irreducible system of S. Next, after establishing a simple canonical form for S and its more important concomitants, we shall show that the 156 forms not included in the list are reducible. Finally, we shall indicate how the actual irreducibility of the set of 65 forms retained may be established.

8. In the list of forms which follows, the symbol $[m, n]$ affixed to a form indicates the orders of this form in x and u. Forms of different orders in y are shown separately, and the arrangement of forms of given order in y is in order of increasing degree. The 65 irreducible forms may be taken as follows.

(i) *Forms independent of y* (27 *forms*).

Degree

0 $u\,[1,1]$;
2 $\gamma\,[2,1]$;
4 $(\Sigma,\Sigma)^2\,[0,4]$, $w\,[2,0]$, $(S,Q)^1\,[3,1]$;
6 $\zeta\,[0,3]$, $((\Delta,\Sigma)^2,S)^1\,[2,2]$, $j\,[3,0]$, $(\Sigma,SQ)^2\,[3,3]$, $(\Delta,S^3)^3\,[6,0]$;
8 $(H,\Sigma)^2\,[0,2]$, $((\Delta,\Sigma)^2,Q)^1\,[1,3]$, $(Q,L)^1\,[3,2]$, $(H,S^2)^2\,[4,0]$;
10 $(H,\Lambda)^2\,[1,2]$, $((\Delta,\Sigma^2)^3,Q)^1\,[1,5]$, $(H,SQ)^2\,[3,1]$, $(\Delta,SQ^2)^3\,[4,2]$;
12 $D\,[0,0]$, $(H,\Gamma)^2\,[1,4]$, $(H,Q^2)^2\,[2,2]$, $(H,SL)^2\,[4,1]$;
14 $((T,\Sigma)^2,Q)^1\,[1,3]$, $((H,\Sigma)^1,Q^2)^2\,[2,4]$, $(T,S^2Q)^3\,[5,1]$;
16 $((T,\Sigma)^2,L)^1\,[2,3]$;
18 $(\Delta T,\Sigma^3)^6\,[0,6]$.

(ii) *Forms linear in y* (21 *forms*)

Degree

1 $S\,[2,0]$;
3 $Q\,[1,1]$, $(\Sigma,S)^1\,[2,2]$;
5 $(\Delta,\Sigma)^2\,[0,2]$, $(\Sigma,Q)^1\,[1,3]$, $L\,[2,1]$, $(\Delta,S^2)^2\,[4,0]$;
7 $(\Delta,\Sigma^2)^3\,[0,4]$, $(\Delta,\Lambda)^2\,[1,2]$, $(H,S)^1\,[2,0]$, $(\Delta,SQ)^2\,[3,1]$;
9 $(H,Q)^1\,[1,1]$, $(\Delta,\Sigma\Lambda)^3\,[1,4]$, $(\Delta,Q^2)^2\,[2,2]$;
11 $(T,\Sigma)^2\,[0,2]$, $((H,\Sigma)^1,Q)^1\,[1,3]$, $(H,L)^1\,[2,1]$, $(T,S^2)^2\,[4,0]$;
13 $(T,\Sigma^2)^3\,[0,4]$, $(T,\Lambda)^2\,[1,2]$, $(T,SQ)^2\,[3,1]$.

(iii) *Forms quadratic in y* (13 *forms*)

Degree

2 $\Sigma\,[0,2]$;
4 $\Lambda\,[1,2]$, $(\Delta,S)^1\,[2,0]$;
6 $H\,[0,0]$, $(\Delta,Q)^1\,[1,1]$, $\Gamma\,[1,4]$;
8 $(H,\Sigma)^1\,[0,2]$, $((\Delta,\Sigma)^1,Q)^1\,[1,3]$, $(\Delta,L)^1\,[2,1]$;
10 $(H,\Lambda)^1\,[1,2]$, $(T,S)^1\,[2,0]$;
12 $(T,Q)^1\,[1,1]$;
14 $(T,L)^1\,[2,1]$.

(iv) *Forms cubic in y* (4 *forms*)

Degree

3 $\Delta\,[0,0]$;
5 $(\Delta,\Sigma)^1\,[0,2]$;
7 $(\Delta,\Lambda)^1\,[1,2]$;
9 $T\,[0,0]$.

9. Further discussion of the system is simplified by the fact that a very simple canonical form exists for which the expressions of all these concomitants are quite concise. To obtain such a form we observe, in the first place, that the coordinate system in the plane can be chosen so that the four base-points of the pencil of conics (assumed to be of general type) have coordinates $(\pm 1, \pm 1, \pm 1)$. The equation of every conic of the pencil is then of the form $a_0x_0^2+a_1x_1^2+a_2x_2^2=0$ with $a_0+a_1+a_2=0$. Next, by a

suitable linear transformation of y_0 and y_1, we can reduce the cubic Δ to a multiple of $y_0^3+y_1^3$. This amounts to selecting as the two conics S_0 and S_1 which define the pencil the (unique) pair of conics of the pencil which are mutually apolar. The equations of these conics are $x_0^2+\omega x_1^2+\omega^2 x_2^2=0$ and $x_0^2+\omega^2 x_1^2+\omega x_2^2=0$, where ω is a primitive cube root of unity. We may therefore take, as the canonical form of S, the expression

$$S\equiv a[y_0(x_0^2+\omega x_1^2+\omega^2 x_2^2)+y_1(x_0^2+\omega^2 x_1^2+\omega x_2^2)]. \tag{19}$$

The introduction of the factor a in (19) is convenient, since it affords an automatic check on the degrees of the forms involved in any syzygy, all forms of degree r having a^r as a factor.

We now introduce the following convention. Let f_0 be a polynomial in $x_0, x_1, x_2, u_0, u_1, u_2$; and let f_1 and f_2 be the expressions derived from f_0 by replacing the suffixes $(0,1,2)$ by $(1,2,0)$ and $(0,2,1)$ respectively. We shall write

$$[f_0]_r=f_0+\omega^r f_1+\omega^{2r} f_2 \quad (r=0,1,2). \tag{20}$$

It will be observed that if f_0 is unaltered by cyclic permutation of the suffixes $(0,1,2)$ then $f_0=f_1=f_2$, and $[f_0]_0=3f_0$, $[f_0]_1=[f_0]_2=0$. We shall also use the abbreviation

$$\epsilon=\omega^2-\omega, \quad \epsilon^2=-3. \tag{21}$$

The calculation of the explicit expressions for the 65 forms listed in § 8 now presents no trouble. The eleven forms of $\{M\}$ are calculated directly, and the rest follow by forming the appropriate transvectants. We give the complete list of forms for reference; in forming this list we have occasionally replaced certain of the forms of the last section by equivalent forms when these have a simpler canonical expression. The complete list of forms is as follows; the order of arrangement is that of § 8.

1. $u \quad = [u_0 x_0]_0.$
2. $\gamma \quad = \epsilon a^2[u_0 x_1 x_2]_0.$
3. $(\Sigma,\Sigma)^2 \quad = 6a^4[u_0^4-2u_1^2 u_2^2]_0.$
4. $w \quad = -a^4[x_0^2]_0.$
5. $(S,Q)^1 \quad = -3a^4[u_0 x_0^3]_0-wu.$
6. $\zeta \quad = -\epsilon a^6 u_0 u_1 u_2.$
7. $((\Delta,\Sigma)^2,S)^1 \quad = 2\epsilon a^6[u_0^2(x_1^2-x_2^2)]_0.$
8. $j \quad = \epsilon a^6 x_0 x_1 x_2.$
9. $(\Sigma,SQ)^2 \quad = 3\epsilon a^6[u_0 x_0(x_1^2-x_2^2)(u_1^2+u_2^2-u_0^2)]_0+((\Delta,\Sigma)^2,S)^1 u.$
10. $(\Delta,S^3)^3 \quad = -3\epsilon a^6(x_1^2-x_2^2)(x_2^2-x_0^2)(x_0^2-x_1^2).$
11. $(H,\Sigma)^2 \quad = 2a^8[u_0^2]_0.$
12. $((\Delta,\Sigma)^2,Q)^1 \quad = -6a^8[u_0^3 x_0]_0+(H,\Sigma)^2 u.$
13. $(Q,L)^1 \quad = 3a^8[x_0 u_1 u_2(x_1^2-x_2^2)]_0.$
14. $(H,S^2)^2 \quad = -2a^8[x_0^4-x_1^2 x_2^2]_0.$
15. $(H,\Lambda)^2 \quad = 4\epsilon a^{10}[x_0 u_1 u_2]_0.$
16. $((\Delta,\Sigma^2)^3,Q)^1 \quad = -6\epsilon a^{10}[u_0 x_0(u_1^2-u_2^2)(u_1^2+u_2^2-u_0^2)]_0.$

17. $(H, SQ)^2 = -\epsilon a^{10}[u_0x_0(x_1^2-x_2^2)]_0.$
18. $(\Delta, SQ^2)^3 = 3\epsilon a^{10}[u_0^2x_0^2(x_1^2-x_2^2)]_0+2(H, SQ)^2u.$
19. $D = -2a^{12}.$
20. $(H, \Gamma)^2 = -12a^{12}[u_1u_2x_0(u_1^2-u_2^2)]_0.$
21. $(H, Q^2)^2 = 2a^{12}[u_0^2x_0^2-u_1u_2x_1x_2]_0.$
22. $(H, SL)^2 = 3a^{12}[u_0x_1x_2(x_1^2-x_2^2)]_0.$
23. $((T, \Sigma)^2, Q)^1 = -2\epsilon a^{14}[u_0x_0(u_1^2-u_2^2)]_0.$
24. $((H, \Sigma)^1, Q^2)^2 = -6\epsilon a^{14}[u_0^2x_0^2(u_1^2-u_2^2)]_0-2((T, \Sigma)^2, Q)^1u.$
25. $(T, S^2Q)^3 = 3\epsilon a^{14}[u_0x_0^3(x_1^2-x_2^2)]_0-(H, SQ)^2w.$
26. $((T, \Sigma)^2, L)^1 = 6a^{16}[u_0x_1x_2(u_1^2-u_2^2)]_0.$
27. $(\Delta T, \Sigma^3)^6 = 24\epsilon a^{18}(u_1^2-u_2^2)(u_2^2-u_0^2)(u_0^2-u_1^2).$
28. $S = a(y_0[x_0^2]_1+y_1[x_0^2]_2).$
29. $Q = a^3(y_0[u_0x_0]_1-y_1[u_0x_0]_2).$
30. $(\Sigma, S)^1 = a^3(y_0[3u_0^2x_0^2+(u_2^2-u_0^2)x_1^2+(u_1^2-u_0^2)x_2^2]_1$
$-y_1[3u_0^2x_0^2+(u_2^2-u_0^2)x_1^2+(u_1^2-u_0^2)x_2^2]_2).$
31. $(\Delta, \Sigma)^2 = 2a^5(y_0[u_0^2]_1+y_1[u_0^2]_2).$
32. $(\Sigma, Q)^1 = -3a^5(y_0[u_1u_2(x_1u_2+x_2u_1)]_1+y_1[u_1u_2(x_1u_2+x_2u_1)]_2)-\frac{1}{2}(\Delta, \Sigma)^2u.$
33. $L = \epsilon a^5(y_0[u_0x_1x_2]_1-y_1[u_0x_1x_2]_2).$
34. $(\Delta, S^2)^2 = a^5(y_0[x_0^4+2x_1^2x_2^2]_1+y_1[x_0^4+2x_1^2x_2^2]_2).$
35. $(\Delta, \Sigma^2)^3 = -6a^7(y_0[u_0^4+u_1^2u_2^2]_1-y_1[u_0^4+u_1^2u_2^2]_2).$
36. $(\Delta, \Lambda)^2 = -2\epsilon a^7(y_0[x_0u_1u_2]_1+y_1[x_0u_1u_2]_2).$
37. $(H, S)^1 = -a^7(y_0[x_0^2]_1-y_1[x_0^2]_2).$
38. $(\Delta, SQ)^2 = -3a^7(y_0[u_0x_0^3]_1-y_1[u_0x_0^3]_2)-(H, S)^1u-Qw.$
39. $(H, Q)^1 = -a^9(y_0[u_0x_0]_1+y_1[u_0x_0]_2).$
40. $(\Delta, \Sigma\Lambda)^3 = -9a^9(y_0[x_0u_1u_2(u_1^2-u_2^2)]_1+y_1[x_0u_1u_2(u_1^2-u_2^2)]_2)-3Q\zeta.$
41. $(\Delta, Q^2)^2 = a^9(y_0[u_0^2x_0^2+2u_1u_2x_1x_2]_1+y_1[u_0^2x_0^2+2u_1u_2x_1x_2]_2).$
42. $(T, \Sigma)^2 = -2a^{11}(y_0[u_0^2]_1-y_1[u_0^2]_2).$
43. $((H, \Sigma)^1, Q)^1 = 6a^{11}(y_0[u_0^3x_0]_1-y_1[u_0^3x_0]_2)-(H, \Sigma)^2Q+(T, \Sigma)^2u.$
44. $(H, L)^1 = -\epsilon a^{11}(y_0[u_0x_1x_2]_1+y_1[u_0x_1x_2]_2).$
45. $(T, S^2)^2 = -a^{11}(y_0[x_0^4+2x_1^2x_2^2]_1-y_1[x_0^4+2x_1^2x_2^2]_2).$
46. $(T, \Sigma^2)^3 = -6a^{13}(y_0[u_0^4+u_1^2u_2^2]_1+y_1[u_0^4+u_1^2u_2^2]_2)+(\Delta, \Sigma)^2(H, \Sigma)^2.$
47. $(T, \Lambda)^2 = 2\epsilon a^{13}(y_0[x_0u_1u_2]_1-y_1[x_0u_1u_2]_2).$
48. $(T, SQ)^2 = 3a^{13}(y_0[u_0x_0^3]_1+y_1[u_0x_0^3]_2)-(H, Q)^1w+\frac{1}{2}SDu.$
49. $\Sigma = a^2(2y_0^2[u_0^2]_2-2y_0y_1[u_0^2]_0+2y_1^2[u_0^2]_1).$
50. $\Lambda = -2\epsilon a^4(y_0^2[x_0u_1u_2]_2+2y_0y_1[x_0u_1u_2]_0+y_1^2[x_0u_1u_2]_1).$
51. $(\Delta, S)^1 = a^4(y_0^2[x_0^2]_2-y_1^2[x_0^2]_1).$
52. $H = 2a^6y_0y_1.$
53. $(\Delta, Q)^1 = -a^6(y_0^2[u_0x_0]_2+y_1^2[u_0x_0]_1).$
54. $\Gamma = 6a^6(y_0^2[x_0u_1u_2(u_1^2-u_2^2)+u_0u_1u_2(x_1u_1-x_2u_2)]_2$
$+2y_0y_1[x_0u_1u_2(u_1^2-u_2^2)]_0+y_1^2[x_0u_1u_2(u_1^2-u_2^2)$
$+u_0u_1u_2(x_1u_1-x_2u_2)]_1).$
55. $(H, \Sigma)^1 = -2a^8(y_0^2[u_0^2]_2-y_1^2[u_0^2]_1).$
56. $((\Delta, \Sigma)^1, Q)^1 = -3a^8(y_0^2[u_1u_2(x_1u_2+x_2u_1)]_2-y_1^2[u_1u_2(x_1u_2+x_2u_1)]_1)$
$+\frac{1}{2}(H, \Sigma)^1u+\frac{2}{3}(\Delta, \Sigma)^2Q.$
57. $(\Delta, L)^1 = -\epsilon a^8(y_0^2[u_0x_1x_2]_2+y_1^2[u_0x_1x_2]_1).$
58. $(H, \Lambda)^1 = 2\epsilon a^{10}(y_0^2[x_0u_1u_2]_2-y_1^2[x_0u_1u_2]_1).$
59. $(T, S)^1 = -a^{10}(y_0^2[x_0^2]_2+y_1^2[x_0^2]_1).$
60. $(T, Q)^1 = a^{12}(y_0^2[u_0x_0]_2-y_1^2[u_0x_0]_1).$
61. $(T, L)^1 = \epsilon a^{14}(y_0^2[u_0x_1x_2]_2-y_1^2[u_0x_1x_2]_1).$
62. $\Delta = a^3(y_0^3+y_1^3).$
63. $(\Delta, \Sigma)^1 = -a^5(y_0^3[u_0^2]_0-2y_0^2y_1[u_0^2]_1+2y_0y_1^2[u_0^2]_2-y_1^3[u_0^2]_0).$
64. $(\Delta, \Lambda)^1 = -2\epsilon a^7(y_0^3[x_0u_1u_2]_0+y_0^2y_1[x_0u_1u_2]_1-y_0y_1^2[x_0u_1u_2]_2-y_1^3[x_0u_1u_2]_0).$
65. $T = -a^9(y_0^3-y_1^3).$

10. We shall now show that the remaining 156 forms of the system can be expressed in terms of the 65 which we have obtained explicitly in § 9. In the case of forty of these forms we do this by writing down explicitly the expression in terms of forms of the irreducible system. Many of these syzygies were somewhat difficult to discover; the procedure being to write down all possible products of forms of the irreducible system of the correct degree and orders in the variables, and to see if the form it is proposed to reduce is linearly expressible in terms of these by using the canonical form. Once obtained, however, the verification of the syzygies is elementary algebra (though in the case of the more complicated relations the calculation is somewhat tedious). It is therefore unnecessary to do more than write the syzygies down, and this is done below in equations numbered (22)–(61). We shall show then that the reducibility of the other 116 forms is a consequence of that of the forty for which the syzygies are given. The forms reduced by syzygy comprise one quadratic [equation (22)], sixteen linear forms [equations (23)–(38)], and twenty-three forms independent of y [equations (39)–(61)]. They are as follows:

$$((\Delta,\Sigma)^1,S)^1 = \tfrac{1}{6}(\Delta,\Sigma)^2 S - \tfrac{1}{2}w\Sigma - Q^2 + 2(\Delta,Q)^1 u + \tfrac{1}{2}Hu^2, \tag{22}$$

$$(\Lambda,S)^1 = -2Lu + 2Q\gamma, \tag{23}$$

$$(\Sigma,L)^1 = -\tfrac{1}{2}(\Delta.\Sigma)^2\gamma + \tfrac{3}{2}(\Delta,\Lambda)^2 u - 3\zeta S, \tag{24}$$

$$(\Lambda,Q)^1 = (\Delta,\Lambda)^2 u - 6\zeta S, \tag{25}$$

$$(\Delta,\Gamma)^2 = -\tfrac{2}{3}(\Delta,\Sigma\Lambda)^3 - 8\zeta Q, \tag{26}$$

$$((H,\Sigma)^1,S)^1 = (H,\Sigma)^2 S - (\Delta,\Sigma)^2 w + 4(H,Q)^1 u - 2(\Delta,Q^2)^2, \tag{27}$$

$$(\Lambda,L)^1 = (\Delta,\Lambda)^2\gamma, \tag{28}$$

$$(\Delta,SL)^2 = -(H,S)^1\gamma - Lw - 3jQ. \tag{29}$$

The form $(\Delta,\Sigma\Gamma)^3$ can be replaced by its term $(\Delta\Gamma)^2(\Delta\Sigma)\Sigma_y = ((\Delta,\Gamma)^2,\Sigma)^1$, and hence, using (26), by $((\Delta,\Sigma\Lambda)^3,\Sigma)^1$. And

$$((\Delta,\Sigma\Lambda)^3,\Sigma)^1 = -\tfrac{3}{4}(\Sigma,\Sigma)^2(\Delta,\Lambda)^2 - 6\zeta(\Sigma,Q)^1, \tag{30}$$

$$(\Delta,\Lambda^2)^3 = 12\zeta L. \tag{31}$$

The form $((\Delta,\Sigma)^1 Q^2)^2$ can be replaced by its term

$$(\Delta\Sigma)(\Delta Q)(\Delta Q')\Sigma_y = ((\Delta,Q^2)^2,\Sigma)^1;$$

and
$$((\Delta,Q^2)^2,\Sigma)^1 = \tfrac{1}{4}(\Sigma,\Sigma)^2(H,S)^1 - \tfrac{3}{4}(H,\Sigma)^2(\Sigma,S)^1 + (H,\Sigma)^2 Qu + 2((H,\Sigma)^1,Q)^1 u - \tfrac{1}{2}w(\Delta,\Sigma^2)^3, \tag{32}$$

$$(\Delta,QL)^2 = \tfrac{1}{2}(\Delta,\Lambda)^2 w - \tfrac{1}{4}(H,\Lambda)^2 S - (H,Q)^1\gamma. \tag{33}$$

The form $(\Delta,\Lambda\Gamma)^3$ can be replaced by its term $(\Delta\Gamma)^2(\Delta\Lambda)\Lambda_y = ((\Delta,\Gamma)^2,\Lambda)^1$, and hence, using (26), by $((\Delta,\Sigma\Lambda)^3,\Lambda)^1$. And

$$((\Delta,\Sigma\Lambda)^3,\Lambda)^1 = 3(\Delta,\Lambda)^2\zeta u - 9(\Delta,\Sigma)^2\zeta\gamma - 18\zeta^2 S, \tag{34}$$

$$(\Delta,L^2)^2 = -(H,L)^1\gamma - \tfrac{3}{2}(\Delta,\Lambda)^2 j. \tag{35}$$

The form $(T, \Sigma\Lambda)^3$ may be replaced by its term $(T\Sigma)^2 (T\Lambda) \Lambda_y = ((T, \Sigma)^2 \Lambda)^1$; and

$$((T, \Sigma)^2, \Lambda)^1 = (\Delta, \Lambda)^2 (H, \Sigma)^2 + 12(H, Q)^1 \zeta, \tag{36}$$

$$(T, Q^2)^2 = \tfrac{1}{4}(H, \Sigma)^2 (H, S)^1 - \tfrac{1}{2}(T, \Sigma)^2 w + \tfrac{1}{4} D(\Sigma, S)^1 - DQu, \tag{37}$$

$$(T, \Lambda^2)^3 = (H, \Lambda)^2 (\Delta, \Lambda)^2 - 12(H, L)^1 \zeta, \tag{38}$$

$$(\Sigma, S^2)^2 = -4(S, Q)^1 u - 2\gamma^2 + 2wu^2, \tag{39}$$

$$(\Sigma, \Lambda)^2 = 12\zeta u, \tag{40}$$

$$(S, L)^1 = -3ju - w\gamma, \tag{41}$$

$$(\Lambda, \Lambda)^2 = 24\zeta\gamma, \tag{42}$$

$$(\Sigma, Q^2)^2 = -((\Delta, \Sigma)^2, Q)^1 u - (H, \Sigma)^2 u^2 + 6\zeta\gamma, \tag{43}$$

$$((\Delta, \Sigma^2)^3, S)^1 = 6((\Delta, \Sigma)^2, Q)^1 u + 3(H, \Sigma)^2 u^2 - 12\zeta\gamma - (\Sigma, \Sigma)^2 w, \tag{44}$$

$$((\Delta, \Lambda)^2, S)^1 = 2(Q, L)^1, \tag{45}$$

$$(\Delta, S^2 Q)^3 = \tfrac{1}{2}(H, S^2)^2 u + w^2 u - (S, Q)^1 w + 3j\gamma, \tag{46}$$

$$((\Delta, \Sigma)^2, L)^1 = -2(H, \Sigma)^2 \gamma + \tfrac{3}{2}(H, \Lambda)^2 u - 6w\zeta, \tag{47}$$

$$((\Delta, \Lambda)^2, Q)^1 = -\tfrac{1}{2}(H, \Lambda)^2 u + 6w\zeta, \tag{48}$$

$$(\Delta^2, \Sigma^3)^6 = \tfrac{9}{5}(H, \Sigma)^2 (\Sigma, \Sigma)^2 - 72\zeta^2, \tag{49}$$

$$((T, \Sigma)^2, S)^1 = -2(H, Q^2)^2 - (H, \Sigma)^2 w + Du^2, \tag{50}$$

$$(\Delta, Q^3)^3 = -\tfrac{3}{2}(H, Q^2)^2 u - \tfrac{1}{2} Du^3 - 9j\zeta, \tag{51}$$

$$((\Delta, \Lambda)^2, L)^1 = -\tfrac{1}{2}(H, \Lambda)^2 \gamma - 18j\zeta, \tag{52}$$

$$(T, S^3)^3 = -\tfrac{3}{2}(H, S^2)^2 w - w^3 + 9j^2. \tag{53}$$

The form $(\Delta^2, \Sigma^2\Lambda)^6$ may be replaced by its term $(\Delta\Sigma)^2 (\Delta\Sigma') (\Delta'\Sigma') (\Delta'\Lambda)^2$ which is equal to $((\Delta, \Lambda)^2, (\Delta, \Sigma^2)^3)^1$. And

$$((\Delta, \Lambda)^2, (\Delta, \Sigma^2)^3)^1 = \tfrac{1}{2}(H, \Lambda)^2 (\Sigma, \Sigma)^2 - 12((\Delta, \Sigma)^2, Q)^1 \zeta - 6(H, \Sigma)^2 \zeta u, \tag{54}$$

$$\begin{aligned} ((T, \Sigma^2)^3, S)^1 = {} & -\tfrac{1}{2}(H, \Sigma)^2 ((\Delta, \Sigma)^2, S)^1 + 2((H, \Sigma)^1, Q^2)^2 \\ & + 4((T, \Sigma)^2, Q)^1 u, \end{aligned} \tag{55}$$

$$(H, QL)^2 = \tfrac{3}{2}(H, \Sigma)^2 j + \tfrac{1}{2} D\gamma u, \tag{56}$$

$$((T, \Lambda)^2, S)^1 = -(H, \Lambda)^2 w + 3(H, \Sigma)^2 j + 3D\gamma u, \tag{57}$$

$$\begin{aligned} ((T, \Sigma^2)^3, Q)^1 = {} & -\tfrac{1}{2}((\Delta, \Sigma)^2, Q)^1 (H, \Sigma)^2 \\ & - \tfrac{3}{4}[(H, \Sigma)^2]^2 u - \tfrac{1}{4} D(\Sigma, \Sigma)^2 u + 3(H, \Lambda)^2 \zeta, \end{aligned} \tag{58}$$

$$((T, \Lambda)^2, Q)^1 = -((T, \Sigma)^2, L)^1, \tag{59}$$

$$\begin{aligned} (H, L^2)^2 = {} & -\tfrac{1}{2}(H, S^2)^2 (H, \Sigma)^2 - (H, \Sigma)^2 w^2 - (H, Q^2)^2 w \\ & - D(S, Q)^1 u + Dwu^2, \end{aligned} \tag{60}$$

$$((T, \Lambda)^2, L)^1 = -((T, \Sigma)^2, Q)^1 w - (H, SQ)^2 (H, \Sigma)^2 + \tfrac{1}{2} D((\Delta, \Sigma)^2, S)^1 u. \tag{61}$$

11. We shall now show how all the 221 forms are accounted for. The four cubic forms in y are irreducible. Of the eighteen quadratics, thirteen are included in the irreducible system, and one is reduced by (22). The other four are $((\Delta, \Sigma)^1, L)^1$ and $((\Delta, \Lambda)^1, l)^1$, where l is S, Q or L.

Now

$$\begin{aligned}((\Delta, \Sigma)^1, L)^1 &= (\Delta\Sigma)(\Sigma L)\Delta_y^2 + \tfrac{2}{3}(\Delta\Sigma)[(\Delta L)\Sigma_y - (\Sigma L)\Delta_y]\Delta_y^2 \\ &= (\Delta, (\Sigma, L)^1)^1 + \tfrac{2}{3}(\Delta, \Sigma)^2 L.\end{aligned}$$

But, by (24), $(\Sigma, L)^1$ is expressed as a sum of products of forms, each product containing a factor independent of y. Hence $(\Delta, (\Sigma, L)^1)^1$ is expressible in terms of products of forms of lower degree or order, and is reducible. Thus $((\Delta, \Sigma)^1, L)^1$ is reducible. Similarly, since the forms $(\Lambda, l)^1$ are reducible by (23), (25), (28), the three forms $((\Delta, \Lambda)^1, l)^1$ are reducible.

Thus the five quadratics not included in the irreducible set are all accounted for.

The 58 linear forms comprise

(a) the 13 forms in (17) together with S, Q, L,

(b) 18 forms $(q', l)^1$, where q' is one of the six quadratics in (17) and l is S, Q or L,

(c) 24 forms $(c, l_2)^2$, where c is one of the four cubics and l_2 is a product of two of S, G, L.

Of the 16 forms (a), ten are included in the irreducible system. The other six are reduced by (26), (30), (31), (34), (36), (38).

Consider next the forms (b). Those in the irreducible system are

$$(\Sigma, S)^1, \quad (\Sigma, Q)^1, \quad (H, l)^1, \quad ((H, \Sigma)^1, Q)^1.$$

The transvectants $(\Sigma, L)^1$, $((H, \Sigma)^1, S)^1$ are reduced by (24) and (27). The transvectant $((H, \Sigma)^1, L)^1$ may be replaced by its term

$$(H\Sigma)(\Sigma L) H_y = (H, (\Sigma, L)^1)^1$$

and hence reduced by (24). The forms $(\Lambda, l)^1$ are reduced by (23), (25), (28). And the forms $((H, \Lambda)^1, l)^1$ may be replaced by $(H, (\Lambda, l)^1)^1$ and are accordingly reducible. Lastly, $(\Gamma, l)^1$ contains the term $(\Sigma\Lambda)(\Lambda l)\Sigma_y = (\Sigma, (\Lambda, l)^1)^1$ and is accordingly reducible. Thus all the forms (b) are accounted for.

Consider next the forms (c). Of the six forms $(\Delta, l_2)^2$ the three in which l_2 contains a factor L are reduced by (29), (33), (35). Moreover, $(T, l_2)^2$ contains the term $(H\Delta)(\Delta l_2)^2 H_y = (H, (\Delta, l_2)^2)^1$ and hence is reducible if $(\Delta, l_2)^2$ is. Thus $(T, l_2)^2$ is reducible if l_2 contains the factor L, and $(T, Q^2)^2$ is reduced by (37). Since $((\Delta, \Lambda)^1, l)^1$ is reducible, so also is $((\Delta, \Lambda)^1, l_2)^2$. Similarly, since $((\Delta, \Sigma)^1, l)^1$ is reducible if l is S or L, the only transvectant $((\Delta, \Sigma)^1, l_2)^2$ which remains for consideration is that in which $l_2 = Q^2$, and

this is reduced by (32). Thus nineteen forms (c) reduce; the other five are listed in the irreducible system.

The 58 linear forms are thus accounted for.

Consider next the remaining 141 forms, independent of y. We may arrange these in the following groups:

(a) The five forms u, γ, w, j, ζ.

(b) The 18 invariants of the set (17).

(c) The three forms $(S, Q)^1$, $(S, L)^1$, $(Q, L)^1$.

(d) 39 forms $(l', l)^1$, where l' is one of the 13 linear forms (17) and l is S, Q, L.

(e) 36 forms $(q', l_2)^2$, where q' is one of the six quadratics (17) and l_2 is a product of two forms S, Q, L.

(f) 40 forms $(c, l_3)^3$, where c is one of the four cubics (17) and l_3 is a power product of S, Q, L of degree 3.

The five forms (a) are included in our irreducible system.

Of the forms (b), the irreducible system includes D, $(\Sigma, \Sigma)^2$, $(H, \Sigma)^2$, $(H, \Lambda)^2$, $(H, \Gamma)^2$ and $(\Delta T, \Sigma^3)^6$, six forms in all. The four forms $(\Sigma, \Lambda)^2$, $(\Lambda, \Lambda)^2$, $(\Delta^2, \Sigma^3)^6$, $(\Delta^2, \Sigma^2\Lambda)^6$ are reduced by (40), (42), (49), (54). The remaining eight forms are $(\Delta^2, q_1\Lambda^2)^6$, $(\Delta^2, q_2\Gamma)^6$, $(\Delta T, q_2\Lambda)^6$, where q_i is a homogeneous product of degree i in Σ and Λ; and these contain, respectively, terms with the symbolic factors $(\Delta\Lambda)^2(\Delta\Lambda')$, $(\Delta\Gamma)^2$, and either $(T\Lambda)^2(T\Sigma)$ or $(T\Lambda)^2(T\Lambda')$. But $(\Delta, \Lambda^2)^3$, $(\Delta, \Gamma)^2$, $(T, \Sigma\Lambda)^3$, $(T, \Lambda^2)^3$ are reducible by (31), (26), (36), (38). Hence all the forms in question are reducible. The forms $(S, Q)^1$, $(Q, L)^1$ are irreducible, but $(S, L)^1$ is reduced by (41).

In (d), we need only consider the cases (seven in number) in which l' is irreducible, i.e. those for which l' is one of the set

$$(\Delta, \Sigma)^2,\ (\Delta, \Sigma^2)^3,\ (\Delta, \Lambda)^2,\ (\Delta, \Sigma\Lambda)^3,\ (T, \Sigma)^2,\ (T, \Sigma^2)^3,\ (T, \Lambda)^2. \quad (62)$$

Since $(\Lambda, l)^1$ is reducible and $(\Delta, \Sigma\Lambda)^3$ contains the term $(\Delta\Sigma)^2(\Delta\Lambda)\Lambda_y$, the three forms $((\Delta, \Sigma\Lambda)^3, l)^1$ are reducible. Similarly, $((\Delta, \Sigma^2)^3, L)^1$ and $((T, \Sigma^2)^3, L)^1$ are reducible since $(\Sigma, L)^1$ is reducible. Other forms $(l', l)^1$ with l' in the set (62) are reduced by (44), (45), (47), (48), (50), (52), (55), (57), (58), (59), (61). Thus, of the 21 forms arising from (62), 16 have been reduced. The remaining five, namely,

$$((\Delta, \Sigma)^2, S)^1, \quad ((\Delta, \Sigma)^2, Q)^1, \quad ((\Delta, \Sigma^2)^3, Q)^1, \quad ((T, \Sigma)^2, Q)^1, \quad ((T, \Sigma)^2, L)^1,$$

appear in our list of irreducible forms.

Consider next the 36 forms (e). Of the six forms $(H, l_2)^2$, $(H, QL)^2$ and $(H, SL)^2$ are reduced by (56) and (60). The other four appear in the irreducible list. Since $(\Sigma, L)^1$ is reducible we need only consider, among the forms $(\Sigma, l_2)^2$, those in which l_2 does not contain a factor L. Of these forms

$(\Sigma, S^2)^2$ and $(\Sigma, Q^2)^2$ are reduced by (39) and (43). The remaining form $(\Sigma, SQ)^2$ is in our irreducible list. Since all the forms $(q', l)^1$ with $q' = \Lambda$, Γ or $(H, \Lambda)^1$ have been reduced, all the corresponding forms $(q', l_2)^2$ are reducible. Finally, since the only irreducible transvectant $((H, \Sigma)^1, l)^1$ is that for which $l = Q$, the only form $((H, \Sigma)^1, l_2)^2$ to be considered is that for which $l_2 = Q^2$, and this in fact appears in our irreducible list. Thus all the forms (*e*) have been considered.

Lastly, consider the 40 forms (*f*). Since the only irreducible forms $(c, l_2)^2$ have been shown to be

$$(\Delta, S^2)^2, \quad (\Delta, SQ)^2, \quad (\Delta, Q^2)^2, \quad (T, S^2)^2, \quad (T, SQ)^2,$$

all the forms $(c, l_3)^3$ are reducible with the possible exception of

$$(\Delta, l_3)^3, \quad (T, S^2 l_1)^3,$$

where l_i is a product of degree i in S and Q only. Of these six forms $(\Delta, S^2Q)^3$, $(\Delta, Q^3)^3$ and $(T, S^3)^3$ are reduced by (46), (51), (53). The other three are included in the list of irreducible forms.

The above examination shows that all the 221 forms have been considered, and that accordingly the system of 65 forms listed is complete.

12. Attempts to express forms of the complete system in terms of products, of the same degree and orders in the variables, of other forms of the system having been unsuccessful, it is believed that the sixty-five forms enumerated above constitute the complete *irreducible* system. An example will illustrate the procedure. We shall prove that $(\Delta, \Sigma\Lambda)^3$ cannot be expressed in terms of the other forms of the system. (The irreducibility of this particular form was in fact doubted, since alone among the linear forms in y it is not accompanied by a correlative form in which the variables x and u are interchanged.) The form in question is of degree 9, and of orders 1, 1, 4 in y, x, u respectively. We select from the list of forms those of degree r and orders n_1, n_2, n_3, where $r \leqslant 9$, $n_1 \leqslant 1$, $n_2 \leqslant 1$, $n_3 \leqslant 4$. Denoting such a form f by the symbol $f(r;\, n_1, n_2, n_3)$ we have

$$u(0;\, 0,1,1), \quad Q(3;\, 1,1,1), \quad (\Sigma, \Sigma)^2\,(4;\, 0,0,4), \quad (\Delta, \Sigma)^2\,(5;\, 1,0,2),$$
$$(\Sigma, Q)^1\,(5;\, 1,1,3), \quad \zeta(6;\, 0,0,3), \quad (\Delta, \Sigma^2)^3\,(7;\, 1,0,4), \quad (\Delta, \Lambda)^2\,(7;\, 1,1,2),$$
$$(H, \Sigma)^2\,(8;\, 0,0,2), \quad ((\Delta, \Sigma)^2, Q)^1\,(8;\, 0,1,3), \quad (H, Q)^1\,(9;\, 1,1,1).$$

The only products of these forms which gives a form $(9;\, 1,1,4)$ is ζQ. Reference to the list of standard forms shows that $(\Delta, \Sigma\Lambda)^3$ is not a multiple of ζQ. Hence it is irreducible. The other forms have been tested in a similar way, and found to be irreducible.

IV. *Geometrical interpretation of the system*

13. We shall conclude by interpreting some of the simpler forms of the system geometrically. The geometrical significance of some of the more complex forms of the system is somewhat obscure. Moreover, the irreducibility of a form is frequently of little geometrical significance. For instance, the form $(\Delta, \Lambda)^1$ is certainly irreducible. On the other hand, the same form multiplied by the invariant D is reducible, since, as may be verified at once from the table of canonical forms,

$$D(\Delta, \Lambda)^1 = -(H, \Lambda)^2 T + (T, \Lambda)^2 H. \tag{63}$$

An investigation in detail of the geometrical meaning of all the forms involved would thus not only be lengthy but pointless. We shall content ourselves with identifying a few of the simpler members of the system with well-known geometrical loci associated with the pencil of conics; and (in particular) showing that all the forms of our system, involving a single variable only, admit of simple geometrical interpretations. The proofs of most of the statements made below are left to the reader; they arise almost immediately by considering the canonical forms.

14. The equation $S = 0$ represents, for each value of (y_0, y_1), a conic of the pencil. The tangential equation of this conic is $\Sigma = 0$. This is a binary quadratic in y_0, y_1, and the discriminant of this quadratic, $(\Sigma, \Sigma)^2$, is a quartic in (u_0, u_1, u_2) which when equated to zero is easily recognized as the tangential equation of the set of base-points of the pencil. The form Δ is the discriminant of S, and the equation $\Delta = 0$ determines the three values of $y_0 : y_1$ for which the conic becomes a line-pair. The form $(\Delta, S^3)^3$, which is a sextic in (x_0, x_1, x_2), is easily verified to be the product of the left-hand members of the equations of the three line-pairs in the pencil.

The canonical form of the hessian H of Δ is a multiple of $y_0 y_1$, so that the two conics whose parameters satisfy $H = 0$ are S_0, S_1 and are thus given by $(H, S^2)^2 = 0$. These two conics are characterized by the property that they are mutually apolar, and the four base-points of the pencil form an equianharmonic set on either of them. The equation $w = 0$ is the harmonic locus of these two conics, and is a well-known combinant of the pencil; in the canonical form under discussion the equation of w is $x_0^2 + x_1^2 + x_2^2 = 0$, so that its tangential equation is $u_0^2 + u_1^2 + u_2^2 = 0$, which is at once identified with $(H, \Sigma)^2 = 0$.

The cubic covariant T of Δ determines the parameters of three other conics of the pencil. It is easy to verify that these conics have double contact with w; the chords of contact forming the sides of the common self-polar triangle of the pencil. They also have, as is well known, the property that the base points of the pencil form a harmonic set of any one of them. The

equations of these conics are given by $(T, S^3)^3 = 0$, a form which has been seen to be reducible.

The single invariant D is the discriminant of Δ, and its vanishing expresses the condition that the base points of the pencil should not all be distinct.

The forms j and ζ are easily identified, $j = 0$ being the equation of the sides of the common self-polar triangle and $\zeta = 0$ the tangential equation of its three vertices.

The only form not already mentioned which includes only a single variable is $(\Delta^2, \Sigma^3)^6$, a sextic in u. A glance at the list of canonical forms shows that $(\Delta^2, \Sigma^3)^6 = 0$ is the tangential equation of the six points whose coordinates are permutations of $(1, -1, 0)$; and these points are the intersections of the sides of the self-polar triangle with joins of pairs of the base-points.

15. It will be noticed that in a large number of cases to each form in the system there corresponds another form whose canonical expression (apart from powers of a and numerical multipliers) is obtained from that of the given form by interchanging (x_0, x_1, x_2) and (u_0, u_1, u_2). Two such forms are evidently polar reciprocals with respect to the combinant conic w. It does not, of course, follow that the reciprocal of an irreducible form is necessarily irreducible, and there are in fact a few cases where the reciprocal form does reduce. None the less, many of the forms of the irreducible system do in fact appear in pairs which are related in this way. Thus, for instance, the form $(\Delta, \Sigma)^2 = 0$ represents a tangential pencil of conics, which is the polar reciprocal of the given pencil with respect to w. It may be verified that the point equation of the envelope $(\Delta, \Sigma)^2 = 0$ is a linear combination of $(T, S)^1$ and Hw.

It will also be observed that a number of forms appear in pairs, one of which is derived from the other by replacing (y_0, y_1) by $(y_0, -y_1)$—again ignoring factors independent of u, x and y. This corresponds to the fact that, with each conic S of the pencil is associated another conic S' given by $(H, S)^1 = 0$, such that the parameters of S, S' harmonically separate the parameters of the mutually apolar conics S_0, S_1.

16. We next give geometrical interpretations of the forms γ, Q, Λ and L. If (u_0, u_1, u_2) are the coordinates of a line l, then $\gamma = 0$ is easily recognized as the conic which is the locus of poles of l with respect to conics of the pencil (the eleven-point conic of l), or, alternatively, if (x_0, x_1, x_2) are coordinates of a point P then $\gamma = 0$ is the tangential equation of the point P' which is conjugate to P with respect to every conic of the pencil. The reciprocal form with respect to w is $(H, \Lambda)^2 = 0$, and may be interpreted similarly.

The interpretation of the form Q is a little less simple. If (x_0, x_1, x_2) are coordinates of a point P and (y_0, y_1) are parameters of a conic S of the pencil,

then $Q = 0$ is the tangential equation of a point P^* such that the point P' conjugate to P with respect to the pencil of conics defined by S_0 and S_1 is conjugate to P^* with respect to the pencil of conics defined by S and w.

The forms Λ and L may be interpreted as follows. If (u_0, u_1, u_2) are the coordinates of a line l and (y_0, y_1) are parameters of a conic S of the pencil, then $\Lambda = 0$ is the equation of the tangent to the eleven-point conic of l with respect to the pencil at the point which is the pole of l with respect to S, and $L = 0$ is the equation of the eleven-point conic of l with respect to the pencil defined by S and w.

17. We conclude by pointing out that the conditions which ensure that the pencil of ternary quadratics belongs to a given one of the known types can all be expressed in terms of the vanishing of forms of the complete system. The actual conditions can, of course, be stated in many cases in a variety of ways. The classification we adopt here is based on the following considerations.

First consider the non-singular case in which $\Delta \not\equiv 0$. Then, if $D \neq 0$ we have the general type of pencil, in which all the base points are distinct. If $D = 0$ and $H \neq 0$ then there are two distinct singular conics in the pencil. Two cases arise, according as the double root of the cubic $\Delta = 0$ represents a line-pair or a repeated line. In the former case, which is the general one, the conics have simple contact; in the latter they have double contact. In this case the form j vanishes identically (and so also does ζ). If $H \equiv 0$ the three roots of $\Delta = 0$ are coincident, and the conics have three point or four point contact according as the singular conic in the pencil is a line-pair or a repeated line. In the latter case the forms j, ζ, w all vanish. Finally, the conics of the pencil are all coincident if γ vanishes identically. Thus the non-singular types can be classified as follows.

General case:	$D \neq 0$.
Simple contact:	$D = 0,\ H \neq 0,\ j \neq 0$.
Double contact:	$j = 0,\ H \neq 0$.
Three-point contact:	$H = 0,\ j \neq 0,\ \Delta \neq 0$.
Four-point contact:	$w = 0,\ \Delta \neq 0$.
Conics coincide:	$\gamma = 0,\ \Delta \neq 0$.

For the singular pencils, in which Δ vanishes identically, it is easy to see that if the conics have a fixed double point all the forms vanish except S, Σ, γ and their transvectants with respect to y. The two cases which arise according as the pencil of line pairs in involution is general or degenerate are distinguished (as a simple calculation shows) by the fact that the form $(\Sigma, \Sigma)^2$

vanishes in the latter case and not in the former. Finally, the conics are coincident if $\gamma = 0$. Thus the following singular cases arise. For brevity we have described them by writing down typical canonical terms; the nature of the pencil is quite clear from any one of these.

$$\begin{array}{ll}
S \equiv y_0 x_0 x_1 + y_1 x_0 x_2, & \Delta = 0,\ Q \neq 0. \\
S \equiv y_0 x_0^2 + y_1 x_1^2, & \Delta = 0,\ Q = 0,\ (\Sigma, \Sigma)^2 \neq 0. \\
S \equiv y_0 x_0^2 + y_1 x_0 x_1, & \Delta = 0,\ (\Sigma, \Sigma)^2 = 0,\ \Sigma \neq 0. \\
S \equiv y_0 x_0 x_1 + y_1 x_0 x_1, & \Delta = 0,\ \gamma = 0,\ \Sigma \neq 0. \\
S = y_0 x_0^2 + y_1 x_0^2, & \Sigma = 0.
\end{array}$$

Trinity College
Cambridge

ON THE MULTIPLICATORS OF SOME CLASSES OF FOURIER TRANSFORMS

By RAOUF DOSS

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Introduction and Definitions

Consider the following classes of functions (real or complex) of the real variable x, which will be supposed to be always absolutely integrable in the sense of Lebesgue in the interval $(-\infty, +\infty)$:

{1} Absolutely integrable in the sense of Lebesgue (without any other restriction).

{2} Bounded in the interval $(-\infty, +\infty)$.

{3} Bounded in the interval $(-\infty, +\infty)$ and integrable in the sense of Riemann in every finite interval.

{4} Uniformly continuous in the interval $(-\infty, +\infty)$.

{5} Of bounded variation in the interval $(-\infty, +\infty)$ (we mean that the variation of the function in every finite interval is bounded by a constant independent of the interval).

{6} Absolutely continuous in the interval $(-\infty, +\infty)$ and of bounded variation in the same interval.

It is clear that every class of functions is contained in the preceding one, except for the class {5} which is not contained in {4} but in {3}.

DEFINITION 1. A real or complex function $g(u)$ of the real variable u defined in $(-\infty, +\infty)$ is said to belong to the class (F_k) $(k = 1, 2, 3, 4, 5, 6)$ when it is the Fourier transform of a function of the class $\{k\}$, i.e. if there exists a function $f(x) \in \{k\}$ such that

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\, e^{-iux}\, dx \quad (i = \sqrt{(-1)}).$$

DEFINITION 2. The real or complex function $\lambda(u)$ of the real variable u defined in $(-\infty, +\infty)$ will be called a multiplicator (j, k) $(1 \leqslant j,\ k \leqslant 6)$ if its product by every function of the class (F_j) is an element of the class (F_k).

We shall be concerned in this paper with the following problems:

(A) The function $g(u)$ defined in $(-\infty, +\infty)$ being given, to find necessary and sufficient conditions for $g(u)$ to belong to a given one of the six classes (F_k).

For the class (F_1) this problem has been solved by A. C. Berry.† In theorem I is given another criterion which seems to be easier to handle in applications. For the five other classes the conditions are given in theorems II, III, IV, V and VI.

(B) The function $\lambda(u)$ being given, to find necessary and sufficient conditions for $\lambda(u)$ to be a multiplicator (j, k) (j, k being fixed).

Using the characteristic properties of the functions $g(u)$ of the classes (F_k) and with the help of an existence theorem of Steinhaus and of an analogous theorem given below (theorem VII), and finally applying a theorem of Verblunsky and other known results grouped in § 1, I give the solution of this problem of multiplicators in thirty-five cases. The conditions obtained are of a very simple form. They express the fact that one or other of the three functions

$$\lambda(u), \quad \frac{i\lambda(u)}{u-i}, \quad \frac{\lambda(u)}{(u-i)^2},$$

with $i = \sqrt{(-1)}$, belongs to one of the six classes (F_k).

	{1}	{2}	{3}	{4}	{5}	{6}
{1}	$\frac{i\lambda(u)}{u-i} \in (F_5)$	$\lambda(u) \in (F_2)$	$\lambda(u) \in (F_2)$	$\lambda(u) \in (F_2)$	$\lambda(u) \in (F_5)$	$\lambda(u) \in (F_5)$
{2}	$\frac{i\lambda(u)}{u-i} \in (F_5)$	$\frac{i\lambda(u)}{u-i} \in (F_5)$	—	$\lambda(u) \in (F_1)$	$\lambda(u) \in (F_5)$	$\lambda(u) \in (F_5)$
{3}	$\frac{i\lambda(u)}{u-i} \in (F_5)$	$\frac{i\lambda(u)}{u-i} \in (F_5)$	$\frac{i\lambda(u)}{u-i} \in (F_5)$	$\lambda(u) \in (F_1)$	$\lambda(u) \in (F_5)$	$\lambda(u) \in (F_5)$
{4}	$\frac{i\lambda(u)}{u-i} \in (F_5)$	$\frac{i\lambda(u)}{u-i} \in (F_5)$	$\frac{i\lambda(u)}{u-i} \in (F_5)$	$\frac{i\lambda(u)}{u-i} \in (F_5)$	$\lambda(u) \in (F_5)$	$\lambda(u) \in (F_5)$
{5}	$\frac{i\lambda(u)}{u-i} \in (F_1)$	$\frac{i\lambda(u)}{u-i} \in (F_2)$	$\frac{i\lambda(u)}{u-i} \in (F_3)$	$\frac{i\lambda(u)}{u-i} \in (F_4)$	$\frac{i\lambda(u)}{u-i} \in (F_5)$	$\lambda(u) \in (F_1)$
{6}	$\frac{i^2\lambda(u)}{(u-i)^2} \in (F_5)$	$\frac{i\lambda(u)}{u-i} \in (F_2)$	$\frac{i\lambda(u)}{u-i} \in (F_2)$	$\frac{i\lambda(u)}{u-i} \in (F_2)$	$\frac{i\lambda(u)}{u-i} \in (F_5)$	$\frac{i\lambda(u)}{u-i} \in (F_5)$

We group them in the above table, where, to find the condition characterizing multiplicators (1, 5) for example, we must take the square corresponding to the first row and fifth column. The condition is

$$\lambda(u) \in (F_5).$$

† See also the footnote to theorem II.

Let us mention that the two analogous problems for the trigonometric series of periodic functions have been studied partially by W. H. Young,† H. Steinhaus,‡ S. Sidon,§ A. Zygmund ‖ and S. Bochner,¶ and more fully by M. Fekete†† and S. Verblunsky.‡‡ The influence of the papers of the last two authors on the methods of the present work will be apparent to the reader.

The problem of multiplicators for Fourier transforms was proposed to me by Professor Fekete, of the Hebrew University of Jerusalem. His suggestions and criticisms were invaluable in shaping my ideas. I owe to him, in particular, the statement of theorem II, which has proved exceedingly helpful. For his kind assistance I should like to express my most profound gratitude.

1. *Some known results*

We shall make use of the following known results:

1. *For every* $a > 0$ *the function* $\frac{1}{2}[1-|u|/a]$, *where*

$$\left[1-\frac{|u|}{a}\right] = \begin{cases} \left(1-\frac{|u|}{a}\right) & \text{if} \quad |u| \leqslant a, \\ 0 & \text{if} \quad |u| \geqslant a, \end{cases}$$

is the Fourier transform of the function $(1-\cos ax)/ax^2$, *absolutely integrable and uniformly continuous in the interval* $(-\infty, +\infty)$.

2. *If* $g(u) \in (F_1)$, $g(u)$ *is continuous and bounded in the interval* $(-\infty, +\infty)$.§§

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† W. H. Young, "On a condition that a trigonometrical series should have a certain form", *Proc. Royal Soc.* 88 (1913), 569–574. Also "On Fourier series and functions of bounded variation", *ibid.* 88 (1913), 561–568.

‡ H. Steinhaus, "Additive und stetige Funktionaloperationen", *Math. Zeitschrift*, 5 (1919), 186–221.

§ S. Sidon, "Reihentheoretische Sätze und ihre Anwendungen in der Theorie der Fourierschen Reihen", *Math. Zeitschrift*, 10 (1921), 121–127.

‖ A. Zygmund, "Remarque sur un théoreme de M. Fekete", *Bull. Acad. Polonaise Sci. Lett.* (1927), 343–347

¶ S. Bochner, "Über Faktorenfolgen für Fouriersche Reihen", *Acta Univ. Szeged*, 4 (1929), 125–129.

†† M. Fekete, "Über Faktorenfolgen welche die Klasse einer Fourierschen Reihe unverändert lassen", *Acta Univ. Szeged*, 1 (1923), 148–166. See also two notes in *Comptes Rendus* 190 (1930), 1486 and 193 (1931), 16.

‡‡ S. Verblunsky, "On some classes of Fourier series", *Proc. London Math. Soc.* (2), 33 (1932), 287–327.

§§ See, for example, S. Bochner, *Vorlesungen über Fouriersche Integrale* (Leipzig, 1932), 46; quoted in the sequel as Bochner.

3. *Theorem of G. H. Hardy.*† *Let $g(u)$ be the Fourier transform of the function $f(x) \in \{1\}$. Then the sequence of functions* ($n = 1, 2, \ldots$)

$$\int_{-\infty}^{+\infty} g(u)\left[1-\frac{|u|}{n}\right] e^{iux}\,du = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x-\xi)\frac{1-\cos n\xi}{n\xi^2}\,d\xi$$

converges to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ whenever this last expression has a meaning, and converges to $f(x)$ almost everywhere.

4. *Let $f(x) \in \{1\}$ and let*

$$f_\nu(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x-\xi)\frac{1-\cos \nu\xi}{\nu\xi^2}\,d\xi \quad (\nu = 1, 2, \ldots);$$

then‡

$$\lim_{\nu\to\infty}\int_{-\infty}^{+\infty} |f(x)-f_\nu(x)|\,dx = 0.$$

5. *Let $f(x) \in \{1\}$ and let its Fourier transform $g(u)$ be absolutely integrable in the interval $(-\infty, +\infty)$. Then we have almost everywhere*

$$f(x) = \int_{-\infty}^{+\infty} g(u)\, e^{iux}\,du,$$

and the equality holds everywhere if $f(x)$ is continuous.§

6. *Let $g_1(u)$ and $g_2(u)$ respectively be the Fourier transforms of $f_1(x) \in \{1\}$ and $f_2(x) \in \{1\}$. Then* ||

$$\int_{-\infty}^{+\infty} f_1(\xi)\, g_2(\xi)\,d\xi = \int_{-\infty}^{+\infty} f_2(\xi)\, g_1(\xi)\,d\xi.$$

7. *Let $g_1(u)$ and $g_2(u)$ respectively be the Fourier transforms of $f_1(x) \in \{1\}$ and $f_2(x) \in \{1\}$. Then $g_1(u)\ g_2(u)$ is the Fourier transform of the absolutely integrable function*

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} f_1(\xi) f_2(x-\xi)\,d\xi = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f_1(x-\xi) f_2(\xi)\,d\xi.\P$$

8. *Theorem of Berry.*†† *A necessary and sufficient condition that $\lambda(u) \in (F_1)$ is that*

(1) *$\lambda(u)$ be bounded and measurable in $(-\infty, +\infty)$,*

(2) *there exists a constant C such that*

$$\left|\int_{-\infty}^{+\infty} \lambda(u) f(u)\,du\right| \leqslant C \sup_{-\infty<u<\infty} |g(u)|$$

for every $f(x) \in \{1\}$ whose Fourier transform is $g(u)$,

† See, for example, Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937), 29.

‡ Titchmarsh, *loc. cit.* p. 35.

§ Bochner, *loc. cit.* p. 51.

|| Titchmarsh, *loc. cit.* p. 54.

¶ Bochner, *loc. cit.* p. 46.

†† A. C. Berry, "Necessary and sufficient conditions in the theory of Fourier transforms", *Annals of Math.* 32 (1932), 830–838.

(3) *to every* $\epsilon > 0$ *we can make correspond a* $\delta > 0$ *such that*

$$\left| \int_{-\infty}^{+\infty} \lambda(u) f(u)\, du \right| \leqslant \epsilon \sup |g(u)|$$

for every function $f(x) \in \{1\}$ *whose Fourier transform* $g(u)$ *is summable in* $(-\infty, +\infty)$ *and verifies the condition*

$$\int_{-\infty}^{+\infty} |g(u)|\, du \leqslant \delta \sup |g(u)|.$$

9. *The function* $i/(u-i)$ *is the Fourier transform of the function* $k(x)$, *absolutely integrable and of bounded variation in* $(-\infty, +\infty)$, *where*

$$k(x) = \begin{cases} -2\pi e^{-x} & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0.\dagger \end{cases}$$

10. *Theorem of Bochner.*‡ *The relation* $g(u) \in (F_6)$ *implies the two relations* $g(u) \in (F_1)$ *and* $iug(u) \in (F_1)$. *Conversely, the last two relations imply the first one.*

2. *On the conditions that a function* $g(u)$ *should be a Fourier transform*

THEOREM I. *A necessary and sufficient condition that* $\lambda(u) \in (F_1)$ *is that* $\lambda(u)[1 - |u|/n]$ *be continuous for every positive integer* n *and that the functions*

$$\sigma_n(x) = \int_{-\infty}^{+\infty} \lambda(u) \left[1 - \frac{|u|}{n}\right] e^{iux}\, du$$

satisfy the two conditions:

(1) *There exists a constant* A *such that*

$$\int_{-\infty}^{+\infty} |\sigma_n(x)|\, dx \leqslant A \quad \textit{for every } n.$$

(2) *To every* $\epsilon > 0$ *we can make correspond a* $\delta > 0$ *such that for every set* E *whose measure is less than* δ

$$\int_E |\sigma_n(x)|\, dx \leqslant \epsilon \quad \textit{for every } n.$$

Proof. Necessity. Let $l(x) \in \{1\}$ be the generatrix of $\lambda(u)$. Since $\lambda(u)$ is continuous (result 2), the function $\lambda(u)[1 - |u|/n]$ is also continuous.

† Bochner, *loc. cit.* p. 89.

‡ Bochner, *loc. cit.* pp. 92–93. We verify at once that the statement of the text is equivalent to the statement given by Bochner in his book if we observe that a necessary and sufficient condition that a function $\phi(x)$ be an indefinite integral of some function of the class $\{1\}$ is that $\phi(x)$ be absolutely continuous and be of bounded variation over $(-\infty, +\infty)$.

We have, on the other hand (result 3),

$$\sigma_n(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} l(x-\xi)\,\frac{1-\cos n\xi}{n\xi^2}\,d\xi.$$

Therefore

$$\int_{-\infty}^{+\infty} |\,\sigma_n(x)\,|\,dx = \frac{1}{\pi}\int_{-\infty}^{+\infty} dx \left|\int_{-\infty}^{+\infty} l(x-\xi)\,\frac{1-\cos n\xi}{n\xi^2}\,d\xi\right|$$

$$\leqslant \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1-\cos n\xi}{n\xi^2}\,d\xi \int_{-\infty}^{+\infty} |\,l(x-\xi)\,|\,dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1-\cos n\xi}{n\xi^2}\,d\xi \int_{-\infty}^{+\infty} |\,l(x)\,|\,dx = \int_{-\infty}^{+\infty} |\,l(x)\,|\,dx = A.$$

Finally, since the indefinite integral of an absolutely integrable function is an absolutely continuous function of a set, to every $\epsilon > 0$ we can make correspond a $\delta > 0$ such that for every set E whose measure is less than δ

$$\int_E |\,l(x)\,|\,dx \leqslant \epsilon.$$

Whence
$$\int_E |\,\sigma_n(x)\,|\,dx \leqslant \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1-\cos n\xi}{n\xi^2}\,d\xi \int_E |\,l(x-\xi)\,|\,dx$$

$$\leqslant \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1-\cos n\xi}{n\xi^2}\,d\xi\,\epsilon = \epsilon.$$

Sufficiency. We shall prove that if the conditions of theorem I are satisfied, then the conditions of Berry's theorem are satisfied too.

(1) Since $\lambda(u)\,[1-|\,u\,|/n]$ is summable in $(-\infty, +\infty)$, and since its Fourier transform $(1/2\pi)\,\sigma_n(-x) \in \{1\}$, we have, by result 5, everywhere, since $\lambda(u)\,[1-|\,u\,|/n]$ is continuous,

$$\left|\lambda(u)\left[1-\frac{|\,u\,|}{n}\right]\right| = \left|\frac{1}{2\pi}\int_{-\infty}^{+\infty} \sigma_n(-x)\,e^{iux}\,dx\right| \leqslant \frac{1}{2\pi}A.$$

This inequality being true for every positive integer n gives

$$|\,\lambda(u)\,| \leqslant \frac{1}{2\pi}A.$$

(2) Let $f(x)$ be an arbitrary function of the class $\{1\}$ and let $g(u)$ be its Fourier transform. As we have just seen, the function $\lambda(u)[1-|\,u\,|/n]$ is the Fourier transform of the absolutely integrable function $\sigma_n(x)$. We have, therefore, by result 6,

$$\left|\int_{-\infty}^{+\infty} \lambda(u)\left[1-\frac{|\,u\,|}{n}\right] f(u)\,du\right| = \left|\int_{-\infty}^{+\infty} \sigma_n(u)\,g(u)\,du\right| \leqslant A \sup_{-\infty<u<\infty} |\,g(u)\,|.$$

When $n\to\infty$, the absolutely integrable functions $\lambda(u)[1-|u|/n]f(u)$ tend to $\lambda(u)f(u)$, being less in modulus than the absolutely integrable function $(1/2\pi)A\,|f(u)|$. Therefore

$$\left|\int_{-\infty}^{+\infty}\lambda(u)f(u)\,du\right| = \lim_{n\to\infty}\left|\int_{-\infty}^{+\infty}\lambda(u)\left[1-\frac{|u|}{n}\right]f(u)\,du\right| \leqslant A\sup|g(u)|.$$

(3) Let $f(x)\in\{1\}$ be any function whose Fourier transform $g(u)$ verifies

$$\int_{-\infty}^{+\infty}|g(u)|\,du\leqslant\delta\sup|g(u)|.$$

We know already that

$$\left|\int_{-\infty}^{+\infty}\lambda(u)f(u)\,du\right| = \lim_{n\to\infty}\left|\int_{-\infty}^{+\infty}g(u)\sigma_n(u)\,du\right|.$$

Write
$$E_{k,n} = \underset{t}{E}\{|\sigma_n(t)|>k\}.$$

We have
$$|E_{k,n}|\leqslant\frac{A}{k},$$

$$\begin{aligned}\left|\int_{-\infty}^{+\infty}g(u)\sigma_n(u)\,du\right| &\leqslant \left|\int_{E_{k,n}}g(u)\sigma_n(u)\,du\right| + \left|\int_{CE_{k,n}}g(u)\sigma_n(u)\,du\right| \\ &\leqslant \sup|g(u)|\int_{E_{k,n}}|\sigma_n(u)|\,du + k\int_{-\infty}^{+\infty}|g(u)|\,du \\ &\leqslant \sup|g(u)|\left\{\int_{E_{k,n}}|\sigma_n(u)|\,du + k\delta\right\}.\end{aligned}$$

Choose a fixed k such that $\int_{E_{k,n}}|\sigma_n(u)|\,du\leqslant\frac{1}{2}\epsilon$ for every n. This is possible by the conditions of our theorem since $|E_{k,n}|\leqslant A/k$. Then choose $\delta>0$ such that $k\delta<\frac{1}{2}\epsilon$. This gives

$$\left|\int_{-\infty}^{+\infty}\lambda(u)f(u)\,du\right|\leqslant\epsilon\sup|g(u)|.$$

The three conditions of Berry's theorem are thus satisfied and the theorem is proved.

THEOREM II. *Necessary and sufficient conditions that $\lambda(u)\in(F_2)$ are that $\lambda(u)[1-|u|/n]$ is continuous for every positive integer n and that the functions*

$$\sigma_n(x) = \int_{-\infty}^{+\infty}\lambda(u)\left[1-\frac{|u|}{n}\right]e^{iux}\,du$$

satisfy the two conditions:

(1) *There exists a constant A such that*

$$\int_{-\infty}^{+\infty}|\sigma_n(x)|\,dx\leqslant A\quad\textit{for every } n.$$

(2) *There exists a constant B such that*

$$|\sigma_n(x)| \leqslant B \quad \textit{for every } n.$$

[The statement of this theorem was kindly communicated to me by Professor Fekete. I first proved it with the help of the following extension of a theorem of S. Banach: "Let the functions $\alpha_n(x)$ be summable and essentially bounded in $(-\infty, +\infty)$ and let them satisfy the two conditions:

$$\int_{-\infty}^{+\infty} |\alpha_n(x)|\,dx \leqslant A, \quad \text{ess. u.b.}\, |\alpha_n(x)| \leqslant B,$$

where A and B are constants independent of n. Then we can extract a partial sequence $\{\alpha_{n_k}(x)\}$, and we can find† a function $\alpha(x) \in \{2\}$ such that for every $f(x) \in \{1\}$

$$\lim_{k\to\infty} \int_{-\infty}^{+\infty} f(x)\,\alpha_{n_k}(x)\,dx = \int_{-\infty}^{+\infty} f(x)\,\alpha(x)\,dx."$$

A direct proof of theorem II, independent both of Berry's result and the above-mentioned theorem on linear operations, has been recently (after the completion of this paper) communicated to me by Professor Fekete, who found also a characterization of the functions of the class (F_1) in terms of the $\sigma_n(x)$. Later on, I found that a statement given by H. Cramer‡ and modified by Gonzalez Dominguez§ seems to coincide with the statement of theorem I].

Proof. Necessity. We have only to prove the necessity of the last condition. Suppose that $l(x) \in \{2\}$ is the generatrix of $\lambda(u)$. We have

$$\begin{aligned} |\sigma_n(x)| &= \frac{1}{\pi}\left|\int_{-\infty}^{+\infty} l(x-\xi)\,\frac{1-\cos n\xi}{n\xi^2}\,d\xi\right| \\ &\leqslant \underset{-\infty<\xi<\infty}{\text{u.b.}} |l(x-\xi)|\,\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1-\cos n\xi}{n\xi^2}\,d\xi = \underset{-\infty<x<\infty}{\text{u.b.}} |l(x)| = B. \end{aligned}$$

Sufficiency. It is evident that condition (2) of theorem II implies condition (2) of theorem I. We have only to choose $\delta = \epsilon/B$. Therefore $\lambda(u)$ is already the Fourier transform of some function $l(x) \in \{1\}$. By Hardy's theorem, we have, almost everywhere,

$$|l(x)| = \lim_{n\to\infty} |\sigma_n(x)| \leqslant B.$$

The function $l(x)$ is therefore essentially bounded. Whence follows the existence of a function $l^*(x) \in \{2\}$ equivalent to $l(x)$ and so having the same Fourier transform $\lambda(u)$. This completes the proof.

† Compare S. Banach, *Théorie des Opérations linéaires* (Varsovie, 1932), 130–131.

‡ *Trans. American Math. Soc.* 46 (1939), 191–201.

§ *Duke Math. J.* 6 (1940), 246–255. I have been unable to consult this paper. Cf. *Math. Reviews* 1, (1940), 226.

THEOREM III. *A necessary and sufficient condition that* $\lambda(u) \in (F_3)$ *is that the conditions of theorem II be realized and that the functions* $\sigma_n(x)$ *be uniformly R-integrable in* $(-\infty, +\infty)$.†

Proof. Necessity. The conditions of theorem II are of course necessary. To prove the necessity of the additional condition suppose that $\lambda(u)$ be the Fourier transform of $\sigma(x) \in \{3\}$. Then

$$\sigma_n(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \sigma(x-\xi)\frac{1-\cos n\xi}{n\xi^2}\,d\xi.$$

Let $\epsilon > 0$ be an arbitrary number and $(-a, +a)$ an arbitrary finite interval. Let M be the upper bound of $|\sigma(x)|$ in $(-\infty, +\infty)$. Since $\left|\frac{1-\cos n\xi}{n\xi^2}\right| \leqslant \frac{2}{n\xi^2}$, we can define a positive number X such that for every $n \geqslant 1$

$$\left|\frac{1}{\pi}\int_{-\infty}^{-X} + \frac{1}{\pi}\int_{X}^{+\infty} 4aM\,\frac{1-\cos n\xi}{n\xi^2}\,d\xi\right| \leqslant \tfrac{1}{2}\epsilon. \tag{1}$$

We have also

$$\frac{1}{\pi}\int_{-X}^{+X} \frac{1-\cos n\xi}{n\xi^2}\,d\xi < 1. \tag{2}$$

On the other hand, since $\sigma(x)$ is R-integrable in the interval $(-a-X, +a+X)$, we can find an

$$\eta = \eta\{\tfrac{1}{2}\epsilon, (-a-X, +a+X)\}$$

such that, $(\alpha_1, \beta_1), \ldots, (\alpha_\nu, \beta_\nu)$ being any finite sequence of non-overlapping intervals contained in $(-a, +a)$ and verifying the relations $0 \leqslant (\beta_i - \alpha_i) \leqslant \eta$ $(i = 1, 2, \ldots, \nu)$, and γ_i, δ_i being any numbers satisfying the conditions $\alpha_i \leqslant \gamma_i \leqslant \delta_i \leqslant \beta_i$, we have for every ξ in $(-X, +X)$

$$\left|\sum_{i=1}^{\nu} \{\sigma(\delta_i - \xi) - \sigma(\gamma_i - \xi)\}(\beta_i - \alpha_i)\right| \leqslant \tfrac{1}{2}\epsilon. \tag{3}$$

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† A function $\sigma(x) \in \{2\}$ is said to be R-integrable in $(-\infty, +\infty)$ if it is R-integrable over every finite interval $I = (-a, +a)$ $(a > 0)$, i.e. if to every $\epsilon > 0$ and to every interval I we can make correspond an $\eta = \eta\{\epsilon, I\}$ such that $(\alpha_1, \beta_1), \ldots, (\alpha_\nu, \beta_\nu)$ being any finite sequence of non-overlapping intervals contained in I and satisfying the condition $0 \leqslant (\beta_i - \alpha_i) \leqslant \eta$ $(i = 1, 2, \ldots, \nu)$ and γ_i, δ_i satisfying the condition $\alpha_i \leqslant \gamma_i \leqslant \delta_i \leqslant \beta_i$, we have

$$\left|\sum_{i=1}^{\nu} \{\sigma(\delta_i) - \sigma(\gamma_i)\}(\beta_i - \alpha_i)\right| \leqslant \epsilon.$$

A sequence of functions $\sigma_n(x) \in \{2\}$ is said to be uniformly R-integrable in $(-\infty, +\infty)$ if the numbers $\eta\{\epsilon, I\}$ can be determined independently of n, so that the preceding inequality holds for every $\sigma_n(x)$.

In a previous redaction of this work I had adopted another definition of R-integrability in $(-\infty, +\infty)$. Professor J. M. Whittaker, of Liverpool, kindly pointed out to me that it was not the usual one.

For ξ outside this interval

$$\left| \sum_{i=1}^{\nu} \{\sigma(\delta_i - \xi) - \sigma(\gamma_i - \xi)\} (\beta_i - \alpha_i) \right| \leqslant 4aM. \tag{4}$$

But we can write

$$\left| \sum_{i=1}^{\nu} \{\sigma_n(\delta_i) - \sigma_n(\gamma_i)\} (\beta_i - \alpha_i) \right|$$
$$\leqslant \frac{1}{\pi} \int_{-\infty}^{+\infty} \left| \sum_{i=1}^{\nu} \{\sigma(\delta_i - \xi) - \sigma(\gamma_i - \xi)\} (\beta_i - \alpha_i) \right| \frac{1 - \cos n\xi}{n\xi^2} d\xi.$$

This last relation shows by (1) and (4) and by (2) and (3) that

$$\left| \sum_{i=1}^{\nu} \{\sigma_n(\delta_i) - \sigma_n(\gamma_i)\} (\beta_i - \alpha_i) \right| \leqslant \epsilon.$$

This proves that the functions $\sigma_n(x) \in \{2\}$ are uniformly R-integrable in $(-\infty, +\infty)$.

Sufficiency. If the conditions of theorem II are satisfied, $\lambda(u)$ is already the Fourier transform of some function $\sigma(x) \in \{2\}$. Let ϵ be any positive number and $I = (-a, +a)$ $(a > 0)$ any finite interval. With the notations and the assumptions of the preceding footnote, we have, by hypothesis, for every n

$$\left| \sum_{i=1}^{\nu} \{\sigma_n(\delta_i) - \sigma_n(\gamma_i)\} (\beta_i - \alpha_i) \right| \leqslant \epsilon.$$

By Hardy's theorem, the sequence $\sigma_n(x)$ tends to $\sigma(x)$ everywhere except perhaps on a set E of measure zero. Suppose the γ_i, δ_i be chosen on the set ICE. Going to the limit

$$\left| \sum_{i=1}^{\nu} \{\sigma(\delta_i) - \sigma(\gamma_i)\} (\beta_i - \alpha_i) \right| \leqslant \epsilon.$$

The function $\sigma(x)$ is therefore essentially R-integrable in $(-a, +a)$. Replace it by the equivalent function $\sigma^*(x)$ defined as follows:†

on CE: $\qquad \sigma^*(x) = \sigma(x)$,

on E: $\qquad \sigma^*(x) = \overline{\lim\limits_{\xi \to x}}\, R\sigma(\xi) + i\, \overline{\lim\limits_{\xi \to x}}\, I\sigma(\xi)$, where $\xi \in CE$.

The function $\sigma^*(x)$ is R-integrable in $(-a, +a)$ and its Fourier transform is $\lambda(u)$. But since $\sigma^*(x) \in \{2\}$, and since the interval $(-a, +a)$ is arbitrary, we have $\sigma^*(x) \in \{3\}$. Whence $\lambda(u) \in (F_3)$.

† See the first-mentioned paper by Fekete, Hilfssatz 1.

THEOREM IV. *A necessary and sufficient condition that $\lambda(u) \in (F_4)$ is that the conditions of theorem II be realized and that the functions $\sigma_n(x)$ be uniformly (in n and x) continuous.*

Proof. Necessity. The conditions of theorem II are, of course, necessary. To prove the necessity of the additional condition suppose that $\lambda(u)$ be the Fourier transform of $\sigma(x) \in \{4\}$. Then

$$\sigma_n(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \sigma(x-\xi)\frac{1-\cos n\xi}{n\xi^2}d\xi.$$

By the properties of $\sigma(x)$, to every $\epsilon > 0$ we can make correspond a $\delta > 0$ such that the relation $|x_1 - x_2| \leqslant \delta$, $(-\infty < x_1, x_2 < \infty)$ implies $|\sigma(x_1) - \sigma(x_2)| \leqslant \epsilon$. Whence

$$|\sigma_n(x_1) - \sigma_n(x_2)| \leqslant \frac{1}{\pi}\int_{-\infty}^{+\infty} |\sigma(x_1-\xi) - \sigma(x_2-\xi)| \frac{1-\cos n\xi}{n\xi^2}d\xi$$
$$\leqslant \epsilon\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1-\cos n\xi}{n\xi^2}d\xi = \epsilon.$$

The functions $\sigma_n(x)$ are thus uniformly (in n and x) continuous.

Sufficiency. If the conditions of theorem II are satisfied, $\lambda(u)$ is already the Fourier transform of some function $\sigma(x) \in \{2\}$. By Hardy's theorem the sequence $\sigma_n(x)$ converges to $\sigma(x)$ everywhere except perhaps on a set E of measure zero. By our assumptions, to every $\epsilon > 0$ we can make correspond a $\delta > 0$ such that $|x_1 - x_2| \leqslant \delta$ implies $|\sigma_n(x_1) - \sigma_n(x_2)| \leqslant \epsilon$ for every n. Suppose x_1, x_2 be chosen on the set CE. Going to the limit $|\sigma(x_1) - \sigma(x_2)| \leqslant \epsilon$. The function $\sigma(x)$ is thus essentially uniformly continuous in $(-\infty, +\infty)$. We conclude that there exists a function $\sigma^*(x)$ equivalent to $\sigma(x)$ and so having the same Fourier transform $\lambda(u)$, which is uniformly continuous in $(-\infty, +\infty)$. Whence $\lambda(u) \in (F_4)$.

THEOREM V. *A necessary and sufficient condition that $\lambda(u) \in (F_5)$ is that the functions $\lambda(u)[1 - |u|/n]$ be continuous for every positive integer n and that the functions*

$$\tau_n(x) = \int_{-\infty}^{+\infty} \lambda(u)\frac{u-i}{i}\left[1 - \frac{|u|}{n}\right]e^{iux}du$$

satisfy the following condition:

There exists a constant C such that for every n

$$\int_{-\infty}^{+\infty} |\tau_n(x)|\,dx \leqslant C. \tag{5}$$

Proof. We shall use the following lemma: its proof is exactly the same as that of theorem IV, and so I omit it.

LEMMA. *A necessary and sufficient condition that $\lambda(u)\in(F_5)$ is that $\lambda(u)[1-|u|/n]$ be continuous for every positive integer n and that the functions*

$$\sigma_n(x)=\int_{-\infty}^{+\infty}\lambda(u)\left[1-\frac{|u|}{n}\right]e^{iux}\,du \tag{6}$$

satisfy the three conditions:

$$\int_{-\infty}^{+\infty}|\sigma_n(x)|\,dx\leqslant A, \tag{7}$$

$$|\sigma_n(x)|\leqslant B,\quad(-\infty<x<\infty), \tag{8}$$

$$V\{\sigma_n(x),(-\infty,+\infty)\}\leqslant V_0, \tag{9}$$

where A, B, V_0 are constants independent of n and $V\{\sigma_n(x),(-\infty,+\infty)\}$ represents the variation of $\sigma_n(x)$ in the interval $(-\infty,+\infty)$.

Observe first that, by a theorem of Lebesgue, condition (9) may be written

$$\int_{-\infty}^{+\infty}|\sigma_n'(x)|\,dx\leqslant V_0, \tag{10}$$

where $\sigma_n'(x)$ is the derivative of $\sigma_n(x)$.

We make use of the function $[e^{-\xi}]$ defined as $e^{-\xi}$ if $\xi\geqslant 0$ and equal to zero if $\xi<0$. We have

$$\frac{i}{u-i}=-\int_{-\infty}^{+\infty}[e^{-\xi}]\,e^{-iu\xi}\,d\xi.$$

Hence
$$\begin{aligned}\sigma_n(x)&=-\int_{-\infty}^{+\infty}\left(\lambda(u)\,\frac{u-i}{i}\left[1-\frac{|u|}{n}\right]\int_{-\infty}^{+\infty}[e^{-\xi}]\,e^{-iu\xi}\,d\xi\right)e^{iux}\,du\\&=-\int_{-\infty}^{+\infty}\left([e^{-\xi}]\int_{-\infty}^{+\infty}\lambda(u)\frac{u-i}{i}\left[1-\frac{|u|}{n}\right]e^{iu(x-\xi)}\,du\right)d\xi\\&=-\int_{-\infty}^{+\infty}[e^{-\xi}]\,\tau_n(x-\xi)\,d\xi.\end{aligned}$$

Sufficiency. Suppose that condition (5) is satisfied. Then

$$|\sigma_n(x)|\leqslant\sup_{-\infty<\xi<\infty}[e^{-\xi}]\int_{-\infty}^{+\infty}|\tau_n(x-\xi)|\,d\xi\leqslant C. \tag{8'}$$

Similarly

$$\begin{aligned}\int_{-\infty}^{+\infty}|\sigma_n(x)|\,dx&\leqslant\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[e^{-\xi}]\,|\tau_n(x-\xi)|\,d\xi\,dx\\&=\int_{-\infty}^{+\infty}[e^{-\xi}]\,d\xi\int_{-\infty}^{+\infty}|\tau_n(x-\xi)|\,dx\leqslant C\int_{-\infty}^{+\infty}[e^{-\xi}]\,d\xi=C.\end{aligned} \tag{7'}$$

Finally, differentiating relation (6) with respect to x (the limits of integration are finite and the derivative of the function under the integral sign is continuous) we get

$$\sigma_n'(x) = \int_{-\infty}^{+\infty} \lambda(u)\, iu \left[1 - \frac{|u|}{n}\right] e^{iux}\, du,$$

whence
$$\tau_n(x) = -\sigma_n(x) - \sigma_n'(x).$$

This gives by (5) and (7′)

$$\int_{-\infty}^{+\infty} |\sigma_n'(x)|\, dx \leqslant 2C. \tag{10′}$$

Conditions (7), (8) and (10) of the lemma are thus satisfied and $\lambda(u) \in (F_5)$.

Necessity. Conversely, suppose that conditions (7), (8) and (10) of the lemma are satisfied. This gives

$$\int_{-\infty}^{+\infty} |\tau_n(x)|\, dx \leqslant \int_{-\infty}^{+\infty} |\sigma_n(x)|\, dx + \int_{-\infty}^{+\infty} |\sigma_n'(x)|\, dx \leqslant B + V_0.$$

The proof is now complete.

THEOREM VI. *The two relations* $g(u) \in (F_6)$ *and* $\frac{u-i}{i} g(u) \in (F_1)$ *are equivalent.*

Proof. Let $g(u)$ be an arbitrary function of the class (F_6). By the theorem of Bochner we have $-g(u) \in (F_1)$ and $-iug(u) \in (F_1)$. Whence, by addition,

$$\frac{u-i}{i} g(u) \in (F_1).$$

Conversely, let
$$\frac{u-i}{i} g(u) \in (F_1). \tag{11}$$

Since $\frac{i}{u-i} \in (F_1)$, we have (result 7)

$$\frac{u-i}{i} g(u) \frac{i}{u-i} = g(u) \in (F_1). \tag{12}$$

By (11) and (12)
$$\frac{u-i}{i} g(u) + g(u) = -iug(u) \in (F_1). \tag{13}$$

By Bochner's theorem (12) and (13) imply

$$g(u) \in (F_6).$$

Remark. We shall use sometimes the following form of theorem VI: The two relations

$$g(u) \in (F_1) \quad \text{and} \quad \frac{ig(u)}{u-i} \in (F_6)$$

are equivalent.

3. *The fundamental theorem*

Before stating our fundamental theorem, let us recall a

Theorem of Steinhaus.† *Let the functions $\sigma_n(x)$ be measurable and essentially bounded in the interval $(-\infty, +\infty)$ and let*

$$\limsup_{n\to\infty} (\text{ess u.b.}\,|\sigma_n(x)|) = \infty.$$

Then there exists a function $f(x) \in \{1\}$ such that

$$\limsup_{n\to\infty} \left| \int_{-\infty}^{+\infty} f(x)\,\sigma_n(x)\,dx \right| = \infty.$$

THEOREM VII. *If the functions $\sigma_n(x)$ are absolutely integrable in the interval $(-\infty, +\infty)$, and if*

$$\limsup_{n\to\infty} \int_{-\infty}^{+\infty} |\sigma_n(x)|\,dx = \infty,$$

then there exists a function $f(x) \in \{4\}$ such that

$$\limsup_{n\to\infty} \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f(x-\xi)\,\sigma_n(\xi)\,d\xi \right| dx = \infty.$$

The proof is given in two parts.

I. If $\sigma(x) \in \{1\}$ we shall show that, given any positive $\eta < \frac{1}{2}$, there exists a function $f(x) \in \{4\}$ satisfying the three conditions

$$\int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f(x-\xi)\,\sigma(\xi)\,d\xi \right| dx > (1-2\eta) \int_{-\infty}^{+\infty} |\sigma(x)|\,dx,$$

$$|f(x)| \leqslant 1, \quad \int_{-\infty}^{+\infty} |f(x)|\,dx = 1.$$

Making use of result 4 with respect to the function $\sigma(x)$, we see that there exists an integer $\nu \geqslant 2$ such that

$$\int_{-\infty}^{+\infty} \left| \sigma(x) - \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1-\cos\nu\xi}{\nu\xi^2}\,\sigma(x-\xi)\,d\xi \right| dx < \eta \int_{-\infty}^{+\infty} |\sigma(\xi)|\,d\xi.$$

Hence

$$\int_{-\infty}^{+\infty} \left| \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1-\cos\nu(x-\xi)}{\nu(x-\xi)^2}\,\sigma(\xi)\,d\xi \right| dx$$

$$= \int_{-\infty}^{+\infty} \left| \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1-\cos\nu\xi}{\nu\xi^2}\,\sigma(x-\xi)\,d\xi \right| dx > (1-\eta) \int_{-\infty}^{+\infty} |\sigma(\xi)|\,d\xi.$$

† H. Steinhaus, *loc. cit.* pp. 219–221. Steinhaus has given his theorem for finite limits, but there is no difficulty in proving it for infinite limits.

Since the integral

$$\int_{-\infty}^{+\infty}\left|\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1-\cos\nu(x-\xi)}{\nu(x-\xi)^2}\sigma(\xi)\,d\xi\right|dx$$

exists, in virtue of the foregoing inequality we can choose $\omega \geqslant 2/\sqrt{\pi}$ so large that

$$\frac{1}{\pi}\int_{-\omega}^{+\omega}\left|\int_{-\infty}^{+\infty}\frac{1-\cos\nu(x-\xi)}{\nu^2(x-\xi)^2}\sigma(\xi)\,d\xi\right|dx > \frac{1-\eta}{\nu}\int_{-\infty}^{+\infty}|\sigma(\xi)|\,d\xi.$$

Hence if E is any interval or sum of intervals which have no point in common with $(-\omega, +\omega)$

$$\frac{1}{\pi}\int_{E}\left|\int_{-\infty}^{+\infty}\frac{1-\cos\nu(x-\xi)}{\nu^2(x-\xi)^2}\sigma(\xi)\,d\xi\right|dx < \frac{\eta}{\nu}\int_{-\infty}^{+\infty}|\sigma(\xi)|\,d\xi.$$

Let $\lambda = 2\omega$ and take $\quad f(x) = \dfrac{1}{\pi}\displaystyle\sum_{k=0}^{\nu-1}\frac{1-\cos\nu(x-k\lambda)}{\nu^2(x-k\lambda)^2}.$

By this choice of λ and ω for $0 \leqslant k \leqslant \nu-1$ we have the inequality

$$\begin{aligned}
&\frac{1}{\pi}\int_{-\omega+k\lambda}^{\omega+k\lambda}\left|\int_{-\infty}^{+\infty}f(x-\xi)\sigma(\xi)\,d\xi\right|dx\\
&\quad\geqslant \frac{1}{\pi}\int_{-\omega+k\lambda}^{\omega+k\lambda}\left|\int_{-\infty}^{+\infty}\frac{1-\cos\nu(x-k\lambda-\xi)}{\nu^2(x-k\lambda-\xi)^2}\sigma(\xi)\,d\xi\right|dx\\
&\qquad\quad -\frac{1}{\pi}\sum_{\substack{j=0\\ j\neq k}}^{\nu-1}\int_{-\omega+k\lambda}^{\omega+k\lambda}\left|\int_{-\infty}^{+\infty}\frac{1-\cos\nu(x-j\lambda-\xi)}{\nu^2(x-j\lambda-\xi)^2}\sigma(\xi)\,d\xi\right|dx\\
&\quad= \frac{1}{\pi}\int_{-\omega}^{+\omega}\left|\int_{-\infty}^{+\infty}\frac{1-\cos(x-\xi)}{\nu^2(x-\xi)^2}\sigma(\xi)\,d\xi\right|dx\\
&\qquad\quad -\frac{1}{\pi}\sum_{\substack{j=0\\ j\neq k}}^{\nu-1}\int_{-\omega+(k-j)\lambda}^{\omega+(k-j)\lambda}\left|\int_{-\infty}^{+\infty}\frac{1-\cos\nu(x-\xi)}{\nu^2(x-\xi)^2}\sigma(\xi)\,d\xi\right|dx\\
&\quad\geqslant \frac{1-2\eta}{\nu}\int_{-\infty}^{+\infty}|\sigma(\xi)|\,d\xi.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{-\infty}^{+\infty}\left|\int_{-\infty}^{+\infty}f(x-\xi)\sigma(\xi)\,d\xi\right|dx &\geqslant \sum_{k=0}^{\nu-1}\int_{-\omega+k\lambda}^{\omega+k\lambda}\left|\int_{-\infty}^{+\infty}f(x-\xi)\sigma(\xi)\,d\xi\right|dx\\
&\geqslant (1-2\eta)\int_{-\infty}^{+\infty}|\sigma(\xi)|\,d\xi.
\end{aligned}$$

Now by the above choice of $f(x)$, ν, λ, ω, we clearly have

$$0 \leqslant f(x) \leqslant \frac{1}{\pi}\frac{2}{\nu}\frac{4}{\omega^2} \leqslant \frac{4}{\pi\omega^2} \leqslant 1$$

if x lies outside all the intervals of diameter ω whose centres coincide with $2k\omega$ $(0 \leqslant k \leqslant \nu-1)$, while

$$0 \leqslant f(x) \leqslant \frac{1}{2\pi}+\frac{1}{\pi}\frac{2}{\nu}\frac{4}{9\omega^2} \leqslant \frac{1}{\pi}\left(\frac{1}{2}+\frac{4}{9\omega^2}\right) \leqslant \frac{1}{\pi}\left(\frac{1}{2}+\frac{\pi}{9}\right) < 1$$

as x ranges over one of these intervals. Thus $0 \leqslant f(x) \leqslant 1$ for $-\infty < x < \infty$. Finally

$$\int_{-\infty}^{+\infty} |f(x)|\, dx = 1.$$

This completes the proof of part I.

II. †Let $\displaystyle\int_{-\infty}^{+\infty} |\sigma_n(\xi)|\, d\xi = \omega_n$, where $\displaystyle\limsup_{n\to\infty} \omega_n = \infty$.

By part I, to every $\sigma_n(\xi)$ we can make correspond a function $f_n(x) \in \{4\}$ such that

$$|f_n(x)| \leqslant 1, \quad \int_{-\infty}^{+\infty} |f_n(x)|\, dx = 1, \quad \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f_n(x-\xi)\, \sigma_n(\xi)\, d\xi \right| dx > \tfrac{3}{4}\omega_n.$$

If any of the functions $f_k(x)$ satisfies the condition

$$\limsup_{n\to\infty} \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f_k(x-\xi)\, \sigma_n(\xi)\, d\xi \right| dx = \infty,$$

the theorem is proved. We may suppose therefore that for every k

$$\int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f_k(x-\xi)\, \sigma_n(\xi)\, d\xi \right| dx \leqslant c_k \quad \text{for every } n,$$

where the c_k are finite constants.

We shall show that we can extract a subsequence $f_{k_p}(x)$ such that the function

$$f(x) = \sum_{p=1}^{\infty} \frac{f_{k_p}(x)}{3^p}$$

verifies the conditions of our theorem.

---

† Many simplifications in this part of the proof are due to Professor Fekete and Dr M. Schiffer of Jerusalem. Another short proof of part II can be given by making use of the following theorem of Banach and Steinhaus:

"Let the $U_n(x)$ be a sequence of linear operations defined in a Banach space E, and let

$$\limsup_{n\to\infty} \| U_n(x) \| < \infty$$

for every $x \in E$. Then the sequence of norms $\| U_n \|$ is bounded."

See the above-mentioned book by Banach (p. 80, theorem 5).

It is evident that whatever be the subsequence $f_{k_p}(x)$, $f(x)\in\{4\}$. Now for every p

$$\int_{-\infty}^{+\infty}\left|\int_{-\infty}^{+\infty}f(x-\xi)\sigma_{k_p}(\xi)\,d\xi\right|dx \geqslant \frac{1}{3^p}\int_{-\infty}^{+\infty}\left|\int_{-\infty}^{+\infty}f_{k_p}(x-\xi)\sigma_{k_p}(\xi)\,d\xi\right|dx$$

$$-\sum_{i<p}\frac{1}{3^i}\int_{-\infty}^{+\infty}\left|\int_{-\infty}^{+\infty}f_{k_i}(x-\xi)\sigma_{k_p}(\xi)\,d\xi\right|dx$$

$$-\sum_{i>p}\frac{1}{3^i}\int_{-\infty}^{+\infty}\left|\int_{-\infty}^{+\infty}f_{k_i}(x-\xi)\sigma_{k_p}(\xi)\,d\xi\right|dx$$

$$\geqslant \frac{3}{4}\frac{\omega_{k_p}}{3^p}-\sum_{i<p}\frac{c_{k_i}}{3^i}-\sum_{i>p}\frac{\omega_{k_p}}{3^i}=\frac{1}{4}\frac{\omega_{k_p}}{3^p}-\sum_{i<p}\frac{c_{k_i}}{3^i}.$$

The subsequence k_i having been chosen up to k_{p-1} we can always choose, owing to $\limsup_{n\to\infty}\omega_n=\infty$, a number k_p such that

$$\frac{1}{4}\frac{\omega_{k_p}}{3^p}-\sum_{i<p}\frac{c_{k_i}}{3^i}>p.$$

This proves the theorem.

4. *The multiplicators for Fourier transforms*

THEOREM VIII. *A necessary and sufficient condition for $\lambda(u)$ to be respectively a multiplicator* (5, 1), (5, 2), (5, 3), (5, 4), (5, 5) *is that $\dfrac{i\lambda(u)}{u-i}$ belongs respectively to the classes* (F_1), (F_2), (F_3), (F_4), (F_5).

Proof. Necessity. Since $\dfrac{i}{u-i}\in(F_5)$ (result 9) the stated condition is necessary, by definition, in every case.

Sufficiency. The proof is very similar in the five cases. We shall give it, by way of example, successively for multiplicators (5, 1), (5, 5) and (5, 3).

For multiplicators (5, 1). Let $\dfrac{i\lambda(u)}{u-i}$ be the Fourier transform of an arbitrary function of the class $\{1\}$, and let

$$s_n(x)=\int_{-\infty}^{+\infty}\frac{i\lambda(u)}{u-i}\left[1-\frac{|u|}{n}\right]e^{iux}\,du. \tag{14}$$

We have, by theorem I,

(1) $$\int_{-\infty}^{+\infty}|s_n(x)|\,dx\leqslant A;$$

(2) to every $\epsilon > 0$ corresponds a $\delta > 0$ such that for any set E of measure less than δ

$$\int_E |s_n(x)|\, dx \leqslant \epsilon.$$

Let now $g(u) \in (F_5)$ be the Fourier transform of the arbitrary function $f(x) \in \{5\}$.

The functions $\dfrac{i\lambda(u)}{u-i}$ and $g(u)$ are continuous (result 2). The same holds for $\lambda(u)$ and therefore

$$g(u)\,\lambda(u)\left[1-\frac{|u|}{n}\right] \text{ is continuous for every positive integer } n. \tag{15}$$

Consider the two functions

$$\sigma_n(x) = \int_{-\infty}^{+\infty} \lambda(u)\left[1-\frac{|u|}{n}\right] e^{iux}\, du, \tag{16}$$

$$S_n(x) = \int_{-\infty}^{+\infty} g(u)\,\lambda(u)\left[1-\frac{|u|}{n}\right] e^{iux}\, du. \tag{17}$$

We have

$$\begin{aligned} S_n(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{\lambda(u)\left[1-\frac{|u|}{n}\right]e^{iux}\int_{-\infty}^{+\infty} f(\xi)\, e^{-iu\xi}\, d\xi\right\} du \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)\, d\xi \int_{-\infty}^{+\infty}\lambda(u)\left[1-\frac{|u|}{n}\right] e^{iu(x-\xi)}\, du \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)\,\sigma_n(x-\xi)\, d\xi = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x-\xi)\,\sigma_n(\xi)\, d\xi. \end{aligned} \tag{18}$$

If we differentiate relation (14) with respect to x (in fact, the limits of integration are finite and the derivative of the function under the integral sign is continuous), we get

$$s_n'(x) = -\int_{-\infty}^{+\infty}\lambda(u)\left[1-\frac{|u|}{n}\right]\frac{u}{u-i}\, e^{iux}\, du.$$

Whence

$$\left.\begin{aligned} \sigma_n(x) &= -s_n(x) - s_n'(x), \\ S_n(x) &= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x-\xi)\,\{s_n(\xi)+s_n'(\xi)\}\, d\xi. \end{aligned}\right\} \tag{19}$$

But we can write

$$\begin{aligned} \int_{-\infty}^{+\infty} f(x-\xi)\, s_n'(\xi)\, d\xi &= [s_n(\xi) f(x-\xi)]_{\xi=-\infty}^{\xi=+\infty} - \int_{-\infty}^{+\infty} s_n(\xi)\, d_\xi f(x-\xi) \\ &= -\int_{-\infty}^{+\infty} s_n(\xi)\, d_\xi f(x-\xi) = -\int_{-\infty}^{+\infty} s_n(x-\xi)\, df(\xi), \end{aligned}$$

for $s_n(\xi)$ is bounded (result 2) and $f(x-\xi)$ tends to zero when $\xi \to \pm\infty$, since $f(\xi)$ is absolutely integrable and of bounded variation in $(-\infty, +\infty)$.†

Hence
$$S_n(x) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)\, s_n(x-\xi)\, d\xi + \frac{1}{2\pi}\int_{-\infty}^{+\infty} s_n(x-\xi)\, df(\xi). \tag{20}$$

We conclude by the property (1) of $s_n(x)$

$$\int_{-\infty}^{+\infty} |S_n(x)|\, dx \leqslant \frac{1}{2\pi} A \int_{-\infty}^{+\infty} |f(\xi)|\, d\xi + \frac{1}{2\pi} A \int_{-\infty}^{+\infty} |df(\xi)| = AK, \tag{21}$$

where K is a finite constant equal to

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |f(\xi)|\, d\xi + \frac{1}{2\pi}\int_{-\infty}^{+\infty} |df(\xi)|.$$

Let now ϵ' be any positive number. Take $\epsilon = \epsilon'/K$, and let $\delta > 0$ be the number corresponding to ϵ. For every set E whose measure is less than δ we have

$$\begin{aligned}\int_E |S_n(x)|\, dx &\leqslant \frac{1}{2\pi}\int_{-\infty}^{+\infty} |f(\xi)|\, d\xi \int_E |s_n(x-\xi)|\, dx \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{+\infty} |df(\xi)| \int_E |s_n(x-\xi)|\, dx \\ &\leqslant \frac{1}{2\pi}\epsilon \int_{-\infty}^{+\infty} |f(\xi)|\, d\xi + \frac{1}{2\pi}\epsilon\int_{-\infty}^{+\infty} |df(\xi)| = \epsilon K = \epsilon'. \end{aligned} \tag{22}$$

Relations (15), (17), (21) and (22) show now, by theorem I, that

$$\lambda(u)\, g(u) \in (F_1).$$

This proves that $\lambda(u)$ is a multiplicator (5, 1).

For multiplicators (5, 5). Let $\dfrac{i\lambda(u)}{u-i} \in (F_5)$ be the Fourier transform of an arbitrary function of the class {5}. We use the same notations as before. It is clear that relation (20) holds. Moreover, by the lemma of theorem V there exists a constant B such that

$$|s_n(x)| \leqslant B \quad \text{for every } n.$$

---

† To see, for example, that $\lim_{\xi \to +\infty} f(\xi) = 0$, we proceed as follows. $f(\xi)$ belonging to the class {5} we can find a number X_1 such that $\int_{X_1}^{\infty} |df(\xi)| < \epsilon$, where ϵ is an arbitrary positive number. But since $f(\xi)$ belongs to the class {1} we can find an $X_2 > X_1$ such that $|f(X_2)| < \epsilon$. Whence, for $x > X_2$,

$$f(x) = \int_{X_2}^{x} df(\xi) + f(X_2),$$

and so
$$|f(x)| < 2\epsilon.$$

We want to show that there exists a constant A such that

$$\int_{-\infty}^{+\infty} |\sigma_n(x)|\,dx \leqslant A \quad \text{for every } n. \tag{31}$$

In fact if (31) does not hold, there exists, by theorem VII,† a function $f(x) \in \{4\}$ such that

$$\limsup_{n\to\infty} \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f(x-\xi)\,\sigma_n(\xi)\,d\xi \right| dx = \infty,$$

contrary to (30). Therefore (31) holds. By theorem V, (29), (16′) and (31) imply $\dfrac{i\lambda(u)}{u-i} \in (F_5)$.

THEOREM X. *A necessary and sufficient condition that* $\lambda(u)$ *be a multiplicator* (1, 2) *is that* $\lambda(u) \in (F_2)$. *The condition is the same for multiplicators* (1, 3) *and* (1, 4).

Proof. Sufficiency. We need only show that whatever be $g(u) \in (F_1)$ and $\lambda(u) \in (F_2)$ then $g(u)\,\lambda(u)$ is the Fourier transform of some function absolutely integrable and uniformly continuous in $(-\infty, +\infty)$.

Let $f(x) \subset \{1\}$ and $l(x) \in \{2\}$ be respectively the generatrices of $g(u)$ and $\lambda(u)$. By result 7, $g(u)\,\lambda(u)$ is the Fourier transform of the absolutely integrable function

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x-\xi)\,l(\xi)\,d\xi. \tag{32}$$

But

$$\left| \int_{-\infty}^{+\infty} \{f(x+\eta-\xi) - f(x-\xi)\}\,l(\xi)\,d\xi \right|$$
$$\leqslant \underset{-\infty<\xi<\infty}{\text{u.b.}} |l(\xi)| \int_{-\infty}^{+\infty} |f(x+\eta-\xi) - f(x-\xi)|\,d\xi$$
$$= \underset{-\infty<\xi<\infty}{\text{u.b.}} |l(\xi)| \int_{-\infty}^{+\infty} |f(t+\eta) - f(t)|\,dt.$$

This last integral tends to zero with η by a theorem of Lebesgue.

The function (32) is thus uniformly continuous.

Necessity. We need only prove that if $\lambda(u)$ is a multiplicator (1, 2) we have necessarily $\lambda(u) \in (F_2)$. To do this, we shall show that all the requirements of theorem II concerning the functions $\lambda(u)[1-|u|/n]$ and $\sigma_n(x)$ defined by (16) are satisfied.

† Every $\sigma_n(x)$ certainly belongs to the class $\{1\}$. To see this, observe that the function $\lambda(u)[1-|u|/n]$ is absolutely integrable in $(-\infty, +\infty)$. Moreover, it is the Fourier transform of some function $\sigma_n^*(x) \in \{1\}$, since $\lambda(u)$ is a multiplicator (4, 1) and since $[1-|u|/n] \in (F_4)$ (result 1). Hence, by result 5,

$$\sigma_n(x) = \sigma_n^*(x)$$

almost everywhere, which proves that $\sigma_n(x)$ belongs to the class $\{1\}$.

Relations (15), (17), (21), (23) and (28) show, by theorem III, that

$$\lambda(u)\,g(u) \in (F_3).$$

This proves that $\lambda(u)$ is a multiplicator (5, 3).

THEOREM IX. *A necessary and sufficient condition that* $\lambda(u)$ *be a multiplicator* (1, 1) *is that* $\dfrac{i\lambda(u)}{u-i} \in (F_5)$. *The condition is the same for multiplicators* (2,1), (2, 2), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3), (4, 4).

Proof. Sufficiency. We need only show that if $\dfrac{i\lambda(u)}{u-i} \in (F_5)$, then $\lambda(u)$ is a multiplicator (1, 1), (2, 2), (3, 3) and (4, 4). The proof is the same in the four cases. We shall give it, by way of example, for the case (3, 3).

Let $g_3(u)$ be an arbitrary function of the class (F_3). By theorem VIII, $\dfrac{u-i}{i}\,g_3(u)$ is a multiplicator (5, 3). But since by hypothesis $\dfrac{i\lambda(u)}{u-i} \in (F_5)$, we have

$$\frac{u-i}{i}\,g_3(u)\,\frac{i\lambda(u)}{u-i} = g_3(u)\,\lambda(u) \in (F_3).$$

This proves that $\lambda(u)$ is a multiplicator (3, 3).

Necessity. It will be enough to show that if $\lambda(u)$ is a multiplicator (4, 1) we have necessarily $\dfrac{i\lambda(u)}{u-i} \in (F_5)$.

In fact since $[1-|u|/n] \in (F_4)$ we have by hypothesis $\lambda(u)[1-|u|/n] \in (F_1)$ which proves by result 2 that $\lambda(u)[1-|u|/n]$ is continuous and hence that

$$\frac{i\lambda(u)}{u-i}\left[1-\frac{|u|}{n}\right] \text{ is continuous (for every } n). \tag{29}$$

Let $g(u) \in (F_4)$ be the Fourier transform of an arbitrary function $f(x) \in \{4\}$. Consider the functions $\sigma_n(x)$ and $S_n(x)$ defined by (16) and (17). By hypothesis $g(u)\,\lambda(u) \in (F_1)$. Whence by theorem I follows the existence of a constant A' (depending on $f(x)$) such that

$$\int_{-\infty}^{+\infty} |S_n(x)|\,dx \leqslant A'. \tag{30}$$

But we have

$$\sigma_n(x) = \int_{-\infty}^{+\infty} \frac{i\lambda(u)}{u-i}\,\frac{u-i}{i}\left[1-\frac{|u|}{n}\right] e^{iux}\,du, \tag{16'}$$

and we write again relation (18)

$$S_n(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x-\xi)\,\sigma_n(\xi)\,d\xi.$$

We want to show that there exists a constant A such that

$$\int_{-\infty}^{+\infty} |\sigma_n(x)|\,dx \leqslant A \quad \text{for every } n. \tag{31}$$

In fact if (31) does not hold, there exists, by theorem VII,† a function $f(x) \in \{4\}$ such that

$$\limsup_{n\to\infty} \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f(x-\xi)\,\sigma_n(\xi)\,d\xi \right| dx = \infty,$$

contrary to (30). Therefore (31) holds. By theorem V, (29), (16′) and (31) imply $\dfrac{i\lambda(u)}{u-i} \in (F_5)$.

THEOREM X. *A necessary and sufficient condition that* $\lambda(u)$ *be a multiplicator* (1, 2) *is that* $\lambda(u) \in (F_2)$. *The condition is the same for multiplicators* (1, 3) *and* (1, 4).

Proof. Sufficiency. We need only show that whatever be $g(u) \in (F_1)$ and $\lambda(u) \in (F_2)$ then $g(u)\,\lambda(u)$ is the Fourier transform of some function absolutely integrable and uniformly continuous in $(-\infty, +\infty)$.

Let $f(x) \subset \{1\}$ and $l(x) \in \{2\}$ be respectively the generatrices of $g(u)$ and $\lambda(u)$. By result 7, $g(u)\,\lambda(u)$ is the Fourier transform of the absolutely integrable function

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x-\xi)\,l(\xi)\,d\xi. \tag{32}$$

But

$$\left| \int_{-\infty}^{+\infty} \{f(x+\eta-\xi)-f(x-\xi)\}\,l(\xi)\,d\xi \right|$$
$$\leqslant \underset{-\infty<\xi<\infty}{\text{u.b.}} |l(\xi)| \int_{-\infty}^{+\infty} |f(x+\eta-\xi)-f(x-\xi)|\,d\xi$$
$$= \underset{-\infty<\xi<\infty}{\text{u.b.}} |l(\xi)| \int_{-\infty}^{+\infty} |f(t+\eta)-f(t)|\,dt.$$

This last integral tends to zero with η by a theorem of Lebesgue.

The function (32) is thus uniformly continuous.

Necessity. We need only prove that if $\lambda(u)$ is a multiplicator (1, 2) we have necessarily $\lambda(u) \in (F_2)$. To do this, we shall show that all the requirements of theorem II concerning the functions $\lambda(u)[1-|u|/n]$ and $\sigma_n(x)$ defined by (16) are satisfied.

† Every $\sigma_n(x)$ certainly belongs to the class {1}. To see this, observe that the function $\lambda(u)[1-|u|/n]$ is absolutely integrable in $(-\infty, +\infty)$. Moreover, it is the Fourier transform of some function $\sigma_n^*(x) \in \{1\}$, since $\lambda(u)$ is a multiplicator (4, 1) and since $[1-|u|/n] \in (F_4)$ (result 1). Hence, by result 5,

$$\sigma_n(x) = \sigma_n^*(x)$$

almost everywhere, which proves that $\sigma_n(x)$ belongs to the class {1}.

In fact, since $\lambda(u)$ is already a multiplicator (2, 2) we have by the preceding theorem $\dfrac{i\lambda(u)}{u-i}\in(F_5)$, which proves that $\dfrac{i\lambda(u)}{u-i}$ is continuous and so that

$$\lambda(u) \text{ is continuous in } (-\infty, +\infty). \tag{33}$$

Also, by theorem V, there exists a constant A such that

$$\int_{-\infty}^{+\infty} |\sigma_n(x)|\,dx \leqslant A \quad \text{for every } n. \tag{34}$$

Let now $g(u)$ be the Fourier transform of the arbitrary function $f(x)\in\{1\}$. We have, as in (17), (18),

$$S_n(x) = \int_{-\infty}^{+\infty} g(u)\,\lambda(u)\left[1-\frac{|u|}{n}\right]e^{iux}\,du$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x-\xi)\,\sigma_n(\xi)\,d\xi.$$

By hypothesis $g(u)\lambda(u)\in(F_2)$. By theorem II, the functions $S_n(x)$ are uniformly bounded in n and x. In particular, they are uniformly bounded at the point $x = 0$. We conclude that there exists a constant B such that

$$|\sigma_n(x)| \leqslant B \quad \text{for every } n \text{ and } x, \tag{35}$$

for, otherwise, we could find, according to the theorem of Steinhaus, a function $f(x)\in\{1\}$ for which the sequence $S_n(0)$ is not bounded. By theorem II, relations (33), (34) and (35) imply $\lambda(u)\in(F_2)$.

Let us now recall a theorem of Verblunsky.†

If in the finite interval (a, b), $(a<b)$, the functions $\alpha_n(t)$ are bounded and if

$$\lim_{n\to\infty}\int_a^b f(t)\,\alpha_n(t)\,dt$$

exists for every $f(t)$ bounded and R-integrable in (a, b), then the sequence of functions

$$\int_0^x \alpha_n(t)\,dt$$

tends to a function absolutely continuous in (a, b).

THEOREM XI. *A necessary and sufficient condition that $\lambda(u)$ be a multiplicator (2, 4) is that $\lambda(u)\in(F_1)$. The condition is the same for multiplicators (3, 4).*

Proof. Sufficiency. We must show that the relations $\lambda(u)\in(F_1)$ and $g_2(u)\in(F_2)$ imply $\lambda(u)\,g_2(u)\in(F_4)$. But this is an immediate consequence of theorem X.

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† S. Verblunsky, *loc. cit.* theorem I.

Necessity. It will be enough to prove that if $\lambda(u)$ is a multiplicator $(3, 4)$ then necessarily $\lambda(u) \in (F_1)$ or what comes to the same, by theorem VI,

$$\frac{i\lambda(u)}{u-i} \in (F_6).$$

Since $\lambda(u)$ is a multiplicator $(3, 3)$, we have already, by theorem IX, $\frac{i\lambda(u)}{u-i} \in (F_5)$. Let then $s(x) \in \{5\}$ be the generatrix of $\frac{i\lambda(u)}{u-i}$, and put

$$s^*(x) = \tfrac{1}{2}\{s(x+0)+s(x-0)\}.$$

Evidently $s^*(x) = s(x)$ almost everywhere. Thus $s^*(x)$ yields another generatrix for $\frac{i\lambda(u)}{u-i}$. We shall prove that $s^*(x) \in \{6\}$.

I. We shall first show that $s^*(x)$ is absolutely continuous in every finite interval $(-a, a)$, $(a > 0)$.

$g(u)$ being the Fourier transform of the arbitrary function $f(x) \in \{3\}$, consider again the functions $s_n(x)$, $\sigma_n(x)$ and $S_n(x)$ defined in formulae (14), (16) and (17). We have, by (19) for every x in $(-a, a)$,

$$\int_0^x \sigma_n(t)\,dt = -\int_0^x s_n(t)\,dt - \int_0^x s_n'(t)\,dt.$$

But for each $n > 0$ the function $s_n(x)$ is continuous in $(-a, a)$ together with its derivative $s_n'(x)$. We conclude

$$\int_0^x s_n'(t)\,dt = s_n(x) - s_n(0).$$

Hence
$$s_n(x) = s_n(0) - \int_0^x s_n(t)\,dt - \int_0^x \sigma_n(t)\,dt.$$

We shall show that

$$\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} s_n(0) - \lim_{n\to\infty}\int_0^x s_n(t)\,dt - \lim_{n\to\infty}\int_0^x \sigma_n(t)\,dt,$$

where the limit of each term of the second member exists and is absolutely continuous in $(-a, a)$. Thus the function

$$s^*(x) = \tfrac{1}{2}\{s(x+0)+s(x-0)\}$$

will have the same property, since, by Hardy's theorem,

$$\lim_{n\to\infty} s_n(x) = \tfrac{1}{2}\{s(x+0)+s(x-0)\}.$$

We first have
$$\lim_{n\to\infty} s_n(0) = \tfrac{1}{2}\{s(+0)+s(-0)\},$$

which is a constant.

On the other hand, $s_n(t)$ tends to $s(t)$ almost everywhere and it remains bounded, by theorem II, since $\frac{i\lambda(u)}{u-i} \in (F_5) \subset (F_2)$. Hence

$$\lim_{n\to\infty} \int_0^x s_n(t)\,dt = \int_0^x s(t)\,dt,$$

and the second member is an absolutely continuous function. Finally, $\int_0^x \sigma_n(t)\,dt$ tends also to a function, absolutely continuous in $(-a, a)$.

To see this, consider the subclass $\{3'\} \subset \{3\}$ of all functions bounded, R-integrable in $(-a, a)$ and vanishing outside this interval. Let $f(x) \in \{3'\}$ and let $g(u)$ be its Fourier transform. We have as usual

$$S_n(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)\,\sigma_n(x-\xi)\,d\xi = \frac{1}{2\pi}\int_{-a}^{+a} f(\xi)\,\sigma_n(x-\xi)\,d\xi.$$

By hypothesis $g(u)\,\lambda(u) \in (F_4)$, so that, by Hardy's theorem, the limit

$$\lim_{n\to\infty} 2\pi S_n(0) = \lim_{n\to\infty}\int_{-a}^{+a} f(\xi)\sigma_n(-\xi)\,d\xi = \lim_{n\to\infty}\int_{-a}^{+a} f(-\xi)\,\sigma_n(\xi)\,d\xi$$

exists for every $f(\xi) \in \{3'\}$. We conclude from the theorem of Verblunsky that $\lim_{n\to\infty}\int_0^x \sigma_n(t)\,dt$ exists and is a function absolutely continuous in $(-a, a)$.

II. We shall prove now that $s^*(x)$ is absolutely continuous in $(-\infty, +\infty$.)

Let ϵ be an arbitrary positive number. Since $s^*(x)$ together with $s(x)$ is of bounded variation in $(-\infty, +\infty)$, and since the variation of a function is an additive function of an interval, we can find a finite interval $(-a, a)$, $(a > 0)$, such that

$$V\{s^*(x), (-\infty, -a)\} + V\{s^*(x), (+a, +\infty)\} \leqslant \tfrac{1}{2}\epsilon. \tag{36}$$

On the other hand, since $s^*(x)$ is absolutely continuous in $(-a, a)$, we can find a number $\eta > 0$ such that,

$$(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \quad (\alpha_i < \beta_i, i = 1, 2, \ldots, n)$$

being any finite sequence of non-overlapping intervals contained in $(-a, a)$, the relation

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) \leqslant \eta$$

implies

$$\sum_{i=1}^{n} |\, s^*(\beta_i) - s^*(\alpha_i)\,| \leqslant \tfrac{1}{2}\epsilon. \tag{37}$$

Let now $(\gamma_1, \delta_1), \ldots, (\gamma_m, \delta_m)$ be any finite sequence of non-overlapping intervals whose total length is less than η. By dividing, if necessary, each of the two intervals containing the points $-a$ and $+a$, we can subdivide our intervals in three groups contained respectively in $(-\infty, -a)$, $(-a, +a)$, $(+a, +\infty)$. We have then by (36) and (37)

$$\sum_{i=1}^{m} | s^*(\delta_i) - s^*(\gamma_i) | \leqslant \epsilon.$$

This shows that $s^*(x)$ is absolutely continuous in $(-\infty, +\infty)$ and completes the proof.

THEOREM XII. *A necessary and sufficient condition that $\lambda(u)$ be a multiplicator* (1, 5) *is that $\lambda(u) \in (F_5)$. The condition is the same for multiplicators* (2, 5), (3, 5), (4, 5).

Proof. Sufficiency. We need only show that if $\lambda(u) \in (F_5)$ then $\lambda(u)$ is a multiplicator (1, 5).

Take an arbitrary function $g_1(u) \in (F_1)$. By theorem VIII, $\frac{u-i}{i} g_1(u)$ is a multiplicator (5, 1). Whence $\lambda(u) g_1(u) \frac{u-i}{i} \in (F_1)$. By theorem VI,

$$\lambda(u) g_1(u) \in (F_6) \subset (F_5).$$

This proves that $\lambda(u)$ is a multiplicator (1, 5).

Necessity. It will be sufficient to show that if $\lambda(u)$ is a multiplicator (4, 5), then necessarily $\lambda(u) \in (F_5)$.

I. We shall first show that if $\lambda(u)$ is a multiplicator (4, 5) it is also a multiplicator (1, 5).

Take an arbitrary $g_1(u) \in (F_1)$. By theorem VIII, $\frac{u-i}{i} g_1(u)$ is a multiplicator (5, 1). Therefore the function $\lambda(u) \frac{u-i}{i} g_1(u)$ is a multiplicator (4, 1). By theorem IX, $\lambda(u) g_1(u) \in (F_5)$, which shows that $\lambda(u)$ is a multiplicator (1, 5).

II. Let now $\lambda(u)$ be a multiplicator (4, 5) and so, by I, a multiplicator (1, 5). Take an arbitrary function $g_4(u) \in (F_4)$. By theorem VIII, $\frac{u-i}{i} g_4(u)$ is a multiplicator (5, 4). Therefore the function $\lambda(u) \frac{u-i}{i} g_4(u)$ is a multiplicator (1, 4). By theorem X, $\lambda(u) \frac{u-i}{i} g_4(u) \in (F_2)$, which proves that $\frac{u-i}{i} \lambda(u)$ is a multiplicator (4, 2). Whence, by theorem IX, $\lambda(u) \in (F_5)$.

THEOREM XIII. *The conditions characterizing multiplicators* $(j, 6)$ $(j = 1, 2, 3, 4, 5, 6)$ *are those characterizing multiplicators* $(j, 1)$ *in which* $\dfrac{\lambda(u)\, i}{u-i}$ *is replaced by* $\lambda(u)$. *The conditions characterizing multiplicators* $(6, k)$ $(k = 1, 2, 3, 4, 5, 6)$ *are those characterizing multiplicators* $(1, k)$ *in which* $\lambda(u)$ *is replaced by* $\dfrac{i\lambda(u)}{u-i}$.

Proof. It is an immediate consequence of theorem VI. We shall give it, for example, for multiplicators $(6, k)$.

Let $\lambda(u)$ be a multiplicator $(6, k)$. Take an arbitrary function $g_1(u) \in (F_1)$. We have

$$\frac{i}{u-i} g_1(u) \in (F_6), \quad g_1(u) \frac{i}{u-i} \lambda(u) \in (F_k),$$

which proves that $\dfrac{i\lambda(u)}{u-i}$ is a multiplicator $(1, k)$.

Conversely, let $\dfrac{i\lambda(u)}{u-i}$ be a multiplicator $(1, k)$. Take an arbitrary function $g_6(u) \in (F_6)$. We have

$$\frac{u-i}{i} g_6(u) \in (F_1), \quad \frac{u-i}{i} g_6(u) \frac{i\lambda(u)}{u-i} = g_6(u)\, \lambda(u) \in (F_k),$$

which proves that $\lambda(u)$ is a multiplicator $(6, k)$.

Farouk I University
Alexandria, Egypt

RATIONAL SOLUTIONS OF THE MATRIX EQUATION $XA = BX$

By H. O. FOULKES

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1. *Introduction*

The problem of determining the commutant of two matrices A and B has been discussed by many writers.* It is well known that a necessary and sufficient condition for $XA = BX$ to have a non-singular solution is that $A - \lambda I$ and $B - \lambda I$ should have the same invariant factors, but we are not necessarily restricted to such cases in this paper. If the $n \times n$ matrices A and B have reduced characteristic functions $a(\lambda)$ and $b(\lambda)$ respectively, then polynomials $f(\lambda)$ and $g(\lambda)$ exist such that

$$a(\lambda) f(\lambda) + b(\lambda) g(\lambda) = h(\lambda),$$

where $h(\lambda)$ is the greatest common divisor of $a(\lambda)$ and $b(\lambda)$. Hence

$$Xh(A) = Xa(A) f(A) + Xb(A) g(A).$$

Clearly, $a(A)$ is zero, and if X is an $n \times n$ matrix satisfying $XA = BX$, then $XA^r = B^r X$ for r any positive integer and so $Xb(A) = b(B) X$, which is also zero since $b(B) = 0$. Hence† $Xh(A) = 0$ and similarly $h(B) X = 0$. In the extreme case, when $a(\lambda)$ and $b(\lambda)$ are relatively prime, $h(\lambda)$ and $h(A)$ are respectively unity and the unit $n \times n$ matrix, so that X is necessarily the null matrix. Otherwise, if A and B have one or more latent roots in common, then X can be non-zero, and it has been proved‡ that the number of linearly independent solutions of $XA = BX$ is Σe_{ij}, where e_{ij} is the degree of the greatest common divisor of the invariant factor p_i of $A - \lambda I$ and the

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* C. E. Cullis, *Matrices and Determinoids*, 3, part I (Cambridge, 1925), Chaps. XXVII, XXVIII. H. W. Turnbull and A. C. Aitken, *An introduction to the theory of canonical matrices* (London, 1932), Chap. X. C. C. MacDuffee, *The theory of matrices* (Berlin, 1933), Chap. VIII. The last-named author gives many other references.

† The converse is not necessarily true; if $Xh(A) = h(B) X = 0$, it does not follow that $XA = BX$.

‡ C. C. MacDuffee, *loc. cit.*, 90–92, ascribes the theorem to F. Cecioni (1909), and G. Frobenius (1910).

invariant factor q_j of $B-\lambda I$. In determining the number of solutions, an invariant factor repeated r times is to be regarded as r invariant factors.

In this paper we suppose A and B are given $n \times n$ matrices with elements rational numbers, so that their characteristic equations and invariant factors can be determined. We give a method of obtaining sets of rational matrices satisfying $XA = BX$, this method giving the exact number of solutions prescribed by the Cecioni-Frobenius theorem.

When A and B have some latent roots in common, as they must have if X is to be non-zero, algebraic equations can be written down which are satisfied by both A and B. Such an equation will contain the reduced characteristic equations of A and B as factors, but need not be their product. In § 6 a method of obtaining sets of rational solutions of $XA = BX$ is given, the solutions being written down in terms of the elements of A and B and the coefficients of the particular algebraic equation chosen. The method is more limited than that of the earlier part of the paper, in that the solutions obtained are not all linearly independent, and the sets may not be complete.

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2. *Outline of method*

If C is a square n-rowed matrix and it is possible to find non-zero matrices ϕ and ψ such that $\phi A = C\phi$ and $B\psi = \psi C$, then

$$\psi\phi A = \psi C\phi = B\psi\phi.$$

Hence $\psi\phi$ will be a solution of $XA = BX$. We will choose C appropriate to the irreducible factors (or the powers of the irreducible factors) of the greatest common divisors d_{ij} of the invariant factors p_i of $A-\lambda I$ and q_j of $B-\lambda I$, and by making C rational we obtain rational matrices ϕ, ψ and $\psi\phi$. We show that every irreducible factor (or power of an irreducible factor) of each d_{ij} gives rise to a number of linearly independent rational solutions $\psi\phi$ equal to the degree of the factor.

Throughout the paper, questions of rationality, reducibility, and linear dependence are to be understood to be relative to the field of the elements of A and B.

3. *Solutions when d_{ij} is a product of rational, unrepeated, linear factors*

When the two invariant factors p_i and q_j have a common rational linear factor $\lambda-\alpha$, we may take C to be the n-rowed square matrix with α in the first place on the leading diagonal and zeroes elsewhere. Then $\phi A = C\phi$ is satisfied by a square n-rowed matrix ϕ which has all its rows zero except

the first, Y_1, which must satisfy $Y_1(A-\alpha I) = 0$. Also $B\psi = \psi C$ is satisfied by a square n-rowed matrix ψ which has all its columns zero except the first, Z_1, which must satisfy $(B-\alpha I)\,Z_1 = 0$. Then Y_1 and Z_1 are respectively a left pole of A and a right pole of B corresponding to the common latent root α, and we have $Y_1 A = \alpha Y_1$ and $BZ_1 = \alpha Z_1$, giving

$$Z_1 Y_1 A = \alpha Z_1 Y_1 = BZ_1 Y_1,$$

thus giving $Z_1 Y_1$ as a solution of $XA = BX$. Such a solution is a product of a column and a row vector and is thus a square matrix of unit rank.

By considering the classical canonical form of A, it can be verified in any given case that the number of linearly independent rows Y_1 is equal to the nullity n_A of $A-\alpha I$, and is equal to the number* of invariant factors of $A-\lambda I$ which contain $\lambda-\alpha$. The number of linearly independent columns Z_1 is n_B, the nullity of $B-\alpha I$, and is equal to the number of invariant factors of $B-\lambda I$ which are divisible by $\lambda-\alpha$. Hence we have $n_A n_B$ rational square matrices $Z_1 Y_1$ satisfying $XA = BX$. These are not null matrices and they are linearly independent, since any linear relation between them implies a linear relation between the n_A independent rows or the n_B independent columns. When α is the only common latent root of A and B and is not repeated in any d_{ij}, then there are $n_A n_B$ divisors d_{ij} each equal to $\lambda-\alpha$ and each contributing one of the independent solutions $Z_1 Y_1$. Hence we obtain the correct number of solutions required by the Cecioni-Frobenius theorem in this case.

Also, when A and B have several distinct, rational, latent roots in common, none appearing as repeated roots of any d_{ij}, we can construct a set of linearly independent solutions corresponding to each distinct root. These sets are linearly independent, for any linear relation between the solutions in different sets implies a linear relation between the poles belonging to different latent roots. It has been proved† that no such relation exists. Furthermore, the $\Sigma n_A n_B$ products $\psi\phi$ (or $Z_1 Y_1$), summed for all common latent roots of A and B, constitute a complete set of solutions of $XA = BX$ in the case when every d_{ij} is the product of rational, unrepeated, linear factors; each d_{ij} contributes e_{ij} solutions, where e_{ij} is the degree of d_{ij}. This may be verified by supposing $A-\lambda I$ has, for example, three invariant factors with latent roots

$$\alpha, \alpha, \beta, \beta, \gamma, \delta; \quad \alpha, \alpha, \beta, \beta, \gamma; \quad \alpha, \beta,$$

and $B-\lambda I$ has four invariant factors with latent roots

$$\alpha, \beta, \gamma, \gamma, \delta, \epsilon, \epsilon; \quad \alpha, \gamma, \gamma; \quad \alpha, \gamma; \quad \gamma.$$

* This is a simple case of a more general result given by C. E. Cullis, *loc. cit.*, 358–359.

† Cullis, *loc. cit.*, 396–398.

Then for $\alpha, \beta, \gamma, \delta$ respectively we have $n_A n_B$ equal to 3×3, 3×1, 2×4, 1×1, giving $\Sigma n_A n_B = 21$. Also,

$$\begin{aligned} e_{11} &= 4, & e_{12} &= 2, & e_{13} &= 2, & e_{14} &= 1,\\ e_{21} &= 3, & e_{22} &= 2, & e_{23} &= 2, & e_{24} &= 1,\\ e_{31} &= 2, & e_{32} &= 1, & e_{33} &= 1, & e_{34} &= 0, \end{aligned}$$

giving $\Sigma e_{ij} = 21$.

If, however, one of the greatest common divisors d_{ij} has repeated linear factors, then $\Sigma n_A n_B < \Sigma e_{ij}$. Thus, if $A-\lambda I$ has the single invariant factor $(\lambda-\alpha)^3$, and $B-\lambda I$ has two invariant factors $(\lambda-\alpha)^2$, $\lambda-\alpha$, then $n_A n_B = 2$ and $\Sigma e_{ij} = 3$. Such cases are dealt with in § 5.

We conclude this section with an example to illustrate the method:

Example i:

$$A = \begin{bmatrix} 1 & -1 & -5 & 5\\ 1 & -1 & -11 & 11\\ -2 & -1 & -4 & 6\\ -2 & -2 & -10 & 12 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -6 & 8 & -2\\ 9 & -4 & 1 & -4\\ 4 & -4 & 5 & -2\\ -8 & -4 & 17 & 4 \end{bmatrix}.$$

There are two invariant factors of $A-\lambda I$, namely, $(\lambda-1)(\lambda-2)(\lambda-3)$ and $\lambda-2$, and two invariant factors of $B-\lambda I$, each being $(\lambda-2)(\lambda-3)$. Hence $A-2I$ and $A-3I$ are respectively of nullities 2 and 1, giving

$$\phi_1 = [0, 2, 8\ -7] \quad\text{and}\quad \phi_2 = [2, 0, 0, -1],$$

corresponding to the latent root 2, and $\phi_3 = [1, 2, 9, -9]$ corresponding to the latent root 3. We write the ϕ matrices as row vectors since all the other rows are zero. Similarly, $B-2I$ and $B-3I$ are each of nullity 2, giving the column vectors $\psi_1 = \{0, 5, 2, -7\}$, $\psi_2 = \{2, -1, 0, 6\}$
and $\psi_3 = \{5, 7, 4, 0\}$, $\psi_4 = \{0, 3, 1, -5\}$

respectively. Hence we have a complete set of six linearly independent solutions of $XA = BX$,

$$\psi_1\phi_1 = \begin{bmatrix} . & . & . & .\\ . & 10 & 40 & -35\\ . & 4 & 16 & -14\\ . & -14 & -56 & 49 \end{bmatrix}, \quad \psi_1\phi_2 = \begin{bmatrix} . & . & . & .\\ 10 & . & . & -5\\ 4 & . & . & -2\\ -14 & . & . & 7 \end{bmatrix},$$

$$\psi_2\phi_1 = \begin{bmatrix} . & 4 & 16 & -14\\ . & -2 & -8 & 7\\ . & . & . & .\\ . & 12 & 48 & -42 \end{bmatrix}, \quad \psi_2\phi_2 = \begin{bmatrix} 4 & . & . & -2\\ -2 & . & . & 1\\ . & . & . & .\\ 12 & . & . & -6 \end{bmatrix},$$

$$\psi_3\phi_3 = \begin{bmatrix} 5 & 10 & 45 & -45\\ 7 & 14 & 63 & -63\\ 4 & 8 & 36 & -36\\ . & . & . & . \end{bmatrix}, \quad \psi_4\phi_3 = \begin{bmatrix} . & . & . & .\\ 3 & 6 & 27 & -27\\ 1 & 2 & 9 & -9\\ -5 & -10 & -45 & 45 \end{bmatrix}.$$

4. *Solutions when d_{ij} has a non-repeated irreducible factor of degree m*

The method of § 3 is applicable when the common roots are irrational, but as the solutions would also be irrational we now consider the case when d_{ij} has a non-repeated irreducible factor

$$f(\lambda) = \lambda^m - b_1\lambda^{m-1} - b_2\lambda^{m-2} - \dots - b_m.$$

We take C of § 2 to be the n-rowed square matrix with the m-rowed square matrix

$$\begin{bmatrix} . & . & \dots & . & b_m \\ 1 & . & \dots & . & b_{m-1} \\ . & 1 & \dots & . & b_{m-2} \\ \dots & \dots & \dots & \dots & \dots \\ . & . & \dots & . & b_1 \end{bmatrix}$$

in the leading position on the principal diagonal and zeroes elsewhere. We find that ϕ has its last $n-m$ rows zero, and its first m rows $Y_1, Y_2, \dots, Y_m$ are given by

$$\begin{aligned} b_m Y_m &= Y_1 A, \\ Y_1 + b_{m-1} Y_m &= Y_2 A, \\ Y_2 + b_{m-2} Y_m &= Y_3 A, \\ &\dots\dots \\ Y_{m-1} + b_1 Y_m &= Y_m A, \end{aligned}$$

from which it readily follows that

$$\begin{aligned} Y_{m-1} &= Y_m(A - b_1), \\ Y_{m-2} &= Y_{m-1}A - b_2 Y_m = Y_m(A^2 - b_1 A - b_2), \\ Y_{m-3} &= Y_{m-2}A - b_3 Y_m = Y_m(A^3 - b_1 A^2 - b_2 A - b_3), \\ &\dots\dots \\ Y_1 &= Y_2 A - b_{m-1} Y_m = Y_m(A^{m-1} - b_1 A^{m-2} - b_2 A^{m-3} - \dots - b_{m-1}). \end{aligned}$$

Also, from § 1, each Y satisfies $Yf(A) = 0$, and since $f(\lambda)$ is irreducible, Y cannot be annihilated by right-hand multiplication by any polynomial in A of degree less than m. It seems most convenient to derive the rows successively from the mth row.

Since $C^k\phi A = CC^k\phi$, from any solution ϕ of $\phi A = C\phi$ we can obtain a set of matrices

$$\phi, \quad C\phi, \quad C^2\phi, \quad \dots, \quad C^{m-1}\phi$$

which will also be solutions. Further, these solutions will be linearly independent, for a linear relation

$$l_1\phi + l_2 C\phi + l_3 C^2\phi + \dots + l_m C^{m-1}\phi = 0$$

leads to

$$\phi(l_1 + l_2 A + l_3 A^2 + \dots + l_m A^{m-1}) = 0,$$

which necessitates the annihilation of each row of ϕ by a polynomial in A of degree less than m.

Similarly, consideration of $B\psi = \psi C$ leads to ψ as a square n-rowed matrix with its last $n-m$ columns zero and its first m columns $Z_1, Z_2, \ldots, Z_m$ satisfying

$$Z_2 = BZ_1, \quad Z_3 = BZ_2, \quad \ldots, \quad Z_m = BZ_{m-1}$$

and

$$b_m Z_1 + b_{m-1} Z_2 + b_{m-2} Z_3 + \ldots + b_1 Z_m = BZ_m,$$

from which we find that each column Z satisfies $f(B)\,Z = 0$. Since $f(B)$ is irreducible, Z cannot be annihilated by left multiplication by a polynomial in B of degree less than m. It follows that

$$\psi, \quad \psi C, \quad \psi C^2, \quad \ldots, \quad \psi C^{m-1}$$

constitute a set of m linearly independent solutions of $B\psi = \psi C$.

We are thus led to a set of products $\psi C^k \phi$, where $k = 0, 1, 2, \ldots, 2(m-1)$, each being, by § 2, a rational solution of $XA = BX$. Of these,

$$\psi\phi, \quad \psi C\phi, \quad \psi C^2\phi, \quad \ldots, \quad \psi C^{m-1}\phi$$

will be linearly independent, for they can be written as*

$$\psi\phi, \quad \psi\phi A, \quad \psi\phi A^2, \quad \ldots, \quad \psi\phi A^{m-1},$$

any linear relation between them involving the annihilation of $\psi\phi$ by post-multiplication by a polynomial in A of degree less than m; since $\psi\phi$ can be considered as a column matrix in which each term is a linear function of $Y_1, \ldots, Y_m$ with scalar coefficients, it is impossible to annihilate such a function by a polynomial in A of degree less than m. Hence, corresponding to the factor $f(\lambda)$ of d_{ij}, we obtain m linearly independent solutions of $XA = BX$.

We note that ψ and ϕ are respectively rectangular $n \times m$ and $m \times n$ matrices in effect, for ψ may be written as $[Z_1, Z_2, \ldots, Z_m]$ and ϕ as $\{Y_1, Y_2, \ldots, Y_m\}$, and their product $\psi\phi$ is a square n-rowed matrix of rank m. It may be verified directly that $\psi\phi A = B\psi\phi$, using the recurrence relations between the row vectors Y and the column vectors Z.

In general, $f(\lambda)$ will be a non-repeated factor of more than one d_{ij}. The number of linearly independent sets of m solutions ϕ of $\phi A = C\phi$ is equal to the number n_A of invariant factors of $A - \lambda I$ which contain $f(\lambda)$, since the total number of linearly independent rows satisfying $Yf(A) = 0$ is equal to mn_A, the nullity of $f(A)$. The number of linearly independent sets of m solutions of $B\psi = \psi C$ is n_B, the number of invariant factors of $B - \lambda I$ which contain $f(\lambda)$. Hence we can obtain $n_A n_B$ sets, each of m products $\psi C^k \phi$ $(k = 0, 1, 2, \ldots, m-1)$, corresponding to $f(\lambda)$, each product being a rational solution of $XA = BX$. We have shown that there is no linear relation between the m products in any set. Any such relation between products in different sets is ruled out by the linear independence of the mn_A solutions of $\phi A = C\phi$ or of the mn_B solutions of $B\psi = \psi C$.

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* They may also be written as $\psi\phi, B\psi\phi, B^2\psi\phi, \ldots, B^{m-1}\psi\phi$.

Further, when there arise several irreducible functions $f(\lambda)$ of various degrees, the sets of solutions corresponding to the different functions will all be linearly independent since they are annihilated by different irreducible polynomials in A or B. Hence, corresponding to every d_{ij} of degree e_{ij}, consisting of a product of irreducible non-repeated factors, there will be e_{ij} linearly independent rational solutions of $XA = BX$, the number required by the Cecioni-Frobenius theorem.

The case when d_{ij} contains a repeated irreducible factor is considered in § 5.

Example ii:

$$A = \begin{bmatrix} -2 & 1 & 2 & 2 \\ 2 & -4 & -8 & -6 \\ 0 & 1 & 3 & 2 \\ -2 & 2 & 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 & 2 & 4 \\ 2 & 2 & 0 & -4 \\ -2 & 1 & 2 & 6 \\ -1 & 0 & 1 & 2 \end{bmatrix}.$$

The invariant factors of $A-\lambda I$ are $(\lambda^2-2\lambda-2)(\lambda+1)$ and $\lambda+1$; those of $B-\lambda I$ are $\lambda^2-2\lambda-2$ repeated. If $f(\lambda) = \lambda^2-2\lambda-2$, then

$$f(A) = \begin{bmatrix} 4 & -2 & -2 & -4 \\ -4 & 4 & 4 & 6 \\ -2 & 1 & 1 & 2 \\ 6 & -4 & -4 & -7 \end{bmatrix} \quad \text{and} \quad f(B) = 0.$$

As a row vector satisfying $Yf(A) = 0$ we may take $Y_2 = [1, 0, 2, 0]$, from which

$$\phi_1 = \begin{bmatrix} -4 & 3 & 4 & 6 \\ 1 & 0 & 2 & 0 \end{bmatrix} \quad \text{and} \quad C\phi_1 = \phi_1 A = \begin{bmatrix} 2 & 0 & 4 & 0 \\ -2 & 3 & 8 & 6 \end{bmatrix}$$

are linearly independent solutions of $\phi A = C\phi$. As a column vector satisfying $f(B)Z = 0$ we may take $Z_1 = \{1, 0, 0, 0\}$, giving

$$\psi_1 = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & -2 & -1 \end{Bmatrix} \quad \text{and} \quad \psi_1 C = B\psi_1 = \begin{Bmatrix} -2 & 2 & -2 & -1 \\ -2 & 4 & -4 & -2 \end{Bmatrix},$$

so that corresponding to d_{11} we have two solutions of $XA = BX$, viz.

$$\psi_1\phi_1 = \begin{bmatrix} -6 & 3 & . & 6 \\ 2 & . & 4 & . \\ -2 & . & -4 & . \\ -1 & . & -2 & . \end{bmatrix},$$

and

$$\psi_1 C\phi_1 = \psi_1\phi_1 A = B\psi_1\phi_1 = \begin{bmatrix} 6 & -6 & -12 & -12 \\ -4 & 6 & 16 & 12 \\ 4 & -6 & -16 & -12 \\ 2 & -3 & -8 & -6 \end{bmatrix}.$$

Corresponding to d_{12} we have ϕ_1 and $C\phi_1$ as above, and the first column of ψ_2 may be taken as $\{0, 1, 0, 0\}$, giving

$$\psi_2 = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{Bmatrix} \quad \text{and} \quad \psi_2 C = B\psi_2 = \begin{Bmatrix} 1 & 2 & 1 & 0 \\ 2 & 6 & 2 & 0 \end{Bmatrix},$$

from which we obtain two solutions

$$\psi_2\phi_1 = \begin{bmatrix} 1 & . & 2 & . \\ -2 & 3 & 8 & 6 \\ 1 & . & 2 & . \\ . & . & . & . \end{bmatrix}$$

and

$$\psi_2 C\phi_1 = \psi_2\phi_1 A = B\psi_2\phi_1 = \begin{bmatrix} -2 & 3 & 8 & 6 \\ -2 & 6 & 20 & 12 \\ -2 & 3 & 8 & 6 \\ . & . & . & . \end{bmatrix},$$

which complete the set of four required in this example.

5. *Solutions when d_{ij} has repeated factors*

If d_{ij} has an irreducible factor $f(\lambda)$, of degree m, repeated k times, e.g.

$$\{f(\lambda)\}^k = \lambda^{mk} - b_1\lambda^{mk-1} - b_2\lambda^{mk-2} - \ldots - b_{mk},$$

then we take C, as in § 4, to be the n-rowed square matrix with the mk-rowed square canonical matrix in the leading position on the principal diagonal and zeroes elsewhere. To bring this case into line with the method of § 4, we have to show (a) that there is a set of mk linearly independent solutions of $XA = BX$ corresponding to each d_{ij} containing $\{f(\lambda)\}^k$, (b) that sets obtained from different d_{ij}'s are linearly independent, (c) that if various d_{ij}'s contain other powers k_1, k_2, etc., of $f(\lambda)$, the sets of mk_1, mk_2, etc. solutions will also be linearly independent, and (d) that these solutions obtained from powers of $f(\lambda)$ are linearly independent of any solutions obtained from powers of other irreducible functions of λ occurring as factors of some of the d_{ij}'s.

As in § 4, each ϕ consists of mk rows, each satisfying $Y\{f(A)\}^k = 0$. We choose Y_{mk} so that $Y_{mk}\{f(A)\}^{k-1} \neq 0$. Then the remaining rows are obtained successively from Y_{mk} as in § 4, and are linearly independent since any linear relation between them involves annihilation of Y_{mk} by post-multiplying by a polynomial in A of degree less than mk. It is therefore possible to find a matrix U, of n rows and mk columns, such that ϕU is the square mk-rowed unit matrix. Similarly, ψ consists of mk columns, each satisfying $\{f(B)\}^k Z = 0$. We choose Z_1 so that $\{f(B)\}^{k-1} Z_1 \neq 0$, and derive the remaining columns successively from Z_1 as in § 4. These columns will be linearly

independent, and so there exists a matrix V, of n columns and mk rows, such that $V\psi$ is the square mk-rowed unit matrix. The mk products

$$\psi\phi, \quad \psi C\phi, \quad \psi C^2\phi, \quad \ldots, \quad \psi C^{mk-1}\phi$$

will all be solutions of $XA = BX$, and will be linearly independent,* since any linear relation between them,

$$\psi(l_0 + l_1 C + l_2 C^2 + \ldots + l_{mk-1} C^{mk-1})\phi = 0,$$

leads to $$V\psi(l_0 + l_1 C + l_2 C^2 + \ldots + l_{mk-1} C^{mk-1})\phi U = 0,$$

which involves the vanishing of a polynomial in C of degree $mk-1$, since $V\psi$ and ϕU are unit matrices. The canonical matrix C cannot satisfy any equation of degree less than mk. We have thus proved (a).

To prove (b) we have to consider the rows of ϕ in greater detail. The case of repeated factors differs from § 4 in that certain linear functions of the mk rows are annihilated by post-multiplying by certain polynomials in A of degree less than mk. None of the m rows

$$Y_{mk}, \quad Y_{mk-1}, \quad \ldots, \quad Y_{mk-m+1},$$

nor any linear function of them, can satisfy $Y\{f(A)\}^{k-1} = 0$, for this would involve annihilation of Y_{mk} by a polynomial in A of degree less than mk, in virtue of the recurrence relations of § 4. The next row, Y_{mk-m}, is, however, such that $$L_1 = l_0 Y_{mk} + l_1 Y_{mk-1} + \ldots + l_{m-1} Y_{mk-m+1} + l_m Y_{mk-m}$$

will satisfy $L_1\{f(A)\}^{k-1} = 0$, if the scalar constants l_i are chosen so that

$$l_0 + l_1(A - b_1) + l_2(A^2 - b_1 A - b_2) + \ldots + l_m(A^m - b_1 A^{m-1} - \ldots - b_m)$$

is identically equal to $f(A)$. Similarly, there exists a linear function of

$$Y_{mk}, \quad Y_{mk-1}, \quad \ldots; \qquad Y_{mk-m}, \quad Y_{mk-m-1},$$

which satisfies $L\{f(A)\}^{k-1} = 0$, and so on for each row up to $Y_{mk-2m+1}$. There can, however, be no linear function of these $2m$ rows which will satisfy $L\{f(A)\}^{k-2} = 0$, because of the initial choice of Y_{mk} as not satisfying $Y_{mk}\{f(A)\}^{k-1} = 0$. Thus the first m rows of ϕ (starting from Y_{mk}) give m vectors satisfying $L\{f(A)\}^{k-1} = 0$ and not satisfying $L\{f(A)\}^{k-2} = 0$; in the same way the third set of m rows of ϕ (starting from Y_{mk-2m}) leads to m vectors satisfying $L\{f(A)\}^{k-2} = 0$ and not satisfying $L\{f(A)\}^{k-3} = 0$, and so on for the remaining rows of ϕ. Each of the mk linearly independent rows of ϕ defines a vector satisfying $L\{f(A)\}^k = 0$, those in the rth set not satisfying $L\{f(A)\}^{k-r} = 0$.

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* This proof of the independence of the mk solutions was suggested by Mr D. E. Littlewood.

Now suppose $\{f(\lambda)\}^k$ occurs in two invariant factors a_1 and a_2 of $A-\lambda I$ and in b_1 of $B-\lambda I$. Then both d_{11} and d_{21} contain $\{f(\lambda)\}^k$, and we should obtain mk independent solutions of $XA = BX$ corresponding to each. We construct those corresponding to d_{11} by determining ψ_1 and ϕ_1 as already described, and forming the products

$$\psi_1 C^i \phi_1 \quad (i = 0, 1, \ldots, mk-1).$$

We next form ϕ_2, corresponding to d_{21}, by choosing an initial Y_{mk} satisfying $Y_{mk}\{f(A)\}^k = 0$ and not satisfying $Y_{mk}\{f(A)\}^{k-1} = 0$, and at the same time linearly independent of the rows of ϕ_1. That this is possible is known from theorems* on the nullity of $\{f(A)\}^k$, which are equivalent to the statement that we can determine a complete set of unconnected solutions of $L\{f(A)\}^k = 0$ of which mn_A are solutions of $L\{f(A)\}^k = 0$ and not of $L\{f(A)\}^{k-1} = 0$, where n_A is the number of invariant factors of $A-\lambda I$ which contain $\{f(A)\}^k$. It is thus possible to choose the initial row Y_{mk} of ϕ_2 to satisfy the above conditions, and when this is done we obtain a further set of mk linearly independent solutions

$$\psi_1 C^i \phi_2 \quad (i = 0, 1, \ldots, mk-1)$$

of $XA = BX$. Similarly, when $\{f(\lambda)\}^k$ occurs in a third invariant factor of $A-\lambda I$, we can, by proper choice of the initial row of ϕ_3, obtain another set of mk linearly independent solutions

$$\psi_1 C^i \phi_3 \quad (i = 0, 1, \ldots, mk-1).$$

Discussion of the columns of ψ follows the same lines as that of the rows of ϕ, and it will follow that by proper choice of the various ϕ's and ψ's as regards the independence of the initial rows Y_{mk} of the ϕ's and the initial columns Z_1 of the ψ's, every d_{ij} containing $\{f(\lambda)\}^k$ gives a set of mk unconnected solutions and the various sets are also independent.

In the same way, d_{ij}'s containing other powers k_1, k_2, etc., of $f(\lambda)$ will give rise to the correct number mk_1, mk_2, etc. of solutions if ϕ and ψ are properly chosen in each case. The ϕ (or ψ) corresponding to any invariant factor must have its initial row Y_{mk} (or column Z_1) linearly independent of any rows (or columns) used for another invariant factor. That there are sufficient unconnected rows (or columns) for the purpose is known from the theorems on the nullity of powers of $f(A)$ and $f(B)$.

Powers of other irreducible polynomials may also occur in some of the d_{ij}'s. The solutions corresponding to these will clearly be independent of those obtained for powers of $f(\lambda)$, since the various sets have as annihilating polynomials powers of different irreducible functions of A or B.

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* Cullis, *loc. cit.*, 362, 387.

6. *Solutions involving the coefficients of an algebraic equation satisfied by A and B*

The method already outlined gives the complete set of rational commutants in any given case. We now indicate a method of obtaining sets of rational commutants when A and B satisfy the same algebraic equation. The matrices obtained are not all linearly independent, but a linearly independent set may sometimes be picked out by inspection in numerical cases of comparatively low order.

If the minimum functions of A and B are divisors of

$$F_m(\lambda) = \lambda^m - b_1\lambda^{m-1} - b_2\lambda^{m-2} - \ldots - b_m \quad (m \leqslant n),$$

then A and B will satisfy the algebraic equation $F_m(\lambda) = 0$, and we may choose C appropriate to $F_m(\lambda)$ instead of an irreducible factor (or power of an irreducible factor) of one of the d_{ij}'s. As in § 4, we choose Y_m to satisfy $Y\{F_m(A)\} = 0$, which means that Y_m is arbitrary. We can take it to be any one of the n unit-row vectors I_i with unity in the ith position and zeroes elsewhere. Having constructed ϕ_1 with I_1 as its mth row, we do not proceed as in § 4 to find $C\phi_1$, $C^2\phi_1$, etc., but we form n linearly independent matrices ϕ_i each with its mth row equal to I_i. From the mode of construction of the successive rows it is clear that the remaining non-zero rows, $Y_{m-1}, Y_{m-2}, \ldots, Y_1$ of the ϕ-matrices are expressed in terms of some of $b_1, b_2, \ldots, b_m$ and the elements of A. Thus, when $m = 3$,

$$\phi_j = \{I_j(A^2 - b_1A - b_2),\ I_j(A - b_1),\ I_j,\ 0,\ \ldots,\ 0\}.$$

Similarly, the first column of ψ, satisfying $\{F_m(B)\}Z = 0$, is arbitrary and may be taken to be any one of the n unit-column vectors J_i. We form n linearly independent matrices ψ_i each with J_i as its first column. Thus, for $m = 3$,

$$\psi_i = [J_i,\ BJ_i,\ B^2J_i,\ 0,\ \ldots,\ 0].$$

Hence we obtain n^2 solutions of $XA = BX$ of the type $\psi_i\phi_j$. They are not all linearly independent. The number of unconnected solutions obtained depends on the relations between the minimum functions of A and B and $F_m(\lambda)$, and has to be investigated in any particular case.

Solutions obtained in this way have the merit of being written down by comparatively simple rules in terms of the elements of A and B and the coefficients of $F_m(\lambda)$. For example, when A and B satisfy the same quadratic equation, we have

$$\psi_i = [J_i, BJ_i, 0, \ldots, 0] \quad \text{and} \quad \phi_j = \{I_j(A - b_1), I_j, 0, \ldots, 0\},$$

This gives a simple rule for writing down X_{ij}, viz. X_{ij} is the sum of two matrices, the first having its ith row equal to the jth row of $A-b_1$ with zeroes elsewhere, and the second having its jth column equal to the ith column of B with zeroes elsewhere.

As an example, take

$$A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ -3 & -1 & -3 & -3 \\ 1 & 1 & 3 & 1 \\ 2 & 2 & 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -6 & 8 & -2 \\ 9 & -4 & 1 & -4 \\ 4 & -4 & 5 & -2 \\ -8 & -4 & 17 & 4 \end{bmatrix}.$$

Each matrix satisfies $\lambda^2-5\lambda+6=0$, the remaining invariant factors of $A-\lambda I$ being $\lambda-2$, $\lambda-2$, and of $B-\lambda I$, $\lambda^2-5\lambda+6$. By the above rule, we find

$$X_{11} = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 9 & . & . & . \\ 4 & . & . & . \\ -8 & . & . & . \end{bmatrix}, \quad X_{12} = \begin{bmatrix} -3 & -1 & -3 & -3 \\ . & 9 & . & . \\ . & 4 & . & . \\ . & -8 & . & . \end{bmatrix},$$

$$X_{13} = \begin{bmatrix} 1 & 1 & 3 & 1 \\ . & . & 9 & . \\ . & . & 4 & . \\ . & . & -8 & . \end{bmatrix}, \quad X_{14} = \begin{bmatrix} 2 & 2 & 2 & 4 \\ . & . & . & 9 \\ . & . & . & 4 \\ . & . & . & -8 \end{bmatrix},$$

$$X_{21} = \begin{bmatrix} -6 & . & . & . \\ -6 & 1 & 1 & 1 \\ -4 & . & . & . \\ -4 & . & . & . \end{bmatrix}, \quad X_{22} = \begin{bmatrix} . & -6 & . & . \\ -3 & -10 & -3 & -3 \\ . & -4 & . & . \\ . & -4 & . & . \end{bmatrix},$$

$$X_{23} = \begin{bmatrix} . & . & -6 & . \\ 1 & 1 & -6 & 1 \\ . & . & -4 & . \\ . & . & -4 & . \end{bmatrix}, \quad X_{24} = \begin{bmatrix} . & . & . & -6 \\ 2 & 2 & 2 & -5 \\ . & . & . & -4 \\ . & . & . & -4 \end{bmatrix},$$

these being seen to be independent by inspection of the last two rows. The remaining eight matrices X_{ij} ($i=3,4$; $j=1,2,3,4$) are linear functions of these, but since eight solutions are required by the Cecioni-Frobenius theorem, our solution is complete.

Similarly, when $m=3$, we have

$$X_{ij} = J_iI_j(A^2-b_1A-b_2)+BJ_iI_j(A-b_1)+B^2J_iI_j.$$

In a numerical case we could obtain X_{ij} by adding three matrices, the first having its ith row equal to the jth row of $A^2-b_1A-b_2$ with zeroes elsewhere, the second being the product of the ith column of B by the jth row of $A-b_1$, and the third having its jth column equal to the ith column of B^2, with zeroes elsewhere.

Applying this rule to Example ii of § 4, we find that

$$\lambda^3-\lambda^2-4\lambda-2\equiv(\lambda^2-2\lambda-2)(\lambda+1)=0$$

is satisfied by A and B, and having written down

$$A^2-A-4=\begin{bmatrix} . & -1 & . & -2 \\ -2 & -2 & -4 & . \\ -2 & 2 & 2 & 4 \\ 4 & -2 & . & -6 \end{bmatrix}, \quad A-1=\begin{bmatrix} -3 & 1 & 2 & 2 \\ 2 & -5 & -8 & -6 \\ . & 1 & 2 & 2 \\ -2 & 2 & 4 & 2 \end{bmatrix},$$

$$B^2=\begin{bmatrix} -2 & 2 & 4 & 8 \\ 4 & 6 & . & -8 \\ -4 & 2 & 6 & 12 \\ -2 & . & 2 & 6 \end{bmatrix},$$

we can write down sixteen solutions of $XA=BX$, these being of the type

$$X_{42}=\begin{bmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ -2 & -2 & -4 & . \end{bmatrix}+\begin{bmatrix} 4 \\ -4 \\ 6 \\ 2 \end{bmatrix}[2\ \ -5\ \ -8\ \ -6]+\begin{bmatrix} . & 8 & . & . \\ . & -8 & . & . \\ . & 12 & . & . \\ . & 6 & . & . \end{bmatrix}$$

$$=\begin{bmatrix} 8 & -12 & -32 & -24 \\ -8 & 12 & 32 & 24 \\ 12 & -18 & -48 & -36 \\ 2 & -6 & -20 & -12 \end{bmatrix}.$$

When A and B satisfy the same quartic equation we find in the same way

$$X_{ij}=J_iI_j(A^3-b_1A^2-b_2A-b_3)+BJ_iI_j(A^2-b_1A-b_2)$$
$$+B^2J_iI_j(A-b_1)+B^3J_iI_j,$$

and a rule can be formulated in which X_{ij} can be written down as the sum of four matrices.

In general, when A and B satisfy $F_m(\lambda)=0$ $(m\leqslant n)$, then

$$X_{ij}=\sum_{p=0}^{m-1}B^pJ_iI_jF_{m-p-1}(A)=\sum_{p=0}^{m-1}F_{m-p-1}(B)J_iI_jA^p,$$

where

$$F_m(\lambda)=\lambda^m-b_1\lambda^{m-1}-b_2\lambda^{m-2}-\ldots-b_m,$$
$$F_{m-1}(\lambda)=\lambda^{m-1}-b_1\lambda^{m-2}-b_2\lambda^{m-3}-\ldots-b_{m-1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$F_1(\lambda)=\lambda-b_1,$$
$$F_0(\lambda)=1.$$

7. *Solutions of $XA = AX$ and $XA = A'X$ when A is an elementary matrix*

It is well known that when A is elementary, its minimum and characteristic equations being identical, the solutions of $XA = AX$ are polynomials in A. If $A_{ij}(\lambda)$ is the expansion as a polynomial in λ of the cofactor of $a_{ii} - \lambda$, when $i = j$, or of a_{ij} when $i \neq j$, in $| A - \lambda I |$, then the solutions X_{ij} found by the method of § 6 can be written as $(-1)^{n+1} A_{ij}(A)$. Thus, when $n = 3$,

$$\begin{aligned}
X_{11} &= A_{11} - (a_{22} + a_{33})\, A + A^2, \\
X_{22} &= A_{22} - (a_{33} + a_{11})\, A + A^2, \\
X_{33} &= A_{33} - (a_{11} + a_{22})\, A + A^2, \\
X_{12} &= A_{12} + a_{21} A, \\
X_{13} &= A_{13} + a_{31} A, \quad \text{etc.,}
\end{aligned}$$

where A_{ij} is the numerical value of the cofactor of a_{ij} in the determinant of A. As a matrix, X_{11} may be written

$$\begin{bmatrix} A_{11} - A_{22} - A_{33} & A_{21} & A_{31} \\ A_{12} & \cdot & \cdot \\ A_{13} & \cdot & \cdot \end{bmatrix} + \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} [a_{11} \quad a_{12} \quad a_{13}].$$

If A' is the transpose of A, solutions of $XA = A'X$ are not in general expressible as polynomials in A, but the solutions obtained by the method of § 6 are closely similar in form to those of $XA = AX$. They have been given in some detail in an earlier paper,* where the similarity was observed.

University College
Swansea

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* H. O. Foulkes, "Collineatory transformation of a square matrix into its transpose", *Journal London Math. Soc.* 17 (1942), 70–80.

ON THE SUMMATION OF MULTIPLE FOURIER SERIES. I

By K. CHANDRASEKHARAN

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1·1. This paper* is the first of a projected set of papers devoted to the study† of multiple Fourier series. While single Fourier series and, to a lesser extent, double Fourier series have been investigated in great detail, multiple Fourier series have not received the same degree of attention. It will be interesting to know how far, and in what way, the results on ordinary Fourier series can be extended to the case of multiple Fourier series. Among the various possible methods of considering the behaviour of this series, the one adopted by S. Bochner (**1**) of studying it as a simple series by taking "spherical" partial sums, instead of the usual rectangular sums, is specially noticeable, not only because it leads to elegant results, but also because it accords with the method of treatment of general classes of expansions in *eigen*-functions of which multiple Fourier series can be viewed as a special case. While the results on the summation of expansions in *eigen*-functions (**7**) are, in a sense, more general, they are much less precise than the corresponding results on multiple Fourier series; in the latter case, however, it should be observed that we impose the condition of periodicity on the function that is expanded.

1·2. In this paper we are concerned with the problem of summability of multiple Fourier series. We connect the Riesz summability of the series at a point with the behaviour of the generalized "spherical mean" of the function near the point. In one dimension, this summability will turn out to be equivalent to Cesàro summability, and the results of Hardy, Littlewood and Bosanquet (**5**, **2**) connecting the Cesàro summability of Fourier series with the Cesàro mean of the function are obtainable as corollaries therefrom.

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* *Added* 22 *October* 1947.—The results of this paper, as well as those of the succeeding paper, were announced in *Proc. Indian Acad. Sci.* XXIV, 2, A (1946), 229–232.

† My thanks are due to Dr S. Minakshisundaram, by whom I was prompted to study this problem.

Let $f(x) = f(x_1, \ldots, x_k)$ be a function of the Lebesgue class L, periodic in each of the k variables, and having the period 2π.

Let $$a_{n_1, \ldots, n_k} = \frac{1}{(2\pi)^k} \int_{-\pi}^{+\pi} \ldots \int_{-\pi}^{+\pi} f(x)\, e^{-i(n_1 x_1 + \ldots + n_k x_k)} dx_1 \ldots dx_k.$$

The series $\Sigma a_{n_1, \ldots, n_k} e^{i(n_1 x_1 + \ldots + n_k x_k)}$ is called the multiple Fourier series of the function $f(x)$, and we write

$$f(x) \sim \Sigma a_{n_1, \ldots, n_k} e^{i(n_1 x_1 + \ldots + n_k x_k)}. \tag{1·21}$$

Let $$S_R(x) = \sum_{\nu \leqslant R} a_{n_1, \ldots, n_k} e^{i(n_1 x_1 + \ldots + n_k x_k)}, \quad \nu^2 = n_1^2 + \ldots + n_k^2, \tag{1·22}$$

denote the "spherical" partial sum of the series (1·21); that is, we shall consider it as a simple series

$$\sum_{j=0}^{\infty} \sum_{\nu = R_j} a_{n_1, \ldots, n_k} e^{i(n_1 x_1 + \ldots + n_k x_k)}, \tag{1·23}$$

where R_j is the sequence of all integers that can be represented as a sum of k squares.

Let $$S_R^\delta = \sum_{\nu \leqslant R} \left(1 - \frac{\nu^2}{R^2}\right)^\delta a_{n_1, \ldots, n_k} e^{i(n_1 x_1 + \ldots + n_k x_k)}, \tag{1·24}$$

so that S_R^δ is the Riesz mean of the series (1·21), of type ν^2 and order δ. If $\lim_{R \to \infty} S_R^\delta$ exists and is finite, then the series (1·21) will be summable (ν^2, δ). If $k = 1$, this becomes summability (n^2, δ), which is equivalent(4) to summability (C, δ).

Let $$f(x, t) = \frac{1}{(2\pi)^{\frac{1}{2}k}} \int_\sigma f(x_1 + t\xi_1, \ldots, x_k + t\xi_k)\, d\sigma_\xi,$$

where σ denotes the unit sphere $\xi_1^2 + \ldots + \xi_k^2 = 1$ and $d\sigma_\xi$ its $(k-1)$-dimensional volume-element, so that $f(x, t)$, which exists almost everywhere, as a function of the single variable t, is integrable in every finite interval, and represents the $(k-1)$-dimensional "spherical mean" (except for the factor $2^{\frac{1}{2}(k-2)}\Gamma(\frac{1}{2}k)$) of the function $f(x)$ over a sphere of radius t with centre at the point x. If $k = 1$, then

$$f(x, t) = \frac{1}{\sqrt{(2\pi)}} [f(x+t) + f(x-t)].$$

Let* $$\begin{aligned} f_p(x, t) &= \frac{2}{B(p, \frac{1}{2}k)\, t^{2p+k-2}} \int_0^t (t^2 - s^2)^{p-1} s^{k-1} f(x, s)\, ds \\ &= \frac{c}{t^k} \int f(y) \left(1 - \frac{s^2}{t^2}\right)^{p-1} dy, \end{aligned} \tag{1·25}$$

where dy is the k-dimensional volume-element and $p > 0$, $\sum_1^k (y_i - x_i)^2 = s^2 \leqslant t^2$.

* c, c_1, c_2, etc., are some constants.

If $p = 0$, we define $f_0(x, t) = f(x, t)$; $f_p(x, t) = f_p(t)$ is called the "spherical mean" of the function $f(x)$, of order p. This exists for almost all t if $0 < p < 1$, and is absolutely continuous in every finite interval excluding the origin if $p \geqslant 1$.

If $f_p(t)$ tends to a limit as $t \to 0$, then $f(x)$ is said to have a "mean limit" of order p at the point x. If $k = 1$,

$$f_p(t) = \frac{c}{t}\int_0^t \left(1 - \frac{s^2}{t^2}\right)^{p-1} f(x, s)\, ds,$$

so that whenever $\lim_{t\to 0} f_p(t)$ exists, the pth Cesàro mean of the function $f(x, t)$ tends to a limit, and conversely.

The connexion between multiple Fourier series and Fourier series of *eigen*-functions will be clear if we consider the boundary-value problem,

$$\frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_k^2} + \lambda u(x_1, \ldots, x_k) = 0, \tag{1·26}$$

with the condition that $u(x)$ is periodic in each of the k-variables, with period 2π. It is easily seen that the elementary exponentials

$$u = e^{i(n_1 x_1 + \ldots + n_k x_k)},$$

where (n_k) are all integers, form a complete set of regular solutions of (1·26). Since the *eigen*-values are given by $\lambda = n_1^2 + \ldots + n_k^2$, the method of writing the series (1·21) in the form (1·23) is simply the arrangement of the terms of the series (1·21) according to the order of magnitude of the *eigen*- values λ.

1·3. We shall state here some of the notations and formulae which will be frequently used:

$$\left.\begin{aligned}
\phi_p(r) &= \frac{1}{2^{p-1}\Gamma(p)}\int_0^r (r^2 - t^2)^{p-1} t^{k-1} f(x, t)\, dt \quad (p > 0),\\
f_p(r) &= \frac{2^p\Gamma(p)}{B(p, \frac{1}{2}k)} \frac{\phi_p(r)}{r^{2p+k-2}},\\
f_0(r) &= f(x, r).
\end{aligned}\right\} \tag{1·31}$$

It is easily verified that

$$\phi_{p+q}(r) = \frac{1}{2^{q-1}\Gamma(q)}\int_0^r (r^2 - t^2)^{q-1} t\phi_p(t)\, dt, \tag{1·32}$$

if $p + q \geqslant 1$, and hence it follows that

$$f_{p+q}(r) = \frac{2\Gamma(p + \frac{1}{2}k + q)}{\Gamma(p + \frac{1}{2}k)\,\Gamma(q)\, r^{2(p+q)+k-2}}\int_0^r (r^2 - t^2)^{q-1} t^{2p+k-1} f_p(t)\, dt. \tag{1·33}$$

Let $J_\mu(x)$ denote the Bessel function of order μ. Let

$$V_\mu(x) = J_\mu(x)/x^\mu.$$

$$J_\mu(x) = O(x^\mu) \text{ as } x \to 0; \quad J_\mu(x) = O(x^{-\frac{1}{2}}) \text{ as } x \to \infty, \tag{1·34}$$

$$J_{n-p-1}(x) = \frac{x^{p+1}}{2^p\Gamma(p+1)}\int_1^\infty \frac{J_n(xt)\,(t^2-1)^p}{t^{n-1}}\,dt, \quad \text{if} \quad n > 2p+\tfrac{1}{2}, \tag{1·35}$$

$$\int_0^\infty \frac{J_\nu(t)}{t^{\nu-\mu+1}}\,dt = \frac{\Gamma(\frac{1}{2}\mu)}{2^{\nu-\mu+1}\Gamma(\nu-\frac{1}{2}\mu+1)}, \quad \text{for} \quad 0<\mu<\nu+\tfrac{3}{2}, \tag{1·36}$$

$$\int_0^\infty \frac{J_\nu(\omega t)\,J_\mu(nt)}{t^{\nu-\mu-1}}\,dt = \frac{n^\mu}{2^{\nu-\mu-1}\Gamma(\nu-\mu)\,\omega^{2\mu-\nu+2}}\left(1-\frac{n^2}{\omega^2}\right)^{\nu-\mu-1}, \quad \text{if} \quad 0<n<\omega,$$

and $$= 0, \quad \text{if} \quad n > \omega. \text{(3)} \tag{1·37}$$

2·1. *Preliminary lemmas*

LEMMA 1: $$\int_0^u t^{k-1}\,|\,f(x,t)\,|\,dt = O(u^k), \quad \textit{as} \quad u\to\infty, \tag{2·11}$$

$$f_p(u) = O(1), \quad \textit{for} \quad p \geqslant 1, \quad \textit{as} \quad u\to\infty, \tag{2·12}$$

$$\phi_p(u) = O(u^{2p+k-2}), \quad \textit{for} \quad p \geqslant 1, \quad \textit{as} \quad u\to\infty. \tag{2·13}$$

(2·11) has been proved by Bochner((**1**), 189 last line); (2·13) follows from (2·12) in virtue of (1·31). (2·12) is proved thus:

$$\begin{aligned}|\,f_1(u)\,| &= \frac{c}{u^k}\left|\int_0^u t^{k-1} f(x,t)\,dt\right| \\ &= O\left(\frac{1}{u^k}\int_0^u t^{k-1}\,|\,f(x,t)\,|\,dt\right) \\ &= O(1), \quad \text{by (2·11)}.\end{aligned}$$

Now, using the formula (1·33), we easily prove (2·12).

LEMMA 2. *If*

$$\Phi_p(u) = \int_0^u t\,|\,\phi_p(t)\,|\,dt, \quad F_p(u) = \int_0^u t^{2p+k-1}\,|\,f_p(t)\,|\,dt,$$

and $$G_p(u) = \int_0^u t^{2p+k-1}\,|\,g_p(t)\,|\,dt, \quad \textit{where} \quad g_p(t) = f_p(t) - l,$$

then $$\Phi_p(u) = O(u^{2p+k}), \quad \textit{as} \quad u\to\infty, \tag{2·14}$$

$$F_p(u) = O(u^{2p+k}), \quad \textit{as} \quad u\to\infty, \tag{2·15}$$

$$G_p(u) = O(u^{2p+k}), \quad \textit{as} \quad u\to\infty. \tag{2·16}$$

(2·14) and (2·16) easily follow from (2·15) which is proved thus:

$$\int_0^u t^{2p+k-1} \mid f_p(t) \mid dt = O\left\{\int_0^u (u^2-s^2)^p s^{k-1} \mid f(x,s) \mid ds\right\}$$

$$= O(u^{2p}) \int_0^u s^{k-1} \mid f(x,s) \mid ds$$

$$= O(u^{2p+k}), \quad \text{by (2·11)}.$$

2·2. Bochner has proved the following fundamental result ((**1**), 176, 189):

$$S_R^\delta(x) = 2^\delta \Gamma(\delta+1) R^k \int_0^\infty t^{k-1} f(x,t) V_{\delta+\frac{1}{2}k}(tR)\, dt, \tag{2·21}$$

provided $\delta > \frac{1}{2}(k-1)$.

If $\eta > 0$, and $\delta > \frac{1}{2}(k-1)$, we have

$$2^\delta \Gamma(\delta+1) R^k \int_\eta^\infty t^{k-1} f(x,t) V_{\delta+\frac{1}{2}k}(tR)\, dt = o(1), \quad \text{as} \quad R \to \infty, \tag{2·22}$$

uniformly for all x. For,

$$\left| R^k \int_\eta^\infty t^{k-1} f(x,t) V_{\delta+\frac{1}{2}k}(tR)\, dt \right| = O\left(\frac{1}{R^{\delta-\frac{1}{2}(k-1)}} \int_\eta^\infty \frac{t^{k-1} \mid f(x,t) \mid dt}{t^{\delta+\frac{1}{2}(k+1)}}\right) \quad \text{by (1·34)}$$

$$= O\left(\frac{1}{R^{\delta-\frac{1}{2}(k-1)}} \int_\eta^\infty \frac{dF}{t^{\delta+\frac{1}{2}(k+1)}}\right)$$

$$= O\left(\frac{1}{R^{\delta-\frac{1}{2}(k-1)}}\right) = o(1).$$

Combining (2·21) and (2·22), we obtain, for $\delta > \frac{1}{2}(k-1)$,

$$S_R^\delta(x) = 2^\delta \Gamma(\delta+1) R^k \int_0^\eta t^{k-1} f(x,t) V_{\delta+\frac{1}{2}k}(tR)\, dt + o(1), \tag{2·23}$$

which shows that *Riesz summability of type ν^2 and order $\delta > \frac{1}{2}(k-1)$ is a local property for multiple Fourier series, when summed "spherically"*. That this property ceases to hold when $\delta = \frac{1}{2}(k-1)$ and $k > 1$, has been shown by Bochner ((**1**), 193, Th. VII).

3·1. *Fundamental relations.* In this section, we generalize the formula (2·21) and prove its reciprocal.

THEOREM 1. *If h is the greatest integer less than p, where $p > 0$, then*

$$S_R^\delta(x) = \frac{2^{\delta-p}\Gamma(\delta+1)\Gamma(\frac{1}{2}k) R^{k+2p}}{\Gamma(p+\frac{1}{2}k)} \int_0^\infty t^{k+2p-1} f_p(t) V_{\delta+p+\frac{1}{2}k}(tR)\, dt, \tag{3·11}$$

provided that $\delta > h + \frac{1}{2}(k-1)$.

Proof. The case $p = 0$ of (1·35) gives $V'_\mu(x) = -xV_{\mu+1}(x)$, and so, integrating the right side of (2·21) by parts $h+1$ times, we get

$$S_R^\delta(x) = 2^\delta\Gamma(\delta+1)\, R^{k+2h+2}\int_0^\infty t\phi_{h+1}(t)\, V_{\delta+\frac{1}{2}k+h+1}(tR)\, dt, \tag{3·12}$$

provided $\delta > h+\frac{1}{2}(k-1)$, on using lemma 1 at each stage of partial integration. If p is an integer, $h+1 = p$, and (3·12) leads directly to the result (3·11). If p is not an integer, substituting for ϕ_{h+1} in terms of ϕ_p as in (1·32), and setting $\beta = \delta+\frac{1}{2}k+h+1$, we write (3·12) as

$$\begin{aligned} S_R^\delta(x) &= \frac{2^\delta\Gamma(\delta+1)}{2^{h-p}\Gamma(h-p+1)} R^{k+2h+2}\int_0^\infty tV_\beta(tR)\, dt\int_0^t (t^2-s^2)^{h-p}\, s\phi_p(s)\, ds \\ &= \frac{2^\delta\Gamma(\delta+1)}{2^{h-p}\Gamma(h-p+1)} R^{k+2h+2}\int_0^\infty s\phi_p(s)\, ds\int_s^\infty (t^2-s^2)^{h-p}\, tV_\beta(tR)\, dt, \end{aligned} \tag{3·13}$$

since

$$\lim_{m\to\infty}\int_0^m s\phi_p(s)\, ds\int_m^\infty (t^2-s^2)^{h-p}\, tV_\beta(tR)\, dt = 0, \quad \text{for} \quad \delta > h+\tfrac{1}{2}(k-1),$$

which is proved as follows: for $s < m$, and a fixed $R > 0$,

$$\begin{aligned} \left|\int_m^\infty (t^2-s^2)^{h-p}\, tV_\beta(tR)\, dt\right| &\leqslant (m^2-s^2)^{h-p}\max_{m'>m}\left|\int_m^{m'} tV_\beta(tR)\, dt\right| \\ &= (m^2-s^2)^{h-p}\, O\left(\frac{1}{m^{\delta+h+\frac{1}{2}(k+1)}}\right), \end{aligned}$$

so that

$$\begin{aligned} &\frac{1}{m^{\delta+h+\frac{1}{2}(k+1)}}\int_0^m s\,|\,\phi_p(s)\,|\,(m^2-s^2)^{h-p}\, ds \\ &= \frac{1}{m^{\delta+h+\frac{1}{2}(k+1)}}\left[\{\Phi_p(s)\,(m^2-s^2)^{h-p}\}_0^m - 2(h-p)\int_0^m \Phi_p(s)\,(m^2-s^2)^{h-p-1}\, s\, ds\right] \\ &= \frac{1}{m^{\delta+h+\frac{1}{2}(k+1)}}\, O(m^{2h+k}) = o(1), \quad \text{if} \quad \delta > h+\tfrac{1}{2}(k-1). \end{aligned}$$

Now (3·13) can be written as

$$\begin{aligned} S_R^\delta(x) &= \frac{2^\delta\Gamma(\delta+1)}{2^{h-p}\Gamma(h-p+1)} R^{k+2h+2}\int_0^\infty s\phi_p(s)\, ds\int_1^\infty (y^2-1)^{h-p}\, s^{2h-2p+2}\, V_\beta(sRy)\, y\, dy \\ &= 2^\delta\Gamma(\delta+1)\, R^{k+2h+2}\int_0^\infty s\phi_p(s)\,\frac{J_{\beta-h+p-1}(sR)}{(sR)^{\beta+h-p+1}}\, s^{2h-2p+2}\, ds, \quad \text{by (1·35)}, \\ &= 2^\delta\Gamma(\delta+1)\, R^{k+2p}\int_0^\infty s\phi_p(s)\, V_{\delta+p+\frac{1}{2}k}(sR)\, ds \\ &= \frac{2^{\delta-p}\Gamma(\delta+1)\,\Gamma(\frac{1}{2}k)}{\Gamma(p+\frac{1}{2}k)} R^{k+2p}\int_0^\infty s^{2p+k-1} f_p(s)\, V_{\delta+p+\frac{1}{2}k}(sR)\, ds. \end{aligned}$$

From this formula it follows that a necessary and sufficient condition for the summability (ν^2, δ) of the multiple Fourier series is that

$$\lim_{R\to\infty} R^{k+2p} \int_0^\infty s^{2p+k-1} f_p(s)\, V_{\delta+p+\frac{1}{2}k}(sR)\, ds$$

should exist, where $\delta > p + \frac{1}{2}(k-1)$.

Remarks. (i) In the above theorem, if $0 < p < 1$, we naturally interpret $h = 0$.

(ii) If $p \geqslant 1$, we can actually prove a relation which is sharper than (3·11). The sharper form will not, however, be needed in this paper, but it leads to an interesting theorem which is connected with the question of absolute summability. For that reason it will be proved as a lemma in the second paper of this series.*

THEOREM II. *If* $p > 1$, *and* $\delta > \frac{1}{2}(k-1)$, *then*

$$f_p(y) = \frac{\Gamma(p+\frac{1}{2}k)}{2^{\delta-p}\Gamma(\delta+1)\,\Gamma(\frac{1}{2}k)} y^{2\delta+2} \int_0^\infty S_R^\delta R^{2\delta+1} V_{p+\delta+\frac{1}{2}k}(yR)\, dR. \qquad (3\cdot 14)$$

Proof. From (1·24) it follows trivially that $S_R^\delta = O(R^k)$, and so the integral

$$J = y^{2\delta+2} \int_0^\infty R^{2\delta+1} S_R^\delta V_\gamma(Ry)\, dR$$

converges for $\gamma > 2\delta + \frac{3}{2}$. Hence, by (2·21),

$$J = 2^\delta \Gamma(\delta+1)\, y^{2\delta+2} \int_0^\infty R^{k+2\delta+1} V_\gamma(Ry)\, dR \int_0^\infty t^{k-1} f(x,t)\, V_{\delta+\frac{1}{2}k}(tR)\, dt$$

$$= \frac{2^\delta\Gamma(\delta+1)}{y^{\gamma-2\delta-2}} \int_0^\infty t^{\frac{1}{2}k-\delta-1} f(x,t)\, dt \int_0^\infty R^{\frac{1}{2}k+\delta+1-\gamma} J_{\delta+\frac{1}{2}k}(tR)\, J_\gamma(yR)\, dR,$$

as the double integral is absolutely convergent. Hence, by (1·37),

$$J = \frac{2^\delta\Gamma(\delta+1)}{\Gamma(\gamma-\delta-\frac{1}{2}k)\, 2^{\gamma-\delta-\frac{1}{2}k-1} y^{2\gamma-2\delta-2}} \int_0^y t^{k-1} f(x,t)\, (y^2-t^2)^{\gamma-\delta-\frac{1}{2}k-1}\, dt,$$

$$= \frac{2^{\delta-p}\Gamma(\delta+1)\,\Gamma(\frac{1}{2}k)}{\Gamma(p+\frac{1}{2}k)} f_p(y),$$

for $p = \gamma - \delta - \frac{1}{2}k$, $\gamma > 2\delta + \frac{3}{2}$, $\delta > \frac{1}{2}(k-1)$.

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* Immediately following this paper in these *Proceedings*.

3·2. *Theorems on summability*

THEOREM III. *If $f_p(t) \to l$ as $t \to 0$, then*

$$\lim_{R\to\infty} S_R^\delta(x) = 2^{\frac{1}{2}(k-2)}\Gamma(\tfrac{1}{2}k)\,l, \quad \textit{for} \quad \delta > p + \tfrac{1}{2}(k-1).$$

Proof. Let

$$g_p(t) = f_p(t) - l,$$

$$I = S_R^\delta(x) - 2^{\frac{1}{2}(k-2)}\Gamma(\tfrac{1}{2}k)\,l$$

$$= \xi R^{k+2p}\int_0^\infty t^{k+2p-1} g_p(t) V_{\delta+p+\frac{1}{2}k}(tR)\,dt, \tag{3·21}$$

where

$$\xi = \frac{2^{\delta-p}\Gamma(\delta+1)\,\Gamma(\frac{1}{2}k)}{\Gamma(p+\frac{1}{2}k)},$$

since

$$R^{k+2p}\int_0^\infty t^{k+2p-1} V_{\delta+p+\frac{1}{2}k}(tR)\,dt = \frac{\Gamma(p+\frac{1}{2}k)}{2^{\delta-p-\frac{1}{2}k+1}\Gamma(\delta+1)}.$$

Let, for $0 < \eta < \infty$,

$$I = \xi R^{k+2p}\left[\int_0^\eta + \int_\eta^\infty\right] = I_1 + I_2, \quad \text{say}, \tag{3·22}$$

$$|I_1| \leqslant \xi R^{k+2p}\int_0^\eta t^{k+2p-1}\,|g_p(t)|\,\frac{|J_{p+\delta+\frac{1}{2}k}(tR)|\,dt}{(tR)^{\delta+p+\frac{1}{2}k}}$$

$$= o(1)\int_0^{\eta R}\frac{y^{2p+k-1}}{y^{\delta+p+\frac{1}{2}k}}\,|J_{p+\delta+\frac{1}{2}k}(y)|\,dy$$

$$= o(1), \quad \text{if} \quad \delta > p + \tfrac{1}{2}(k-1). \tag{3·23}$$

Using (1·34),

$$|I_2| = O\left(\frac{1}{R^{\delta-\frac{1}{2}k-p+\frac{1}{2}}}\right)\int_\eta^\infty \frac{t^{k+2p-1}\,|g_p(t)|\,dt}{t^{\delta+p+\frac{1}{2}(k+1)}}$$

$$= O\left(\frac{1}{R^{\delta-\frac{1}{2}k-p+\frac{1}{2}}}\right)\left[\left\{\frac{G_p(t)}{t^{\delta+p+\frac{1}{2}(k+1)}}\right\}_\eta^\infty + c\int_\eta^\infty \frac{G_p(t)\,dt}{t^{\delta+p+\frac{1}{2}(k+3)}}\right]$$

$$= O\left(\frac{1}{R^{\delta-\frac{1}{2}k-p+\frac{1}{2}}}\right)$$

$$= o(1), \quad \text{if} \quad \delta > p + \tfrac{1}{2}(k-1). \tag{3·24}$$

From (3·23) and (3·24) it follows that $|I| = o(1)$ for $\delta > p + \frac{1}{2}(k-1)$, from which results theorem III.

Remarks. (i) At a point of continuity of $f(x)$, we have $S_R^\delta(x) \to f(x)$ as $R \to \infty$, for $\delta > \frac{1}{2}(k-1)$, since $f(x,t) \to \{2^{\frac{1}{2}(k-2)}\Gamma(\frac{1}{2}k)\}^{-1} f(x)$, as $t \to 0$.

(ii) If $f(x)$ is a continuous periodic function, $f(x,t)$ converges uniformly to $\dfrac{f(x)}{2^{\frac{1}{2}(k-2)}\Gamma(\frac{1}{2}k)}$ as $t \to 0$ for all x; and hence S_R^δ converges uniformly to $f(x)$.

Theorem III is a generalization of a well-known result of Bosanquet (**2**).

THEOREM IV. *If*

(i) $$f_{p+1}(t) - l = o(1), \quad as \quad t \to 0,$$

(ii) $$\frac{1}{t^{k+2p}} \int_0^t s^{k+2p-1} \, | f_p(s) | \, ds = O(1), \quad as \quad t \to 0,$$

then $$\lim_{R \to \infty} S_R^\delta(x) = L,$$

where $\delta > p + \frac{1}{2}(k-1)$ *and* $L = 2^{\frac{1}{2}(k-2)} \Gamma(\frac{1}{2}k) \, l.$

Proof. From (3·21) we have

$$S_R^\delta(x) - 2^{\frac{1}{2}(k-2)} \Gamma(\tfrac{1}{2}k) \, l = \xi R^{k+2p} \int_0^\infty t^{k+2p-1} g_p(t) \, V_{\delta+p+\frac{1}{2}k}(tR) \, dt$$

$$= \xi R^{k+2p} \left[\int_0^{1/R} + \int_{1/R}^\infty \right]$$

$$= J_1 + J_2, \quad \text{say},$$

$$| J_1 | = O\left(R^{k+2p} \int_0^{1/R} t^{k+2p-1} \, | g_p(t) | \, dt \right) = O(1), \quad \text{by hypothesis (ii);}$$

$$| J_2 | = O\left(R^{k+2p} \int_{1/R}^\infty \frac{t^{k+2p-1} \, | g_p(t) | \, dt}{(tR)^{\delta+p+\frac{1}{2}(k+1)}} \right), \quad \text{on using (1·34),}$$

$$= O\left(\frac{1}{R^{\delta-p-\frac{1}{2}(k-1)}} \int_{1/R}^\infty \frac{dG_p}{t^{\delta+p+\frac{1}{2}(k+1)}} \right)$$

$$= O(1),$$

if $\delta > p + \frac{1}{2}(k-1)$, on using (2·16) in partial integration.

Hence the multiple Fourier series is bounded (ν^2, δ), for $\delta > p + \frac{1}{2}(k-1)$; by a well-known result, it follows that the series is either summable for every $\delta > p + \frac{1}{2}(k-1)$, or none at all (6); but, by hypothesis (i) and theorem III, it is summable (ν^2, δ) for $\delta > p + 1 + \frac{1}{2}(k-1)$ to the sum $2^{\frac{1}{2}(k-2)} \Gamma(\frac{1}{2}k) \, l$. Hence,

$$\lim_{R \to \infty} S_R^\delta(x) = 2^{\frac{1}{2}(k-2)} \Gamma(\tfrac{1}{2}k), \quad \text{for} \quad \delta > p + \tfrac{1}{2}(k-1).$$

Remarks. (i) In the above theorem, conditions (i) and (ii) could be replaced by

$$\frac{1}{t^{k+2p}} \int_0^t r^{k+2p-1} \, | g_p(r) | \, dr = o(1), \quad \text{as} \quad t \to 0.$$

(ii) If $p = 0$ and $k = 1$, we obtain Lebesgue's criterion for the summability of ordinary Fourier series.

(iii) Instead of $p+1$ occurring in condition (i), we may have any $q > p$. For, condition (ii) of the theorem alone implies that the series is bounded (ν^2, δ) for $\delta > p + \frac{1}{2}(k-1)$, so that for the summability (ν^2, δ) of the series, it is sufficient if the series is summable for *some* δ and this will be the case if $\lim_{t \to 0} f_q(t)$ exists for any $q > p$, by theorem III.

THEOREM V. *If* $f_p(t)-l = O(t^{\alpha})$, *as* $t\to 0$, $\alpha>0$, *then, for*

$$\delta = p+\tfrac{1}{2}(k-1)+\beta, \quad 0<\beta,$$

we have

$$S_R^{\delta}(x)-L = \begin{cases} O\left(\dfrac{1}{R^{\alpha}}\right), & \textit{if} \quad \beta>\alpha, \\ O\left(\dfrac{\log R}{R^{\alpha}}\right), & \textit{if} \quad \beta=\alpha, \\ O\left(\dfrac{1}{R^{\beta}}\right), & \textit{if} \quad \beta<\alpha, \end{cases}$$

where $L = 2^{\frac{1}{2}(k-2)}\Gamma(\tfrac{1}{2}k)\,l$.

Proof. From (3·21) we have

$$I = S_R^{\delta}(x)-L = \xi R^{k+2p}\int_0^{\infty} t^{k+2p-1}g_p(t)\,V_{\delta+p+\frac{1}{2}k}(tR)\,dt$$

$$= \xi R^{k+2p}\left[\int_0^{\eta}+\int_{\eta}^{\infty}\right] = I_1+I_2, \quad \text{say.}$$

As in (3·24) we have $\quad |I_2| = O\left(\dfrac{1}{R^{\delta-\frac{1}{2}k-p+\frac{1}{2}}}\right) = O\left(\dfrac{1}{R^{\beta}}\right).$

A fortiori

$$|I_2| = \begin{cases} O\left(\dfrac{1}{R^{\alpha}}\right), & \text{if} \quad \beta>\alpha, \\ O\left(\dfrac{\log R}{R^{\alpha}}\right), & \text{if} \quad \beta=\alpha, \\ O\left(\dfrac{1}{R^{\beta}}\right), & \text{if} \quad \beta<\alpha. \end{cases} \tag{3·25}$$

We may consider I_1 as

$$I_1 = \xi R^{k+2p}\left[\int_0^{1/R}+\int_{1/R}^{\eta}\right]$$

$$= I_{1,1}+I_{1,2}, \quad \text{say.}$$

$$|I_{1,1}| = O\left(R^{k+2p}\int_0^{1/R} t^{k+2p-1}\,|g_p(t)|\,dt\right)$$

$$= O\left(\frac{1}{R^{\alpha}}\right).$$

$$|I_{1,2}| = O\left(R^{k+2p}\int_{1/R}^{\eta}\frac{t^{k+2p-1}\,|g_p(t)|\,dt}{(tR)^{\delta+p+\frac{1}{2}(k-1)}}\right)$$

$$= O\left(\frac{1}{R^{\delta-p-\frac{1}{2}k+\frac{1}{2}}}\int_{1/R}^{\eta}\frac{dG_p}{t^{\delta+p-\frac{1}{2}(k+1)}}\right)$$

$$= O\left(\frac{1}{R^{\delta-p-\frac{1}{2}k+\frac{1}{2}}}\right)\left[\left\{\frac{G(t_p)}{t^{\delta+p+\frac{1}{2}(k+1)}}\right\}_{1/R}^{\eta}+c_1\int_{1/R}^{\eta}\frac{G_p(t)\,dt}{t^{\delta+p+\frac{1}{2}(k+3)}}\right]$$

$$= O\left(\frac{1}{R^{\delta-p-\frac{1}{2}k+\frac{1}{2}}}\right)\left[O(1)+O(R^{\delta-p-\alpha-\frac{1}{2}(k-1)})+O\left(\int_{1/R}^{\eta}\frac{dt}{t^{\delta-p-\alpha-\frac{1}{2}k+\frac{3}{2}}}\right)\right], \tag{3·26}$$

since
$$G_p(t) = O(t^{k+2p+\alpha}), \quad \text{as} \quad t \to 0, \tag{3·27}$$
by hypothesis. Now

$$O\left(\int_{1/R}^{\eta} \frac{dt}{t^{\delta-p-\alpha-\frac{1}{2}k+\frac{1}{2}}}\right) = O(1)+O(R^{\beta-\alpha}), \quad \text{if} \quad \beta \neq \alpha,$$
$$= O(1)+O(\log R), \quad \text{if} \quad \beta = \alpha,$$

so that
$$|I_{1,2}| = O\left(\frac{1}{R^\beta}\right)[O(1)+O(R^{\beta-\alpha})], \quad \text{if} \quad \beta \neq \alpha,$$

that is
$$= O\left(\frac{1}{R^\beta}\right)+O\left(\frac{1}{R^\alpha}\right), \quad \text{if} \quad \beta \neq \alpha$$

and
$$= O\left(\frac{1}{R^\alpha}\right)[O(1)+O(\log R)], \quad \text{if} \quad \beta = \alpha.$$

Hence
$$|I_{1,2}| = \begin{cases} O\left(\dfrac{1}{R^\alpha}\right), & \text{if} \quad \beta > \alpha, \\ O\left(\dfrac{1}{R^\beta}\right), & \text{if} \quad \beta < \alpha, \\ O\left(\dfrac{\log R}{R^\alpha}\right), & \text{if} \quad \beta = \alpha. \end{cases} \tag{3·28}$$

Combining (3·25), (3·26) and (3·28), we get the required result. It is clear that our hypothesis may be replaced by the more general condition (3·27).

THEOREM VI. *If* $\dfrac{f_p(t)-l}{t^\alpha} \to s_\alpha$, *then*

$$R^\alpha[S_R^\delta - 2^{\frac{1}{2}(k-2)}\Gamma(\tfrac{1}{2}k)\,l] \to l_\alpha, \quad \textit{if} \quad \delta > p+\tfrac{1}{2}(k-1)+\alpha,$$

where

$$l_\alpha = s_\alpha . 2^{\frac{1}{2}(k-2)+\alpha}\Gamma(\delta+1)\,\Gamma(p+\tfrac{1}{2}\alpha+\tfrac{1}{2}k)\{\Gamma(p+\tfrac{1}{2}k)\,\Gamma(\delta+1-\tfrac{1}{2}\alpha)\}^{-1}.$$

This is a generalization of a theorem of Szász (**8**), and the proof can be easily constructed on the same lines as in our earlier theorems.

3·3. *Theorems on the "mean limit" of* $f(x)$

THEOREM VII. *If* $S_R^\gamma(x) \to s$ *as* $R \to \infty$, *then*

$$f_p(y) \to \frac{s}{2^{\frac{1}{2}(k-2)}\Gamma(\frac{1}{2}k)} \quad \textit{as} \quad y \to 0,$$

provided that
$$p > \max(1, \gamma - \tfrac{1}{2}(k-3)).$$

Proof. We prove the theorem using the formula (3·14),

$$I = f_p(y) - \frac{s}{2^{\frac{1}{2}(k-1)}\Gamma(\frac{1}{2}k)} = \frac{\Gamma(p+\frac{1}{2}k)\,y^{2\delta+2}}{2^{\delta-p}\Gamma(\delta+1)\,\Gamma(\frac{1}{2}k)}\int_0^\infty (S_R^\delta - s)\,R^{2\delta+1}V_{\delta+p+\frac{1}{2}k}(yR)\,dR,$$

which holds only for $p > 1$ and $\delta > \frac{1}{2}(k-1)$. If $\gamma > \frac{1}{2}(k-1)$, we choose $\delta = \gamma$; if $\gamma \leqslant \frac{1}{2}(k-1)$ choose $\delta = \frac{1}{2}(k-1)+\theta$ where $\theta > 0$ is arbitrary.

Setting $$I = cy^{2\delta+2}\left[\int_0^\omega + \int_\omega^\infty\right] = I_1 + I_2, \quad \text{say},$$

we obtain $$|I_1| = O\left(\int_0^{\omega y} z^{2\delta+1}\frac{|J_{p+\frac{1}{2}k+\delta}(z)|\,dz}{z^{p+\delta+\frac{1}{2}k}}\right)$$

$$= o(1), \quad \text{as} \quad y \to 0, \tag{3·31}$$

$$|I_2| = o\left(y^{2\delta+2}\int_\omega^\infty R^{2\delta+1}\,|V_{p+\delta+\frac{1}{2}k}(yR)|\,dR\right)$$

$$= o\left(\int_{y\omega}^\infty z^{2\delta+1}\frac{|J_{p+\delta+\frac{1}{2}k}(z)|\,dz}{z^{p+\delta+\frac{1}{2}k}}\right)$$

$$= o(1), \quad \text{if} \quad p > \delta - \tfrac{1}{2}(k-3). \tag{3·32}$$

From (3·31) and (3·32) the result follows.

This again is a generalization of a theorem of Bosanquet (2).

THEOREM VIII. *If* $S_R^\gamma(x) - s = O\left(\frac{1}{R^\alpha}\right)$, *as* $R \to \infty$, $0 < \alpha < 2$, *then*

$$f_p(y) - \frac{s}{2^{\frac{1}{2}(k-2)}\Gamma(\frac{1}{2}k)} = O(y^\alpha),$$

provided that $$p > \max(1, \gamma - \tfrac{1}{2}(k-3) + \alpha).$$

Proof. Let δ and I have the same significance as in Theorem VII. Then $S_R^\gamma - s = O\left(\frac{1}{R^\alpha}\right)$ implies $S_R^\delta - s = O\left(\frac{1}{R^\alpha}\right)$ for $\delta > \gamma$, since $0 < \alpha < 2$. Now

$$|I_1| = O\left(y^{2\delta+2}\int_0^\omega R^{2\delta+1}\,|V_{p+\frac{1}{2}k+\delta}(yR)|\,dR\right)$$

$$= O\left(\int_0^{\omega y} z^{2\delta+1}\,|V_{p+\frac{1}{2}k+\delta}(z)|\,dz\right)$$

$$= O(y^{2\delta+2}) = O(y^\alpha), \quad \text{since} \quad 0 < \alpha < 2; \tag{3·33}$$

$$|I_2| = O\left(y^{2\delta+2}\int_\omega^\infty R^{2\delta+1-\alpha}\,|V_{p+\frac{1}{2}k+\delta}(yR)|\,dR\right)$$

$$= O\left(y^\alpha\int_{\omega y}^\infty z^{2\delta+1-\alpha}\,|V_{p+\frac{1}{2}k+\delta}(z)|\,dz\right)$$

$$= O(y^\alpha). \tag{3·34}$$

Combining (3·33) and (3·34) we get

$$|I| = O(y^\alpha). \tag{3·35}$$

Theorems VII and VIII are converse to theorems III and V.

3·4. Combining theorems III and VII we can state the following analogue of a theorem of Hardy and Littlewood (**5**):

THEOREM IX. *A necessary and sufficient condition that the multiple Fourier series of a function $f(x)$ should be summable* (spherically) at a point x is that the mean limit of some order of the function exists at the point.*

Again, combining theorems IV and VII, we state the following theorem (Bosanquet (**2**)):

THEOREM X. *If*

$$\frac{1}{t^{k+2p}}\int_0^t s^{k+2p-1}\,|\,f_p(s)\,|\,ds = O(1), \quad \text{as} \quad t \to 0,$$

or, in particular, if $f_p(s) = O(1)$, then the multiple Fourier series of $f(x)$ is either summable (ν^2, δ) for every $\delta > p + \frac{1}{2}(k-1)$ or for no δ; a necessary and sufficient condition for it to be summable is that $f_q(t) \to l$, as $t \to 0$, for $q > p+1$.

In conclusion we may point out that the method of this paper applies *mutatis mutandis* to multiple Fourier integrals (cf. Bochner, (**1**)).

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Madras, India

* Summability of the type and order specified in (1·2).

ON THE SUMMATION OF MULTIPLE FOURIER SERIES. II

By K. CHANDRASEKHARAN

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1. In paper I* (4) we discussed the summability of multiple Fourier series by Riesz's means of suitable type. Here we discuss the absolute summability of the series, and obtain results which are sharper than those on series of eigenfunctions (3) and which lead to results equivalent to those of Bosanquet (1) in one dimension.

Let $f(x) = f(x_1, \ldots, x_k)$ be a function of the Lebesgue class L, having the period 2π in each of the k variables. Let

$$f(x) \sim \sum a_{n_1,\ldots,n_k} e^{i(n_1 x_1 + \ldots + n_k x_k)},$$

$$S_R^\delta(x) = \sum_{\nu \leqslant R} \left(1 - \frac{\nu^2}{R^2}\right)^\delta a_{n_1,\ldots,n_k} e^{i(n_1 x_1 + \ldots + n_k x_k)},$$

$$f_p(x,t) = f_p(t) = \frac{c}{t^k} \int f(y) \left(1 - \frac{s^2}{t^2}\right)^{p-1} dy,\dagger$$

where $p > 0$, and c is a constant, dy is the k-dimensional volume-element and

$$\sum_1^k (y_i - x_i)^2 = s^2 \leqslant t^2,$$

$$\phi_p(t) = \frac{B(p, \frac{1}{2}k)}{2^p \Gamma(p)} f_p(t)\, t^{2p+k-2},$$

$$f_0(x,t) = f(x,t) = \frac{1}{(2\pi)^{\frac{1}{2}k}} \int_\sigma f(x_1 + t\xi_1, \ldots, x_k + t\xi_k)\, d\sigma_\xi,$$

where σ denotes the unit sphere $\xi_1^2 + \ldots + \xi_k^2 = 1$ and $d\sigma_\xi$ its $(k-1)$-dimensional volume-element.

$f_p(t)$ is the spherical mean of order p of the function $f(x)$ at the point x.

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* Published in these *Proceedings*.

† Throughout this paper, the c's stand for some constants.

In theorems III and VII of Paper I (**4**) we proved the following:

(*a*) *If* $f_p(t) \to l$ *as* $t \to 0$, *then* $\lim_{R\to\infty} S_R^\delta(x) = 2^{\frac{1}{2}(k-2)}\Gamma(\frac{1}{2}k)\, l$, *for* $\delta > p + \frac{1}{2}(k-1)$.

(*b*) *If* $S_R^\gamma(x) \to l$ *as* $R \to \infty$, *then* $\lim_{y\to 0} f_p(y) = l/2^{\frac{1}{2}(k-2)}\Gamma(\frac{1}{2}k)$, *for*

$$p > \max(1, \gamma - \tfrac{1}{2}(k-3)).$$

It is our object to prove that, if $f_p(t)$ is of bounded variation in $0 < t < \infty$, then $S_R^\delta(x)$ is of bounded variation in $0 < R < \infty$; in other words, the multiple Fourier series is absolutely summable (ν^2, δ), or summable $|\nu^2, \delta|$, and conversely. If $k = 1$, summability $|\nu^2, \delta|$ is known to be equivalent to absolute Cesàro summability $|C, \delta|$ (**2**).

We prove besides that Riesz's summability $|\nu^2, \delta|$ is a local property for multiple Fourier series for $\delta > \frac{1}{2}(k+1)$, thus generalizing a known result on single Fourier series, and also establish a connexion between summability $|\nu^2, \delta|$ and summability (ν^2, δ).

2. Let $J_\mu(t)$ be the Bessel function of order μ; $V_\mu(t) = J_\mu(t)/t^\mu$. Then we have the following formulae (**5**):

$$V_\mu(t) = \frac{\cos(t+c)}{t^{\mu+\frac{1}{2}}} + O\left(\frac{1}{t^{\mu+\frac{3}{2}}}\right), \quad \text{as} \quad t \to \infty, \tag{2·1}$$

$$\int_0^\infty \frac{J_\nu(t)\,dt}{t^{\nu-\mu+1}} = \frac{\Gamma(\frac{1}{2}\mu)}{2^{\nu-\mu+1}\Gamma(\nu-\frac{1}{2}\mu+1)}, \quad \text{for} \quad 0 < \mu < \nu + \tfrac{3}{2}. \tag{2·2}$$

LEMMA* 1. *If* $p \geqslant 1$, $f_p(t) = O(1)$, *as* $t \to \infty$.

LEMMA 2. *If* $p \geqslant 1$, *then* $f_p'(t) = \frac{1}{t}[c_1 f_{p-1}(t) + c_2 f_{p-2}(t)]$, *for almost all* t.

For, in this case, $$f_p(t) = c\phi_p(t)/t^{k+2p-2}$$

is absolutely continuous in every finite interval excluding the origin, so that almost everywhere,

$$f_p'(t) = c\frac{\phi_p'(t)}{t^{k+2p-2}} + c_1\frac{\phi_p(t)}{t^{k+2p-1}}.$$

But $$\phi_p'(t) = t\phi_{p-1}(t).$$

Hence $$f_p'(t) = c\frac{\phi_{p-1}(t)}{t^{k+2p-3}} + c_1\frac{\phi_p(t)}{t^{k+2p-1}} = c_2\frac{f_{p-1}(t)}{t} + c_3\frac{f_p(t)}{t}.$$

The next lemma relates to a sharpening of relation (3·11) of Paper I (cf. remarks at the end of theorem I of that paper). It should be noted, however, that the present lemma does not entirely replace the previous

* (**4**), (2·12).

theorem, because the sharper result of this lemma holds only under an additional restriction, namely, $p \geqslant 1$. Thus we have:

LEMMA 3. *If* $\delta > p-1+\frac{1}{2}(k-1)$, $p \geqslant 1$, *then*

$$S_R^\delta(x) = R^{k+2p}\int_0^\infty t^{k+2p-1}f_p(t)\,V_{p+\frac{1}{2}k+\delta}(tR)\,dt. \tag{2·3}$$

Proof. Case (i). Suppose $p \geqslant 1$ and p is an integer. Then this lemma is a direct consequence of theorem I of the previous paper.

Case (ii). Suppose $p \geqslant 1$ and p is not an integer. Let $p = q+r$, where $0<q<1$ and r is an integer. Then, using theorem I of the previous paper, we can at once say

$$S_R^\delta(x) = 2^\delta\Gamma(\delta+1)\,R^{k+2q}\int_0^\infty t\phi_q(t)\,V_{\delta+q+\frac{1}{2}k}(tR)\,dt, \tag{2·4}$$

provided that $\delta > \frac{1}{2}(k-1)$.

We now integrate the right-hand side of (2·4) by parts r times, and each time use the relation

$$\phi_p(t) = O(t^{k+2p-2}), \quad p \geqslant 1, \quad \text{as} \quad t \to \infty,$$

(cf. lemma 1) and obtain

$$cS_R^\delta(x) = R^{k+2p}\int_0^\infty t^{k+2p-1}f_p(t)\,V_{p+\frac{1}{2}k+\delta}(tR)\,dt,$$

provided that $\delta > q+r-1+\frac{1}{2}(k-1)$, which is the required formula.

3·1. THEOREM I. *If $f_p(t)$ is of bounded variation in $0<t<\infty$, then $S_R^\delta(x)$ is of bounded variation in $0<R<\infty$, for $\delta > p+\frac{1}{2}(k-1)$.*

Proof. Write

$$\psi(u) = u^{k+2p-1}V_{p+\frac{1}{2}k+\delta}(u),$$

$$\psi_1(u) = \int_0^u \psi(v)\,dv.$$

Then, if $\delta > p+\frac{1}{2}(k-1)$,

$$\psi(u) = O(1/u), \quad \text{as} \quad u \to \infty. \tag{3·11}$$

Now

$$\begin{aligned} cS_R^\delta(x) &= R\int_0^\infty \psi(tR)f_p(t)\,dt = \int_0^\infty f_p(t)\,d\psi_1(tR) \\ &= [f_p(t)\,\psi_1(tR)]_0^\infty - \int_0^\infty \psi_1(tR)\,df_p(t) \\ &= -\int_0^\infty df_p(t)\int_0^{tR}\psi(u)\,du \\ &= -\int_0^\infty df_p(t)\int_0^R \psi(ut)\,du \\ &= -\int_0^R du\int_0^\infty t\psi(ut)\,df_p(t). \end{aligned} \tag{3·12}$$

The last step is permissible since, in view of (3·11), the integral is absolutely convergent.

From (3·12), substituting for ψ, it follows that

$$C\int_0^\infty |\,dS_R^\delta\,| \leqslant \int_0^\infty u^{k+2p-1}\,du \left|\int_0^\infty t^{k+2p} V_{p+\frac{1}{2}k+\delta}(tu)\,df_p(t)\right|$$

$$\leqslant \int_0^\infty u^{k+2p-1}\,du \int_0^\infty t^{k+2p}\,|\,V_{p+\frac{1}{2}k+\delta}(tu)\,|\,|\,df_p\,|$$

$$= \int_0^\infty t^{k+2p}\,|\,df_p\,| \int_0^\infty u^{k+2p-1}\,|\,V_{p+\frac{1}{2}k+\delta}(tu)\,|\,du, \qquad (3·13)$$

which is justified since $\delta > p+\frac{1}{2}(k-1)$.

Let $\quad J = \int_0^\infty u^{k+2p-1}\,|\,V_{p+\frac{1}{2}k+\delta}(ut)\,|\,du = \int_0^{1/t} + \int_{1/t}^\infty = J_1 + J_2, \quad$ say.

Then
$$|\,J_1\,| = O\left(\int_0^{1/t} u^{k+2p-1}\,du\right) = O\left(\frac{1}{t^{k+2p}}\right),$$

and
$$|\,J_2\,| = O\left(\frac{1}{t^{p+\frac{1}{2}k+\delta+\frac{1}{2}}}\int_{1/t}^\infty u^{p+\frac{1}{2}k-\delta-\frac{3}{2}}\,du\right) = O\left(\frac{1}{t^{k+2p}}\right),$$

so that
$$|\,J\,| = O\left(\frac{1}{t^{k+2p}}\right).$$

Using this in (3·13) we obtain

$$\int_0^\infty |\,dS_R^\delta\,| = O\left(\int_0^\infty |\,df_p(t)\,|\right) = O(1),$$

which is the required result.

THEOREM II. *If S_R^δ is of bounded variation in $0 < R < \infty$, then $f_p(t)$ is of bounded variation in $0 < t < \infty$ for $p > \max(1, \delta - \frac{1}{2}(k-3))$.*

Proof. Using
$$f_p(t) = ct^{2\delta+2}\int_0^\infty S_R^\delta R^{2\delta+1} V_{p+\delta+\frac{1}{2}k}(tR)\,dR,$$

which holds for $p > 1$ and $\delta > \frac{1}{2}(k-1)$, and applying the same argument as in theorem I, we get the result.

3·2. By theorem I it follows that the multiple Fourier series is absolutely summable (ν^2, δ), for $\delta > p + \frac{1}{2}(k-1)$, whenever $f_p(t)$ is of bounded variation, and *ipso facto* summable (ν^2, δ); in fact, we can state something more. We can lower the order of summability to $\delta > p - 1 + \frac{1}{2}(k-1)$, if only $p > 1$.

THEOREM III. *If $f_p(t)$ is of bounded variation in $0<t<\infty$, and $p \geqslant 1$, then*

$$\lim_{R\to\infty} S_R^\delta(x) = 2^{\frac{1}{2}(k-2)}\Gamma(\tfrac{1}{2}k) f_p(+0),$$

for $\delta > p-1+\frac{1}{2}(k-1)$.

Proof. $f_p(t+0)$ exists for every $t>0$, since $f_p(t)$ is of bounded variation in $0<t<\infty$. Also, to begin with, we may assume that $f_p(t)$ is bounded and monotonic and $f_p(+0)$ is zero. This involves no loss of generality, since a function of bounded variation can be expressed as the difference of two bounded monotonic functions.

From (2·3) we have, for $p>1$,

$$S_R^\delta(x) = cR^{k+2p}\int_0^\infty t^{k+2p-1} f_p(t)\, V_{p+\frac{1}{2}k+\delta}(tR)\,dt,$$

for $\delta > p-1+\frac{1}{2}(k-1)$.

Let
$$I = cR^{k+2p}\left[\int_0^\eta + \int_\eta^\omega + \int_\omega^\infty\right] = I_1+I_2+I_3, \tag{3·21}$$

where η is such that $|f_p(t)|<\epsilon$ for $|t|\leqslant \eta$ and any arbitrary $\epsilon>0$, and ω is sufficiently large.

Applying the second mean-value theorem to I_1, we have

$$I_1 = cR^{k+2p}\left[f_p(+0)\int_0^{\eta_1} t^{k+2p-1}V_{p+\frac{1}{2}k+\delta}(tR)\,dt + f_p(\eta-0)\int_{\eta_1}^{\eta} t^{k+2p-1}V_{p+\frac{1}{2}k+\delta}(tR)\,dt\right]$$

$$= c\left[f_p(\eta-0)\int_{\eta_1 R}^{\eta R} z^{k+2p-1}V_{p+\frac{1}{2}k+\delta}(z)\,dz\right],$$

where $0<\eta_1<\eta$.

The integral on the right-hand side is finite, and $|f_p(\eta-0)|<\epsilon$ so that

$$|I_1| < c_1\epsilon. \tag{3·22}$$

Similarly

$$I_3 = c\left[f_p(\omega+0)\int_{\omega R}^{\omega_1 R} z^{k+2p-1}V_{p+\frac{1}{2}k+\delta}(z)\,dz + f_p(\infty)\int_{\omega_1 R}^{\infty} z^{k+2p-1}V_{p+\frac{1}{2}k+\delta}(z)\,dz\right], \tag{3·23}$$

where $\omega<\omega_1<\infty$.

We know that $f_p(\omega+0)$ and $f_p(\infty)$ are finite, and that

$$\int_0^\infty z^{k+2p-1}V_{p+\frac{1}{2}k+\delta}(z)\,dz$$

converges for $\delta>p-1+\frac{1}{2}(k-1)$ by lemma 2, so that, for a fixed ω the integrals on the right-hand side of (3·23) can be made less than an arbitrary $\epsilon>0$ in absolute value, for $R>R_1$. Hence

$$|I_3| < c_2\epsilon. \tag{3·24}$$

Again,

$$I_2 = cR^{k+2p}\left[\int_\eta^\omega t^{k+2p-1} f_p(t)\, V_{p+\frac{1}{2}k+\delta}(tR)\, dt\right]$$

$$= cR^{k+2p}\left[\int_\eta^\omega f_p(t)\, t^{k+2p-1}\frac{\cos(tR+c_4)}{(tR)^{p+\frac{1}{2}k+\delta+\frac{1}{2}}}\, dt\right]$$

$$+ O\left(R^{k+2p}\int_\eta^\omega |f_p(t)|\frac{t^{k+2p-1}}{(tR)^{p+\frac{1}{2}k+\delta+\frac{3}{2}}}\, dt\right)$$

$$= I_{2,1} + I_{2,2}, \quad \text{say},$$

$$I_{2,1} = \frac{c}{R^{\delta-\frac{1}{2}k-p+\frac{1}{2}}}\left[f_p(\eta+0)\int_\eta^{\omega_2}\frac{\cos(tR+c_4)}{t^{\delta-p-\frac{1}{2}k+\frac{3}{2}}}\, dt + f_p(\omega_2-0)\int_{\omega_2}^{\omega}\frac{\cos(tR+c_4)}{t^{\delta-p-\frac{1}{2}k+\frac{3}{2}}}\, dt\right]$$

$$= c\left[f_p(\eta+0)\int_{\eta R}^{\omega_2 R}\frac{\cos(x+c_4)}{x^{\delta-p-\frac{1}{2}k+\frac{3}{2}}}\, dx + f_p(\omega_2-0)\int_{\omega_2 R}^{\omega R}\frac{\cos(x+c_4)}{x^{\delta-p-\frac{1}{2}k+\frac{3}{2}}}\, dx\right].$$

The two integrals on the right-hand side can be made less than any arbitrary $\epsilon > 0$ in absolute value, provided R is sufficiently large. Thus

$$|I_{2,1}| < c_3\epsilon, \quad \text{for} \quad R > R_2. \tag{3·25}$$

Also

$$|I_{2,2}| \leqslant \left(\frac{c_4}{R^{\delta-\frac{1}{2}k-p+\frac{3}{2}}}\int_\eta^\omega |f_p(t)|\frac{dt}{t^{\delta-p-\frac{1}{2}k+\frac{5}{2}}}\right) \tag{3·26}$$

$$< c_4\epsilon,$$

for $R > R_3$, if $\delta > \frac{1}{2}(k-3)+p$.

Combining (3·25) and (3·26) we obtain $|I_2| = o(1)$.

3·3. We shall now prove the local property of absolute summability (ν^2, δ) for multiple Fourier series.

THEOREM IV. *Summability* $|\nu^2, \delta|$, *for* $\delta > \frac{1}{2}(k+1)$, *of the multiple Fourier series of* $f(x)$ *at any point depends only on the behaviour of the function in the neighbourhood of that point.*

Proof. If $p > 1$, using the reasoning in lemma 2, we obtain, in virtue of relation (3·12),

$$c\int |dS_R^\delta| = \int dR\left|R^{k+2p-1}\int_0^\infty t^{k+2p}V_{p+\frac{1}{2}k+\delta}(tR)\, df_p\right|$$

$$= \int dR\left|R^{k+2p-1}\left(\int_0^\eta + \int_\eta^\infty\right)\right|$$

$$= \int dR\,|J_1 + J_2|, \quad \text{say}. \tag{3·31}$$

Now, if $p > 1$, $f_p'(t)$ exists for almost all t, so that, by lemma 2, we have

$$J_2 = c_1 R^{k+2p-1} \int_\eta^\infty t^{k+2p-1} f_{p-1}(t) V_{p+\frac{1}{2}k+\delta}(tR)\, dt$$

$$+ c_2 R^{k+2p-1} \int_\eta^\infty t^{k+2p-1} f_p(t) V_{p+\frac{1}{2}k+\delta}(tR)\, dt$$

$$= J_{2,1} + J_{2,2}, \quad \text{say,} \tag{3·32}$$

$$|J_{2,2}| = O\left(\frac{1}{R^{\delta-p-\frac{1}{2}k+\frac{3}{2}}}\right) \int_\eta^\infty \frac{dt}{t^{\delta-p-\frac{1}{2}k+\frac{3}{2}}}, \quad \text{by lemma 1,}$$

$$= O\left(\frac{1}{R^{\delta-p-\frac{1}{2}k+\frac{3}{2}}}\right), \quad \text{if} \quad \delta > p + \tfrac{1}{2}(k-1). \tag{3·33}$$

If $$F_{p-1}(t) = \int_0^t s^{k+2p-3} |f_{p-1}(s)|\, ds,$$

then $$\int_\eta^\infty \frac{t^{k+2p-3} |f_{p-1}(t)|\, dt}{t^{\delta+p+\frac{1}{2}(k-3)}} = \left[\frac{F_{p-1}(t)}{t^{\delta+p+\frac{1}{2}(k-3)}}\right]_\eta^\infty + c_3 \int_\eta^\infty \frac{F_{p-1}(t)\, dt}{t^{\delta+p+\frac{1}{2}(k-1)}} = O(1),$$

by lemma 1, provided $\delta > p + \frac{1}{2}(k-1)$. Therefore

$$|J_{2,1}| = O\left(\frac{1}{R^{\delta-p-\frac{1}{2}k+\frac{3}{2}}}\right), \quad \text{for} \quad \delta > p + \tfrac{1}{2}(k-1); \tag{3·34}$$

and so, by (3·32), it follows that

$$\int_\eta^\infty |J_2|\, dR < \infty. \tag{3·35}$$

From (3·31) and (3·35) it follows that a necessary and sufficient condition for summability $|\nu^2, \delta|$, $\delta > \frac{1}{2}(k-1)$, is that $\int_\eta^\infty |J_1|\, dR < \infty$, which proves the theorem.

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Madras, India

CANONICAL FACTORIZATION OF PSEUDO-UNITARY MATRICES

By H. C. LEE

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Introduction

A square matrix A is said to be *pseudo-unitary* if

$$A^*\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}A = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \tag{1}$$

(where a star denotes a conjugate transpose). The integers p and q will be fixed once for all. When A is real it is called *pseudo-orthogonal*. Pseudo-orthogonal matrices have been studied by many writers.† Extending Hsu's method after arriving at the splitting invariants of a pseudo-unitary matrix (§ 1), we shall obtain in a very short way a preliminary factorization for such a matrix (§ 2), a result similar to one of Autonne's. Introducing the nucleus (§ 3) we then elaborate a canonical factorization (§ 4) which is unique when certain submatrices are non-singular (§ 5). The paper ends with an application to the theory of plane factorization (§ 6).

1. *The splitting invariants*

Split A in the block form

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \tag{2}$$

where A_1 is $p \times p$ and A_4 is $q \times q$. By (1) we have

$$A_1^* A_1 = I_p + A_3^* A_3, \quad A_4^* A_4 = I_q + A_2^* A_2, \quad A_1^* A_2 = A_3^* A_4.$$

The first two relations imply that A_1 and A_4 are non-singular, since

† A. Brill, *Das Relativitätsprinzip*, Teubner (1912), 31. L. Autonne, *Ann. Univ. Lyon*, 38 (1915), 1–70. H. C. Lee, "On plane factorizations of pseudo-Euclidean rotations", *Quart. J. of Math.* (Oxford series), 15 (1944), 7–10. P. L. Hsu, "On a factorization of pseudo-orthogonal matrices" (to appear elsewhere). I am indebted to Dr Hsu for communicating to me his paper in manuscript form.

$I_p + A_3^* A_3$, for example, is the matrix of a positive definite Hermitian form; whence the third relation may be written

$$A_2 = MA_4, \quad A_3 = M^* A_1, \tag{3}$$

where M is a $p \times q$ matrix. Substituting these in the first two relations we get

$$(A_1 A_1^*)^{-1} = I_p - MM^*, \quad (A_4 A_4^*)^{-1} = I_q - M^* M. \tag{4}$$

We have now to make use of

LEMMA 1. *M being any $p \times q$ matrix of rank r, there exist a unitary matrix U of order p and a unitary matrix V of order q such that*

$$U^* M V = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix},$$

where Δ is a diagonal matrix with r diagonal elements which are the square roots of the non-zero (and so positive) characteristic values of the positive semi-definite hermitian matrix M^*M (of order q, rank r).†

Now the non-zero characteristic values of M^*M being necessarily less than 1 by (4), we may write them in the form $\tanh^2 \theta_i$ $(i = 1, \ldots, r)$, where the θ's are real and positive. Lemma 1 then gives

$$M = U \begin{pmatrix} \tanh \theta_1 \dotplus \ldots \dotplus \tanh \theta_r & 0 \\ 0 & 0 \end{pmatrix} V^*. \tag{5}$$

We find by (4)

$$\left.\begin{aligned} A_1 A_1^* &= U \begin{pmatrix} \cosh^2 \theta_1 \dotplus \ldots \dotplus \cosh^2 \theta_r & 0 \\ 0 & I_{p-r} \end{pmatrix} U^*, \\ A_4 A_4^* &= V \begin{pmatrix} \cosh^2 \theta_1 \dotplus \ldots \dotplus \cosh^2 \theta_r & 0 \\ 0 & I_{q-r} \end{pmatrix} V^*. \end{aligned}\right\} \tag{6}$$

We observe immediately from (6) that we have

THEOREM 1. *The characteristic values of the hermitian matrices $A_1 A_1^*$ and $A_4 A_4^*$ (of orders p and q respectively) are all $\geqq 1$, and those of these values which are > 1, namely, $\cosh^2 \theta_1, \ldots, \cosh^2 \theta_r$, are the same for the two matrices.*

The numbers $\theta_1, \ldots, \theta_r$, as well as the integer r, which were first defined by means of the auxiliary matrix M, receive now an invariant meaning, being intrinsically connected with the matrix A in the manner described in theorem 1. We therefore introduce the

DEFINITION. The real and positive numbers $\theta_1, \ldots, \theta_r$ introduced in theorem 1 are called the *splitting invariants* of the pseudo-unitary matrix A.

In particular, the integer r is a numerical invariant of A, and

$$r \leqq \min(p, q). \tag{7}$$

† This is an extension of a known result; see C. C. MacDuffee, *The theory of matrices* (1933), 78. We defer the proof to Appendix 1.

2. *Preliminary factorization*

Returning now to (6) we easily see that A_1 and A_4 are of the form

$$\left.\begin{aligned} A_1 &= U\begin{pmatrix}\cosh\theta_1 \dotplus \ldots \dotplus \cosh\theta_r & 0 \\ 0 & I_{p-r}\end{pmatrix} X, \\ A_4 &= V\begin{pmatrix}\cosh\theta_1 \dotplus \ldots \dotplus \cosh\theta_r & 0 \\ 0 & I_{q-r}\end{pmatrix} Y, \end{aligned}\right\} \tag{8}$$

where X and Y are unitary matrices of orders p and q respectively. By (3) we find that

$$\left.\begin{aligned} A_2 &= U\begin{pmatrix}\sinh\theta_1 \dotplus \ldots \dotplus \sinh\theta_r & 0 \\ 0 & 0\end{pmatrix} Y, \\ A_3 &= V\begin{pmatrix}\sinh\theta_1 \dotplus \ldots \dotplus \sinh\theta_r & 0 \\ 0 & 0\end{pmatrix} X. \end{aligned}\right\} \tag{9}$$

Hence (2) may be written

$$A = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} T & 0 & S & 0 \\ 0 & I_{p-r} & 0 & 0 \\ S & 0 & T & 0 \\ 0 & 0 & 0 & I_{q-r} \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \tag{10}$$

where $\quad T = \cosh\theta_1 \dotplus \ldots \dotplus \cosh\theta_r, \quad S = \sinh\theta_1 \dotplus \ldots \dotplus \sinh\theta_r, \qquad (11)$

and we have

THEOREM 2. *A factorization of the pseudo-unitary matrix A, whose splitting invariants are $\theta_1, \ldots, \theta_r$, is given by* (10), *where T and S are defined by* (11) *and U, X are unitary matrices of order p, and V, Y are unitary matrices of order q.*†

3. *The nucleus*

We may push on to a deeper analysis of the factorization. Let $\theta_1, \ldots, \theta_s$ be the *distinct* members among the splitting invariants (thus $s \leqq r$) and let their *multiplicities* be $r_1, \ldots, r_s$ respectively, so that

$$r_1 + \ldots + r_s = r. \tag{12}$$

Then we may write (11) in the form

$$T = \cosh\theta_1 I_{r_1} \dotplus \ldots \dotplus \cosh\theta_s I_{r_s}, \quad S = \sinh\theta_1 I_{r_1} \dotplus \ldots \dotplus \sinh\theta_s I_{r_s}. \tag{13}$$

Though really unnecessary, we shall for aesthetic reasons transform the middle matrix on the right of (10) by interchanging the first two (block) rows and the first two (block) columns, thus obtaining a matrix of Autonne's form

$$N = \begin{pmatrix} I_{p-r} & 0 & 0 & 0 \\ 0 & T & S & 0 \\ 0 & S & T & 0 \\ 0 & 0 & 0 & I_{q-r} \end{pmatrix}. \tag{14}$$

† Cf. C. C. MacDuffee, *loc. cit.*, 68.

In doing so the form of (10) is preserved, since only the columns of U and the rows of X are permuted.

DEFINITION. The real matrix N in (14), where T and S are defined by (13) and where $\theta_1, \ldots, \theta_s$ are the distinct splitting invariants of A in increasing order and with respective multiplicities $r_1, \ldots, r_s$, is called the *nucleus* of A.

The nucleus N, being invariantly connected with A, is evidently *uniquely determined*. It is (like A) a pseudo-unitary matrix, satisfying (1), as one can see at once by direct verification. Since N is real it is also a pseudo-orthogonal matrix, of a special form.

Let us now inquire how (to what extent) N is unaltered when pre-multiplied by a unitary matrix L and post-multiplied by a unitary matrix R^*

$$LNR^* = N. \tag{15}$$

When this is so, the right-hand side of (10) may be written

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} LNR^* \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}. \tag{16}$$

We prescribe the choice of L so that when it multiplies into the first factor of (16) it does not disturb the simplicity of that factor as a direct sum, and likewise choose R^* so that its product with the last factor of (16) is a similar direct sum. This is equivalent to supposing that L and R are themselves similar direct sums (i.e. the first direct summand is $p \times p$ and the second $q \times q$). Therefore we write

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \dotplus \begin{pmatrix} L_{33} & L_{34} \\ L_{43} & L_{44} \end{pmatrix}, \quad R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \dotplus \begin{pmatrix} R_{33} & R_{34} \\ R_{43} & R_{44} \end{pmatrix}, \tag{17}$$

where, for L, L_{11} is $(p-r) \times (p-r)$, L_{22} is $r \times r$, L_{33} is $r \times r$, L_{44} is $(q-r) \times (q-r)$, and similarly for R.

Now write (15) in the form

$$LN = NR. \tag{18}$$

By (14) and (17), (18) may be written

$$\begin{pmatrix} L_{11} & L_{12} & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ 0 & 0 & L_{33} & L_{34} \\ 0 & 0 & L_{43} & L_{44} \end{pmatrix} \begin{pmatrix} I_{p-r} & 0 & 0 & 0 \\ 0 & T & S & 0 \\ 0 & S & T & 0 \\ 0 & 0 & 0 & I_{q-r} \end{pmatrix}$$
$$= \begin{pmatrix} I_{p-r} & 0 & 0 & 0 \\ 0 & T & S & 0 \\ 0 & S & T & 0 \\ 0 & 0 & 0 & I_{q-r} \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} & 0 & 0 \\ R_{21} & R_{22} & 0 & 0 \\ 0 & 0 & R_{33} & R_{34} \\ 0 & 0 & R_{43} & R_{4} \end{pmatrix},$$

whence, multiplying and equating, we have

$$\begin{array}{llll} L_{11} = R_{11}, & L_{12}T = R_{12}, & L_{12}S = 0, & 0 = 0, \\ L_{21} = TR_{21}, & L_{22}T = TR_{22}, & L_{22}S = SR_{33}, & 0 = SR_{34}, \\ 0 = SR_{21}, & L_{33}S = SR_{22}, & L_{33}T = TR_{33}, & L_{34} = TR_{34}, \\ 0 = 0, & L_{43}S = 0, & L_{43}T = R_{43}, & L_{44} = R_{44}. \end{array}$$

Since S is non-singular, four of these relations imply at once $L_{12} = 0$, $R_{34} = 0$, $R_{21} = 0$, $L_{43} = 0$; then another four give $R_{12} = 0$, $L_{21} = 0$, $L_{34} = 0$, $R_{43} = 0$; thus L and R reduce to block diagonal form

$$L = L_{11} \dotplus L_{22} \dotplus L_{33} \dotplus L_{44}, \quad R = R_{11} \dotplus R_{22} \dotplus R_{33} \dotplus R_{44}. \tag{19}$$

Since T is also non-singular, the remaining relations can be solved for the R's in terms of the L's;

$$R_{11} = L_{11}, \quad \begin{cases} R_{22} = T^{-1}L_{22}T = S^{-1}L_{33}S, \\ R_{33} = S^{-1}L_{22}S = T^{-1}L_{33}T, \end{cases} \quad R_{44} = L_{44}. \tag{20}$$

Then
$$L_{33} = ST^{-1}L_{22}TS^{-1} = TS^{-1}L_{22}ST^{-1}. \tag{21}$$

Since T and S being diagonal are permutable, we have from (21) $GL_{22} = L_{22}G$, where
$$G = S^2T^{-2} = \tanh^2\theta_1 I_{r_1} \dotplus \dots \dotplus \tanh^2\theta_s I_{r_s}.$$

If therefore we write

$$L_{22} = \begin{pmatrix} L_{22,11} & \dots & L_{22,1s} \\ \vdots & & \vdots \\ L_{22,s1} & \dots & L_{22,ss} \end{pmatrix},$$

where $L_{22,ii}$ is $r_i \times r_i$ $(i = 1, \dots, s)$, on account of the distinctness of $\tanh^2\theta_1, \dots, \tanh^2\theta_s$, the equation $GL_{22} = L_{22}G$ implies immediately that $L_{22,ik} = 0$ $(i \neq k;\ i, k = 1, \dots, s)$, so that L_{22} has block diagonal form

$$L_{22} = L_1 \dotplus \dots \dotplus L_s, \tag{22}$$

where for brevity we write L_i for $L_{22,ii}$ which is $r_i \times r_i$ $(i = 1, \dots, s)$. Then (21) becomes $L_{33} = L_{22}$, and (20) gives $R_{\rho\rho} = L_{\rho\rho}$ $(\rho = 1, 2, 3, 4)$. Thus, by (19), we have

$$L = L_{11} \dotplus (L_1 \dotplus \dots \dotplus L_s) \dotplus (L_1 \dotplus \dots \dotplus L_s) \dotplus L_{44}, \quad R = L. \tag{23}$$

We state the above result in the following

THEOREM 3. *If L and R are unitary matrices, of the form* (17), *which verify the relation* (18), *then they are of the form* (23), *where L_{11}, L_i $(i = 1, \dots, s)$ and L_{44} are unitary matrices* (*of orders already indicated*).

4. *Canonical factorization*

A further factorization can now be elaborated from (16) by properly choosing the matrices L_{11}, L_i $(i = 1, \ldots, s)$ and L_{44} which appear in the constitution of L and R in the manner of (23). In (16), the matrix R^*, which equals L^* by (23), may be absorbed into the last factor resulting in a factor of the same form (with X and Y changed), while the matrix L multiplies the first factor changing U, V into

$$U\begin{pmatrix} L_{11} & 0 \\ 0 & L_{22} \end{pmatrix}, \quad V\begin{pmatrix} L_{33} & 0 \\ 0 & L_{44} \end{pmatrix}. \tag{24}$$

To perform this reduction in a more specific fashion, we need

LEMMA 2. *Every square matrix can be post-multiplied by a unitary matrix so that the product is left-triangular (i.e. having zero elements on the right of the principal diagonal) and has real and non-negative diagonal elements.*†

Then write

$$U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} = \begin{pmatrix} U_1 & & U_2 & \\ & U_{4,11} & \cdots & U_{4,1s} \\ U_3 & \vdots & & \vdots \\ & U_{4,s1} & & U_{4,ss} \end{pmatrix}, \quad V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}, \tag{25}$$

where U_1 is $(p-r)\times(p-r)$, U_4 is $r \times r$, $U_{4,ii}$ is $r_i \times r_i$ $(i = 1, \ldots, s)$, V_1 is $r \times r$, and V_4 is $(q-r)\times(q-r)$. The products in (24) can now be written

$$\begin{pmatrix} U_1 L_{11} & & U_2 L_{22} & \\ & U_{4,11} L_1 & \cdots & U_{4,1s} L_s \\ U_3 L_{11} & \vdots & & \vdots \\ & U_{4,s1} L_1 & \cdots & U_{4,ss} L_s \end{pmatrix}, \quad \begin{pmatrix} V_1 L_{33} & V_2 L_{44} \\ V_3 L_{33} & V_4 L_{44} \end{pmatrix}.$$

We may then choose the unitary matrices L_{11}, L_i $(i = 1, \ldots, s)$ and L_{44} in accordance with lemma 2 such that the products $U_1 L_{11}$, $U_{4,ii} L_i$ $(i = 1, \ldots, s)$ and $V_4 L_{44}$ are left-triangular and have real and non-negative diagonal elements. We suppose this reduction already effected, i.e. in (25) the square blocks U_1, $U_{4,ii}$ $(i = 1, \ldots, s)$ and V_4 are left-triangular matrices having real and non-negative diagonal elements. Hence

THEOREM 4. *A canonical factorization of the pseudo-unitary matrix A, whose nucleus is N, is given by*

$$A = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} N \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \tag{26}$$

† We again defer the proof to Appendix 2. Cf. also H. W. Turnbull and A. C. Aitken, *An introduction to the theory of canonical matrices* (1932), 96 (Schmidt's theorem).

where U, V, X, Y are unitary matrices (of orders p, q, p, q respectively), and if we write U, V in the form (25) *the blocks*

$$U_1, \quad U_{4,ii} \quad (i = 1, \ldots, s), \quad V_4 \tag{27}$$

are left-triangular matrices having real and non-negative diagonal elements.

To count the number of real parameters in the canonical factorization (26), note that a general unitary matrix of order n depends on n^2 real parameters. Thus X and Y together depend on p^2+q^2 real parameters. If U and V were general, they would also involve p^2+q^2 real parameters; but we have used L_{11}, L_i $(i = 1, \ldots, s)$ and L_{44}, which together contain $(p-r)^2 + \sum_{i=1}^{s} r_i^2 + (q-r)^2$ real parameters, to remove the same number of real parameters in U_1, $U_{4,ii}$ $(i = 1, \ldots, s)$ and V_4. Finally, the nucleus N contains s real parameters (namely, $\theta_1, \ldots, \theta_s$). Hence

COROLLARY 1. *The canonical factorization* (26) *depends on*

$$\begin{aligned}\{2p^2+2q^2+s\} - \left\{(p-r)^2 + \sum_{i=1}^{s} r_i^2 + (q-r)^2\right\} \\ = (p+q)^2 - 2(p-r)(q-r) - \sum_{i=1}^{s} (r_i^2 - 1)\end{aligned} \tag{28}$$

real parameters.

In the extreme case when $r = \min(p, q)$ and the splitting invariants are all distinct, $s = r$, $r_i = 1$ $(i = 1, \ldots, s)$, the number (28) attains its maximum value $(p+q)^2$ which is the number of real parameters in a general pseudo-unitary matrix (of order $p+q$).

While formal results in the complex case are valid also in the real case, the counting of parameters is very different since in the real case an orthogonal matrix of order n depends on $\frac{1}{2}n(n-1)$ parameters only. Hence, instead of corollary 1, we now have

COROLLARY 2. *In the real case the canonical factorization* (26) *depends on*

$$\begin{aligned}&\{2 . \tfrac{1}{2}p(p-1) + 2 . \tfrac{1}{2}q(q-1) + s\} \\ &\quad - \left\{\tfrac{1}{2}(p-r)(p-r-1) + \sum_{i=1}^{s} \tfrac{1}{2}r_i(r_i-1) + \tfrac{1}{2}(q-r)(q-r-1)\right\} \\ &= \tfrac{1}{2}(p+q)(p+q-1) - (p-r)(q-r) - \sum_{i=1}^{s} \tfrac{1}{2}(r_i+2)(r_i-1)\end{aligned} \tag{29}$$

parameters.

In the extreme case mentioned above, the number (29) reaches its maximum $\frac{1}{2}(p+q)(p+q-1)$ which is the number of parameters in a general pseudo-orthogonal matrix (of order $p+q$).

5. *Uniqueness*

We wish to prove

THEOREM 5. *The canonical factorization* (26) *in theorem* 4 *is unique when the triangular blocks* (27) *are non-singular.*

We have seen that the nucleus N is unique. Let

$$A = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} N \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = \begin{pmatrix} \tilde{U} & 0 \\ 0 & \tilde{V} \end{pmatrix} N \begin{pmatrix} \tilde{X} & 0 \\ 0 & \tilde{Y} \end{pmatrix},$$

where $\tilde{U}$, $\tilde{V}$, $\tilde{X}$, $\tilde{Y}$ are subject to the same conditions as U, V, X, Y in theorem 4. Then we have a relation of the form (18), where now

$$\left.\begin{aligned} L &= \begin{pmatrix} \tilde{U} & 0 \\ 0 & \tilde{V} \end{pmatrix}^* \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} \tilde{U}^*U & 0 \\ 0 & \tilde{V}^*V \end{pmatrix}, \\ R &= \begin{pmatrix} \tilde{X} & 0 \\ 0 & \tilde{Y} \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}^* = \begin{pmatrix} \tilde{X}X^* & 0 \\ 0 & \tilde{Y}Y^* \end{pmatrix}. \end{aligned}\right\} \tag{30}$$

The uniqueness will be proved if we can show that both L and R here reduce to the identity matrix.

Since by (30) L and R are unitary matrices of the form (17), theorem 3 implies that they have the form (23). Comparing (30) with (17) we obtain

$$\tilde{U}^*U = \begin{pmatrix} L_{11} & 0 \\ 0 & L_{22} \end{pmatrix}, \quad \tilde{V}^*V = \begin{pmatrix} L_{33} & 0 \\ 0 & L_{44} \end{pmatrix},$$

whence

$$\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} = \begin{pmatrix} \tilde{U}_1 & \tilde{U}_2 \\ \tilde{U}_3 & \tilde{U}_4 \end{pmatrix} \begin{pmatrix} L_{11} & 0 \\ 0 & L_{22} \end{pmatrix} = \begin{pmatrix} \tilde{U}_1 L_{11} & \tilde{U}_2 L_{22} \\ \tilde{U}_3 L_{11} & \tilde{U}_4 L_{22} \end{pmatrix},$$

$$\begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} = \begin{pmatrix} \tilde{V}_1 & \tilde{V}_2 \\ \tilde{V}_3 & \tilde{V}_4 \end{pmatrix} \begin{pmatrix} L_{33} & 0 \\ 0 & L_{44} \end{pmatrix} = \begin{pmatrix} \tilde{V}_1 L_{33} & \tilde{V}_2 L_{44} \\ \tilde{V}_3 L_{33} & \tilde{V}_4 L_{44} \end{pmatrix}.$$

We have in particular $\quad U_1 = \tilde{U}_1 L_{11}, \quad V_4 = \tilde{V}_4 L_{44},$ (31)

and also $U_4 = \tilde{U}_4 L_{22}$ which is further analysed into

$$\begin{pmatrix} U_{4,11} & \cdots & U_{4,1s} \\ \vdots & & \vdots \\ U_{4,s1} & \cdots & U_{4,ss} \end{pmatrix} = \begin{pmatrix} \tilde{U}_{4,11} & \cdots & \tilde{U}_{4,1s} \\ \vdots & & \vdots \\ \tilde{U}_{4,s1} & \cdots & \tilde{U}_{4,ss} \end{pmatrix} \begin{pmatrix} L_1 & & \\ & \ddots & \\ & & L_s \end{pmatrix} = \begin{pmatrix} \tilde{U}_{4,11} L_1 & \cdots & \tilde{U}_{4,1s} L_s \\ \vdots & & \vdots \\ \tilde{U}_{4,s1} L_1 & \cdots & \tilde{U}_{4,ss} L_s \end{pmatrix},$$

which gives in particular

$$U_{4,ii} = \tilde{U}_{4,ii} L_i \quad (i = 1, \ldots, s). \tag{32}$$

Remember that in (31) and (32) the blocks L_{11}, L_i $(i = 1, \ldots, s)$ and L_{44} are unitary matrices, and the other blocks are just the left-triangular matrices which we assume to be non-singular and which therefore have real and *positive* diagonal elements. At this point we need

LEMMA 3. *Let W and $\tilde{W}$ be two left-triangular matrices of equal order, each having real and positive diagonal elements. If K is a unitary matrix*

of the same order such that $W = \tilde{W}K$, *then* K *can only be the identity matrix.*†

Applying this lemma to (31) and (32) we see that L_{11}, L_i $(i = 1, \ldots, s)$ and L_{44} all reduce to identity matrices. So do L and R by (23). This completes the proof.

6. *Application to the theory of plane factorization*

In the real case there is a worked theory of plane factorization which asserts that *every orthogonal matrix of order n, determinant* $+1$, *is factorable into a product of* $\frac{1}{2}n(n-1)$ *plane rotations*,‡ and that *every pseudo-orthogonal matrix A of type* (1), *determinant* $+1$, *is factorable into a product of*

$$\tfrac{1}{2}(p+q)(p+q-1)-\min(p,q) \tag{33}$$

plane rotations and

$$\min(p,q) \tag{34}$$

pseudo-plane rotations.§ However, the factors can be reduced in number according to the structure of the given orthogonal or pseudo-orthogonal matrix, this being clear from the method of factorization|| shortened by the theorem in Appendix 2. Now, we propose to see how much the numbers of factors can be reduced for the pseudo-orthogonal matrix A basing on the canonical factorization (26).

The matrices U, V, X, Y in (26) are in the present case orthogonal matrices and have therefore determinants ± 1. We wish first to point out that *these determinants can be made all equal to* $+1$, possibly at the expense of changing the sign of at most two rows and two columns of the nucleus N on the right of (26). To see this let $I_n^{(\pm 1)}$ stand for the matrix obtained from I_n by replacing its first diagonal element 1 by ± 1, thus

$$\det I_n^{(\pm 1)} = \pm 1, \quad I_n^{(\pm 1)} I_n^{(\pm 1)} = I_n.$$

If we denote

$$\det U = u = \pm 1, \quad \det X = x = \pm 1, \quad \det V = v = \pm 1, \quad \det Y = y = \pm 1,$$

we may write (26) in the form

$$A = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} I_p^{(u)} & 0 \\ 0 & I_q^{(v)} \end{pmatrix} \begin{pmatrix} I_p^{(u)} & 0 \\ 0 & I_q^{(v)} \end{pmatrix} N \begin{pmatrix} I_p^{(x)} & 0 \\ 0 & I_q^{(y)} \end{pmatrix} \begin{pmatrix} I_p^{(x)} & 0 \\ 0 & I_q^{(y)} \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$$

$$= \begin{pmatrix} U_0 & 0 \\ 0 & V_0 \end{pmatrix} N_0 \begin{pmatrix} X_0 & 0 \\ 0 & Y_0 \end{pmatrix}, \tag{35}$$

† For proof see Appendix 3.

‡ F. D. Murnaghan, *The theory of group representations* (1938), 320.

§ H. C. Lee, "On plane factorizations of pseudo-Euclidean rotations", *Quart. J. of Math.* (Oxford series), 15 (1944), 8.

|| F. D. Murnaghan, *loc. cit.* 319–320; Lee, *loc. cit.* 8–10.

where $$U_0 = UI_p^{(u)},\quad V_0 = VI_q^{(v)},\quad X_0 = I_p^{(x)}X,\quad Y_0 = I_q^{(y)}Y, \tag{36}$$

$$N_0 = \begin{pmatrix} I_p^{(u)} & 0 \\ 0 & I_q^{(v)} \end{pmatrix} N \begin{pmatrix} I_p^{(x)} & 0 \\ 0 & I_q^{(y)} \end{pmatrix}. \tag{37}$$

It is clear from (36) that U_0, V_0, X_0, Y_0 are orthogonal matrices of determinant $+1$, and from (37) that N_0 is obtained from N by at most changing the sign of the first and the $(p+1)$th rows and the first and the $(p+1)$th columns.

N_0, like N, is also a special pseudo-orthogonal matrix. By (35) we have $\det A = \det N_0$. A as a pseudo-orthogonal matrix has determinant ± 1. We shall suppose $\det A = +1$, thus also $\det N_0 = +1$.

What is essential for us for the present is only the fact that N_0, like N, has only r non-zero elements on the right of the principal diagonal, and these elements are all found after the pth column. Hence, the method of factorization (shortened by the theorem of Appendix 2) tells us that N_0 *is a product of exactly r pseudo-plane rotations.*

Moreover, if we write (35) in the form

$$\begin{pmatrix} U_0 & 0 \\ 0 & I_q \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & V_0 \end{pmatrix} N_0 \begin{pmatrix} X_0 & 0 \\ 0 & I_q \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & Y_0 \end{pmatrix}, \tag{38}$$

and take account of the special shapes of U_0 and V_0 (see (36) and (25) and remember the triangular blocks), the same shortened method informs us that the number of plane rotations in factorization is at most

$$\begin{array}{ll}
\tfrac{1}{2}p(p-1) - \tfrac{1}{2}(p-r)(p-r-1) - \sum_{i=1}^{s} \tfrac{1}{2}r_i(r_i-1) & \text{for } U_0, \\
\tfrac{1}{2}q(q-1) - \tfrac{1}{2}(q-r)(q-r-1) & \text{for } V_0, \\
\tfrac{1}{2}p(p-1) & \text{for } X_0, \\
\tfrac{1}{2}q(q-1) & \text{for } Y_0.
\end{array}$$

These are respectively the numbers of factors when the four matrices of (38) besides N_0 are factorized. Hence

THEOREM 6. *The pseudo-orthogonal matrix A of determinant unity, whose invariant multiplicities are $r_1, \ldots, r_s$ $(r_1 + \ldots + r_s = r)$, can be factorized into a product of at most*

$$\begin{aligned}
&\{2 \cdot \tfrac{1}{2}p(p-1) + 2 \cdot \tfrac{1}{2}q(q-1)\} \\
&\qquad - \left\{\tfrac{1}{2}(p-r)(p-r-1) + \sum_{i=1}^{s} \tfrac{1}{2}r_i(r_i-1) + \tfrac{1}{2}(q-r)(q-r-1)\right\} \\
&= \tfrac{1}{2}(p+q)(p+q-1) - (p-r)(q-r) - \sum_{i=1}^{s} \tfrac{1}{2}r_i(r_i-1) - r \\
&= \tfrac{1}{2}(p+q)(p+q-1) - (p-r)(q-r) - \sum_{i=1}^{s} \tfrac{1}{2}r_i(r_i+1)
\end{aligned} \tag{39}$$

plane rotations and $\qquad r \qquad$ (40)

pseudo-plane rotations.

The two integers (39) and (40) do not exceed the two integers (33) and (34) respectively. Theorem 6 is therefore an invariantive refinement of the known result. In the extreme case when $r = \min(p, q)$ and the splitting invariants are all distinct, $s = r$, $r_i = 1$ $(i = 1, \ldots, s)$, the two numbers (39) and (40) reach their maxima (33) and (34) respectively.

Finally, we add that a parallel theory can also be developed for the complex case.

Appendix 1

Proof of lemma 1. There exists a unitary matrix V of order q such that

$$V^*M^*MV = \begin{pmatrix} \Delta^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

If Φ and Ψ denote the matrices formed by the first r and last $q-r$ columns of V respectively: $V = (\Phi, \Psi)$, then

$$\begin{aligned} V^*M^*MV &= \begin{pmatrix} \Phi^* \\ \Psi^* \end{pmatrix} M^*M(\Phi, \Psi) \\ &= \begin{pmatrix} \Phi^*M^*M \\ \Psi^*M^*M \end{pmatrix}(\Phi, \Psi) = \begin{pmatrix} \Phi^*M^*M\Phi & \Phi^*M^*M\Psi \\ \Psi^*M^*M\Phi & \Psi^*M^*M\Psi \end{pmatrix}, \end{aligned}$$

whence

$$\Phi^*M^*M\Phi = \Delta^2, \quad \Phi^*M^*M\Psi = 0, \quad \Psi^*M^*M\Phi = 0, \quad \Psi^*M^*M\Psi = 0.$$

Of these four relations the last may be written $(M\Psi)^*(M\Psi) = 0$, implying

$$M\Psi = 0 \quad \text{(and so also } \Psi^*M^* = 0);$$

the middle two are then automatically satisfied, and the first may be written

$$\Omega^*\Omega = I_r, \quad \text{where} \quad \Phi = M\Phi\Delta^{-1}.$$

Thus Ω is a $p \times r$ matrix whose r columns are unitary-orthogonal to one another. Let U be a unitary matrix of order p whose first r columns are just those of Ω: $U = (\Omega, \Gamma)$. Then

$$\begin{aligned} U^*MV &= \begin{pmatrix} \Omega^* \\ \Gamma^* \end{pmatrix} M(\Phi, \Psi) = \begin{pmatrix} \Omega^* \\ \Gamma^* \end{pmatrix} (M\Phi, 0) \\ &= \begin{pmatrix} \Omega^*M\Phi & 0 \\ \Gamma^*M\Phi & 0 \end{pmatrix} = \begin{pmatrix} \Omega^*\Omega\Delta & 0 \\ \Gamma^*\Omega\Delta & 0 \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

since $\Gamma^*\Omega$ vanishes by the unitary property. Lemma 1 is proved.

Appendix 2

For simplicity here and for applications to § 6 we first confine ourselves to the real case. We call an $n \times n$ orthogonal matrix P of determinant $+1$ a *plane rotation* if the corresponding linear transformation affects only two variables.

THEOREM. *Let Z be a real square matrix of order n having N non-zero elements on the right of its principal diagonal. There exist N plane rotations $P_1, \ldots, P_N$ such that the product $ZP_1 \ldots P_N$ has exclusively zero elements on the right of its principal diagonal.*

The proof of this theorem is essentially given by Murnaghan.† The non-zero elements of Z on the right of its principal diagonal are annulled successively by properly choosing the P's.

Proof of lemma 2. The above theorem can evidently be carried over to the complex case, with the result that Z (now complex) can be post-multiplied by a unitary matrix so that the product is left-triangular. If the diagonal elements of this product are written in the usual polar form $\rho_1 e^{i\varphi_1}, \ldots, \rho_n e^{i\varphi_n}$, a further post-multiplication by the diagonal unitary matrix $e^{-i\varphi_1} \dotplus \ldots \dotplus e^{-i\varphi_n}$ makes the diagonal elements equal to $\rho_1, \ldots, \rho_n$ which are real and non-negative. Lemma 2 is proved.

Appendix 3

Proof of lemma 3. Since $\tilde{W}$ is non-singular, $\tilde{W}^{-1}$ exists and is likewise a left-triangular matrix having real and positive diagonal elements. The product $K = \tilde{W}^{-1}W$ of two such matrices is again such a matrix, and such a matrix if unitary must be the identity matrix. Lemma 3 is proved.

Institute of Mathematics
Academia Sinica
Shanghai

† F. D. Murnaghan, *loc. cit.* 319.

AN EXTENSION OF A TAUBERIAN THEOREM OF L. J. MORDELL

By S. MINAKSHISUNDARAM *and* C. T. RAJAGOPAL

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1. This paper seeks to extend a Tauberian theorem of L. J. Mordell involving the first-order Cesàro mean (**9**)* by substituting for the Cesàro mean a Riesz mean of any order $r > 0$. Three points regarding the extension may be mentioned at the outset. (i) The particular case $r = 1$ of the extension seems to be superior to a similar extension by N. Higaki (**5**, theorem IV), in that it serves to amplify the familiar Hardy-Landau convergence theorem for series summable by the first Riesz mean (**6**, 31f.). (ii) The general case $r > 0$ of our extension shows how one of K. Ananda Rau's theorems on series (**1**, theorem 4 and **8**, theorem 1; also **2**, theorem 2) may be restated with a one-sided restriction on the terms of the series instead of his two-sided restriction. (iii) The case $r > 1$ of the extension is proved by a method which H. D. Kloosterman (**7**) suggests for obtaining a generalization of Mordell's theorem with a Cesàro mean of any integral order $r > 1$.†

1·1. *Notation.* $\sum_{n=0}^{\infty} a_n$ is a real series with partial sums

$$s_n = \sum_{\nu=0}^{n} a_\nu \quad (n = 0, 1, 2, \ldots).$$

$\{\lambda_n\}$ is a sequence such that

$$0 < \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty \quad (n \to \infty).$$

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* Arabic numerals in thick type enclosed within brackets indicate references given at the end of the paper. We are indebted to the referee for drawing our attention to the paper of Bosanquet (**2**) and Doetsch (**3**).

† Perhaps the chief interest of the proof lies in the fact that it preserves Mordell's ideas in Kloosterman's form. Every other consideration is in favour of the neat alternative proof, based on some identities of Bosanquet (**2**), which has been kindly suggested to us by the referee. This proof, outlined in a subsequent footnote, dispenses with the need for separate analyses in the cases $0 \leqslant r < 1$ and $r > 1$.

$\sigma^r(\omega)$ is the rth Riesz mean of $\{s_n\}$ with respect to $\{\lambda_n\}$, defined by

$$\sigma^r(\omega) = \frac{A^r(\omega)}{\omega^r} = \sum_{\lambda_n \leqslant \omega} \left(1 - \frac{\lambda_n}{\omega}\right)^r a_n$$

$$= \frac{r}{\omega^r} \int_0^\omega (\omega - t)^{r-1} A(t)\, dt \quad (r > 0),$$

where we have written $A(t) = A^0(t) = s_n$ $(\lambda_n \leqslant t < \lambda_{n+1})$.

For any function $F(x)$ of x and $\varphi > 0$ we set

$$\Delta_\varphi^n F(x) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} F(x + (n-\nu)\varphi),$$

$$\Delta_{-\varphi}^n F(x) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} F(x - \nu\varphi).$$

2. THEOREM 1. *Hypotheses:*

(i) $\phi(x)$, $\psi(x)$ *are positive functions of* $x > 0$ *such that*

$$\theta(x) = \{\phi(x)/\psi(x)\}^{1/(r+1)} \leqslant x \quad (r > 0). \tag{2·1}$$

(ii) $\phi(x)$ *is non-decreasing; there are constants* h, H *such that*

$$\frac{\phi(x')}{\phi(x)} \leqslant H \quad (H \geqslant 1), \tag{2·2}$$

$$0 < h \leqslant \frac{\psi(x')}{\psi(x)} \leqslant H, \tag{2·3}$$

for $|x' - x| \leqslant \eta\theta(x)$, $\eta < 1$ *being a given constant.*

(iii) $$\lambda_n - \lambda_{n-1} \leqslant \eta\theta(\lambda_n). \tag{2·4}$$

(iv) $$a_n/(\lambda_n - \lambda_{n-1}) \leqslant K\psi(\lambda_n), \tag{2·5}$$

(v) $$|A^r(\omega)| \leqslant \delta\phi(\omega), \tag{2·6}$$

where K, δ *are positive constants.*

Conclusion: If $\vartheta(x)$ *is defined by*

$$\vartheta(x) = \{\phi(x)\}^{1/(r+1)} \{\psi(x)\}^{r/(r+1)},$$

then

$$A(\omega) = O[\vartheta(\omega)] \quad (\omega \to \infty). \tag{2·7}$$

2·1. *Some lemmas.* The proof of theorem 1 requires the following lemmas, of which the first is taken from Hardy and Riesz (**4**, lemma 8):

LEMMA 1. *If* $0 \leqslant \xi \leqslant \omega$ *and* p *denotes the greatest integer less than* $r > 0$, *then*

$$\frac{\Gamma(r+1)}{\Gamma(r-p)\,\Gamma(p+1)} \left| \int_0^\xi (\omega - t)^{r-p-1} A^p(t)\, dt \right| \leqslant \max_{0 \leqslant t \leqslant \xi} |A^r(t)|.$$

LEMMA 2. *Under the hypotheses* (2·1), (2·2), (2·3), (2·4) *and* (2·5) *of theorem* 1,

$$A(\omega)-A(t)<KH(\epsilon+\eta)\vartheta(\lambda_m) \tag{2·11}$$

provided that $\quad \lambda_m-\epsilon\theta(\lambda_m)\leqslant t<\lambda_m\leqslant\omega<\lambda_{m+1} \quad (0<\epsilon\leqslant\eta).$

Proof. Let

$$\left.\begin{aligned}&\lambda_q\leqslant\lambda_m-\epsilon\theta(\lambda_m)<\lambda_{q+1} \quad (q<m),\\ &\lambda_\nu\leqslant t<\lambda_{\nu+1} \quad (q+1\leqslant\nu+1\leqslant m).\end{aligned}\right\} \tag{2·12}$$

Then, if $\nu\geqslant q+1$, we have, using in succession (2·5), (2·3), (2·12),

$$\begin{aligned}A(\omega)-A(t)&=s_m-s_\nu=a_{\nu+1}+a_{\nu+2}+\ldots+a_m\\ &\leqslant K\{(\lambda_{\nu+1}-\lambda_\nu)\psi(\lambda_{\nu+1})+\ldots+(\lambda_m-\lambda_{m-1})\psi(\lambda_m)\}\\ &\leqslant KH(\lambda_m-\lambda_\nu)\psi(\lambda_m)\leqslant KH(\lambda_m-\lambda_{q+1})\psi(\lambda_m)\\ &<KH\epsilon\vartheta(\lambda_m).\end{aligned} \tag{2·11a}$$

Next, if $\nu=q$, using successively (2·11a), (2·5), (2·4), (2·2), (2·3), we get

$$\begin{aligned}A(\omega)-A(t)&=s_m-s_{q+1}+a_{q+1}\\ &<KH\epsilon\vartheta(\lambda_m)+K(\lambda_{q+1}-\lambda_q)\psi(\lambda_{q+1})\\ &\leqslant KH\epsilon\vartheta(\lambda_m)+K\eta\vartheta(\lambda_{q+1})\\ &\leqslant KH\epsilon\vartheta(\lambda_m)+KH\eta\vartheta(\lambda_m),\end{aligned} \tag{2·11b}$$

since, on account of (2·2) and (2·3), $\phi(\lambda_{q+1})\leqslant H\phi(\lambda_m)$ and $\psi(\lambda_{q+1})\leqslant H\psi(\lambda_m)$ which taken together give $\vartheta(\lambda_{q+1})\leqslant H\vartheta(\lambda_m)$.

(2·11a) and (2·11b) together establish (2·11).

It may be remarked that (2·11) is *a fortiori* true when the proviso attached to it is replaced by

$$\lambda_m-\epsilon\theta(\lambda_m)\leqslant t<\omega<\lambda_m.$$

LEMMA 3. *Under the hypotheses* (2·1), (2·3) *and* (2·5),

$$A(t)-A(\omega)\leqslant KH\epsilon\vartheta(\lambda_m) \tag{2·13}$$

provided that $\quad \lambda_m\leqslant\omega<t\leqslant\lambda_m+\epsilon\theta(\lambda_m) \quad (0<\epsilon\leqslant\eta).$

Proof. Suppose (as we may without loss of generality) that $\omega<\lambda_{m+1}$. Suppose further that

$$\left.\begin{aligned}&\lambda_q\leqslant\lambda_m+\epsilon\theta(\lambda_m)<\lambda_{q+1} \quad (m\leqslant q),\\ &\lambda_\nu\leqslant t<\lambda_{\nu+1} \quad (m\leqslant\nu\leqslant q).\end{aligned}\right\} \tag{2·14}$$

Then, if $m+1 \leqslant \nu \leqslant q$, using successively (2·5), (2·3), (2·14), we find

$$\begin{aligned} A(t)-A(\omega) = s_\nu - s_m &= a_{m+1}+a_{m+2}+\dots+a_\nu \\ &\leqslant K\{(\lambda_{m+1}-\lambda_m)\,\psi(\lambda_{m+1})+\dots+(\lambda_\nu-\lambda_{\nu-1})\,\psi(\lambda_\nu)\} \\ &\leqslant KH(\lambda_\nu-\lambda_m)\,\psi(\lambda_m) \leqslant KH(\lambda_q-\lambda_m)\,\psi(\lambda_m) \\ &\leqslant KH\epsilon\vartheta(\lambda_m). \end{aligned}$$

Since this relation is trivial for $\nu = m$, the proof of (2·13) is complete.

LEMMA 4.

$$\int_{x-\varphi}^{x} F(x-t)\,\Delta_{-\varphi}^{\nu} G(t)\,dt = \Delta_{-\varphi}^{\nu}\int_{x-\varphi}^{x} F(x-t)\,G(t)\,dt, \tag{2·15}$$

$$\int_{x}^{x+\varphi} F(x+\varphi-t)\,\Delta_{\varphi}^{\nu} G(t)\,dt = \Delta_{\varphi}^{\nu}\int_{x}^{x+\varphi} F(x+\varphi-t)\,G(t)\,dt, \tag{2·16}$$

where $F(t)$, $G(t)$ are integrable functions for the values of t in question.
The lemma is easily proved by induction.

LEMMA 5. *Under the hypotheses* (2·1), (2·2), (2·3), (2·4) *and* (2·5),

$$\Delta_{-\varphi}^{\nu+1} A^\nu(x) < \Gamma(\nu+1)\,KH(\epsilon+\eta)\,\varphi^\nu\vartheta(\lambda_m), \tag{2·17}$$

provided that $\quad \lambda_m - \epsilon\theta(\lambda_m) \leqslant x-(\nu+1)\,\varphi < x \leqslant \lambda_m;$

and further

$$\Delta_{\varphi}^{\nu+1} A^\nu(x) \leqslant \Gamma(\nu+1)\,KH\epsilon\varphi^\nu\vartheta(\lambda_m), \tag{2·18}$$

provided that $\quad \lambda_m \leqslant x < x+(\nu+1)\,\varphi \leqslant \lambda_m+\epsilon\theta(\lambda_m).$

Proof. (2·17) and (2·18) follow by induction from (2·11) and (2·13) respectively, since

$$\Delta_{-\varphi}^{\nu+1} A^\nu(x) = \Delta_{-\varphi}^{\nu}\,\nu\int_{x-\varphi}^{x} A^{\nu-1}(t)\,dt = \frac{\Gamma(\nu+1)}{\Gamma(\nu)}\int_{x-\varphi}^{x}\Delta_{-\varphi}^{\nu} A^{\nu-1}(t)\,dt,$$

$$\Delta_{\varphi}^{\nu+1} A^\nu(x) = \Delta_{\varphi}^{\nu}\,\nu\int_{x}^{x+\varphi} A^{\nu-1}(t)\,dt = \frac{\Gamma(\nu+1)}{\Gamma(\nu)}\int_{x}^{x+\varphi}\Delta_{\phi}^{\nu} A^{\nu-1}(t)\,dt.$$

LEMMA 6. *For any positive integer p,*

$$\begin{aligned} \Delta_{-\varphi}^{p} A^p(x) = \Gamma(p+1)\,\varphi^{p-1}&\int_{x-\varphi}^{x} A(t)\,dt \\ &- \int_{x-\varphi}^{x}(t-x+\varphi)\sum_{\nu=0}^{p-2}\frac{\Gamma(p+1)}{\Gamma(\nu+1)}\,\varphi^{p-\nu-2}\Delta_{-\varphi}^{\nu+1}A^\nu(t)\,dt, \end{aligned}$$

where the second expression on the right side is absent if $p = 1$.

Proof.

$$\begin{aligned}\Delta^{p}_{-\varphi} A^{p}(x) &= \Delta^{p-1}_{-\varphi} p \int_{x-\varphi}^{x} A^{p-1}(t)\,dt \\ &= \Delta^{p-2}_{-\varphi} p \int_{x-\varphi}^{x} \Delta_{-\varphi} A^{p-1}(t)\,d(t-x+\varphi) \quad (p>1) \\ &= \Delta^{p-2}_{-\varphi} p\varphi \Delta_{-\varphi} A^{p-1}(x) - \Delta^{p-2}_{-\varphi} p(p-1) \int_{x-\varphi}^{x} (t-x+\varphi)\,\Delta_{-\varphi} A^{p-2}(t)\,dt \\ &= \frac{\Gamma(p+1)}{\Gamma(p)} \varphi \Delta^{p-1}_{-\varphi} A^{p-1}(x) - \frac{\Gamma(p+1)}{\Gamma(p-1)} \int_{x-\varphi}^{x} (t-x+\varphi)\,\Delta^{p-1}_{-\varphi} A^{p-2}(t)\,dt. \end{aligned} \tag{2·19}$$

The lemma is now established by using (2·19) as a reduction formula.

LEMMA 7. *For any positive integer p,*

$$\Delta^{p}_{\varphi} A^{p}(x) = \Gamma(p+1)\,\varphi^{p-1} \int_{x}^{x+\varphi} A(t)\,dt + \int_{x}^{x+\varphi} (x+\varphi-t) \sum_{\nu=0}^{p-2} \frac{\Gamma(p+1)}{\Gamma(\nu+1)} \varphi^{p-\nu-2} \Delta^{\nu+1}_{\varphi} A^{\nu}(t)\,dt, \tag{2·20}$$

the second member of the right side being absent when $p = 1$.

The proof is similar to that of lemma 6.

2·2. *Proof of theorem* 1 *for* $0 < r \leqslant 1$. First we prove that $A(\omega) = O_R[\vartheta(\omega)]$ as $\omega \to \infty$.

If $\lambda_m \leqslant \omega < \lambda_{m+1}$ and $\varphi = \epsilon\theta(\lambda_m)$, we have

$$\begin{aligned}\frac{\varphi^r}{r} A(\omega) &= \int_{\lambda_m-\varphi}^{\lambda_m} (\lambda_m - t)^{r-1} A(t)\,dt + \int_{\lambda_m-\varphi}^{\lambda_m} (\lambda_m - t)^{r-1} \{A(\omega) - A(t)\}\,dt \\ &= J_1 + J_2 \quad \text{(say).}\end{aligned} \tag{2·21}$$

From lemma 1 with $p = 0$, (2·6), and the fact that $\phi(x)$ is non-decreasing, we get

$$\begin{aligned}|J_1| &\leqslant \frac{1}{r}\left\{ \max_{0\leqslant t\leqslant\lambda_m} |A^r(t)| + \max_{0\leqslant t\leqslant\lambda_m-\varphi} |A^r(t)| \right\} \\ &\leqslant \frac{\delta}{r}\{\phi(\lambda_m) + \phi(\lambda_m - \varphi)\} \\ &\leqslant 2\frac{\delta}{r}\phi(\lambda_m) \leqslant 2H\frac{\delta}{r}\phi(\lambda_m).\end{aligned} \tag{2·22}$$

By lemma 2,
$$J_2 < KH(\epsilon+\eta)\frac{\varphi^r}{r}\vartheta(\lambda_m). \tag{2·23}$$

Employing (2·22), (2·23) in (2·21) we get

$$A(\omega) < 2H\frac{\delta}{\varphi^r}\phi(\lambda_m) + KH(\epsilon+\eta)\vartheta(\lambda_m)$$
$$= H\left\{2\frac{\delta}{\epsilon^r}\vartheta(\lambda_m) + K(\epsilon+\eta)\vartheta(\lambda_m)\right\}. \tag{2·24}$$

Now the restriction on ω in conjunction with (2·4) gives

$$\lambda_{m+1} - \eta\theta(\lambda_{m+1}) \leqslant \lambda_m \leqslant \omega < \lambda_{m+1},$$

and so, on account of (2·3),

$$\psi(\lambda_m) = \frac{\psi(\lambda_m)}{\psi(\lambda_{m+1})}\frac{\psi(\lambda_{m+1})}{\psi(\omega)}\psi(\omega) \leqslant \frac{H}{h}\psi(\omega).$$

This, taken along with the fact that $\phi(\lambda_m) \leqslant \phi(\omega)$, gives

$$\vartheta(\lambda_m) \leqslant (H/h)^{r/(r+1)}\vartheta(\omega),$$

which, employed in (2·24), ensures that

$$A(\omega) < H\left(\frac{H}{h}\right)^{r/(r+1)}\left\{2\frac{\delta}{\epsilon^r} + K(\epsilon+\eta)\right\}\vartheta(\omega). \tag{2·25}$$

After this it is sufficient to prove that $A(\omega) = O_L[\vartheta(\omega)]$. If, as before, $\lambda_m \leqslant \omega < \lambda_{m+1}$ and $\varphi = \epsilon\theta(\lambda_m)$, we have

$$\frac{\varphi^r}{r}A(\omega) = \int_{\lambda_m}^{\lambda_m+\varphi}(\lambda_m+\varphi-t)^{r-1}A(t)\,dt - \int_{\lambda_m}^{\lambda_m+\varphi}(\lambda_m+\varphi-t)^{r-1}\{A(t)-A(\omega)\}\,dt$$
$$= J_3 - J_4 \quad \text{(say)}, \tag{2·26}$$

where, in virtue of lemma 1 with $p = 0$, (2·6) and (2·2),

$$|J_3| \leqslant \frac{\delta}{r}\{\phi(\lambda_m) + \phi(\lambda_m+\varphi)\}$$
$$\leqslant \frac{\delta}{r}\{\phi(\lambda_m) + H\phi(\lambda_m)\} \leqslant 2H\frac{\delta}{r}\phi(\lambda_m). \tag{2·27}$$

Also, on account of lemma 3,

$$J_4 \leqslant KH\epsilon\frac{\varphi^r}{r}\vartheta(\lambda_m). \tag{2·28}$$

From (2·26), (2·27) and (2·28) we obtain, arguing as before,

$$A(\omega) \geqslant -H\left(\frac{H}{h}\right)^{r/(r+1)}\left\{2\frac{\delta}{\epsilon^r} + K\epsilon\right\}\vartheta(\omega). \tag{2·29}$$

2·3. *Proof of theorem* 1 *for* $r > 1$. In this case, taking the greatest integer less than r to be p, we have $p \geqslant 1$ and $r - p > 0$.

First we can show that $A(\omega) = O_R[\vartheta(\omega)]$ starting from the following relation equivalent to lemma 6:

$$\varphi^p A(\omega) = \frac{\Delta^p_{-\varphi} A^p(x)}{\Gamma(p+1)} + \varphi^{p-1} \int_{x-\varphi}^{x} \{A(\omega) - A(t)\}\, dt$$
$$+ \int_{x-\varphi}^{x} (t - x + \varphi) \sum_{\nu=0}^{p-2} \frac{\varphi^{p-\nu-2} \Delta^{\nu+1}_{-\varphi} A^\nu(t)}{\Gamma(\nu+1)}\, dt, \qquad (2{\cdot}31)$$

where we take $\lambda_m \leqslant \omega < \lambda_{m+1}$ and $(p+1)\varphi = \epsilon\theta(\lambda_m)$. If we multiply both sides of (2·31) by $(\lambda_m - x)^{r-p-1}$ and integrate with respect to x from $\lambda_m - \varphi$ to λ_m, we have

$$\frac{\varphi^r A(\omega)}{r-p} = \frac{1}{\Gamma(p+1)} \int_{\lambda_m-\varphi}^{\lambda_m} (\lambda_m - x)^{r-p-1} \Delta^p_{-\varphi} A^p(x)\, dx$$
$$+ \varphi^{p-1} \int_{\lambda_m-\varphi}^{\lambda_m} (\lambda_m - x)^{r-p-1}\, dx \int_{x-\varphi}^{x} \{A(\omega) - A(t)\}\, dt$$
$$+ \int_{\lambda_m-\varphi}^{\lambda_m} (\lambda_m - x)^{r-p-1}\, dx \int_{x-\varphi}^{x} (t - x + \varphi) \sum_{\nu=0}^{p-2} \frac{\varphi^{p-\nu-2} \Delta^{\nu+1}_{-\varphi} A^\nu(t)}{\Gamma(\nu+1)}\, dt$$
$$= J_1 + J_2 + I \quad \text{(say)}. \qquad (2{\cdot}32)$$

In virtue of (2·15),

$$J_1 = \frac{\Delta^p_{-\varphi}}{\Gamma(p+1)} \int_{\lambda_m-\varphi}^{\lambda_m} (\lambda_m - x)^{r-p-1} A^p(x)\, dx$$
$$= \frac{1}{\Gamma(p+1)} \sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} \int_{\lambda_m-(\nu+1)\varphi}^{\lambda_m-\nu\varphi} (\lambda_m - \nu\varphi - x)^{r-p-1} A^p(x)\, dx.$$

Treating each term in this expression as we did J_1 in § 2·2, we find that lemma 1 helps to establish

$$|J_1| \leqslant \frac{\Gamma(r-p)}{\Gamma(r+1)} 2\delta \sum_{\nu=0}^{p} \binom{p}{\nu} \phi(\lambda_m - \nu\varphi)$$
$$\leqslant \frac{\Gamma(r-p)}{\Gamma(r+1)} 2\delta \sum_{\nu=0}^{p} \binom{p}{\nu} \phi(\lambda_m) \leqslant \frac{\Gamma(r-p)}{\Gamma(r+1)} 2^{p+1} H \delta\phi(\lambda_m). \qquad (2{\cdot}33)$$

By (2·11), $$J_2 < KH(\epsilon + \eta) \frac{\varphi^r}{r-p} \vartheta(\lambda_m), \qquad (2{\cdot}34)$$

and by (2·17), $$I \leqslant KH(\epsilon + \eta) \frac{p-1}{2(r-p)} \varphi^r \vartheta(\lambda_m). \qquad (2{\cdot}35)$$

From (2·32), (2·33), (2·34) and (2·35), we get

$$\frac{\varphi^r A(\omega)}{r-p} < 2^{p+1}\frac{\Gamma(r-p)}{\Gamma(r+1)} H\,\delta\phi(\lambda_m) + \frac{p+1}{2(r-p)} KH(\epsilon+\eta)\,\varphi^r\vartheta(\lambda_m),$$

whence, remembering that $\varphi = \epsilon\theta(\lambda_m)/(p+1)$, we are led first to

$$A(\omega) < 2^{p+1}(p+1)^r \frac{\Gamma(r-p+1)}{\Gamma(r+1)} H\frac{\delta}{\epsilon^r}\vartheta(\lambda_m) + \frac{p+1}{2} KH(\epsilon+\eta)\,\vartheta(\lambda_m),$$

and then, as a result of $\vartheta(\lambda_m) \leqslant (H/h)^{r/(r+1)}\vartheta(\omega)$, to

$$A(\omega) < H\left(\frac{H}{h}\right)^{r/(r+1)}\left\{2^{p+1}(p+1)^r\frac{\Gamma(r-p+1)}{\Gamma(r+1)}\frac{\delta}{\epsilon^r} + \frac{p+1}{2}K(\epsilon+\eta)\right\}\vartheta(\omega). \tag{2·36}$$

To prove that $A(\omega) = O_L[\vartheta(\omega)]$ we start from the following immediate consequence of lemma 7:

$$\varphi^p A(\omega) = \frac{\Delta_\varphi^p A^p(x)}{\Gamma(p+1)} - \varphi^{p-1}\int_x^{x+\varphi}\{A(t)-A(\omega)\}\,dt$$
$$-\int_x^{x+\varphi}(x+\varphi-t)\sum_{\nu=0}^{p-2}\frac{\varphi^{p-\nu-2}\Delta_\varphi^{\nu+1}A^\nu(t)}{\Gamma(\nu+1)}\,dt, \tag{2·37}$$

where we choose $\lambda_m \leqslant \omega < \lambda_{m+1}$ and $(p+1)\varphi = \epsilon\theta(\lambda_m)$. Multiplying both sides of (2·37) by $(\lambda_m+\varphi-x)^{r-p-1}$ and integrating with respect to x from λ_m to $\lambda_m+\varphi$ we then obtain

$$\frac{\varphi^r A(\omega)}{r-p} = \frac{1}{\Gamma(p+1)}\int_{\lambda_m}^{\lambda_m+\varphi}(\lambda_m+\varphi-x)^{r-p-1}\Delta_\varphi^p A^p(x)\,dx$$
$$-\varphi^{p-1}\int_{\lambda_m}^{\lambda_m+\varphi}(\lambda_m+\varphi-x)^{r-p-1}\,dx\int_x^{x+\varphi}\{A(t)-A(\omega)\}\,dt$$
$$-\int_{\lambda_m}^{\lambda_m+\varphi}(\lambda_m+\varphi-x)^{r-p-1}\,dx\int_x^{x+\varphi}(x+\varphi-t)\sum_{\nu=0}^{p-2}\frac{\varphi^{p-\nu-2}\Delta_\varphi^{\nu+1}A^\nu(t)}{\Gamma(\nu+1)}\,dt$$
$$= J_3 - J_4 - I' \quad \text{(say)}. \tag{2·38}$$

From this, reasoning as in the first half of our proof, but using (2·16) in place of (2·15), (2·13) in place of (2·11) and finally (2·18) in place of (2·17), it is not difficult to conclude that

$$A(\omega) \geqslant -H\left(\frac{H}{h}\right)^{r/(r+1)}\left\{2^{p+1}(p+1)^r\frac{\Gamma(r-p+1)}{\Gamma(r+1)}\frac{\delta}{\epsilon^r} + \frac{p+1}{2}K\epsilon\right\}\vartheta(\omega). \tag{2·39}$$

It may be observed that, if $p = 1$, I is absent from (2·32) and I' from (2·38); nevertheless, (2·36) and (2·39) continue to hold with $p = 1$.*

2·4. *Modifications of theorem* 1. (i) *Dropping the condition involving h in* (2·3), *we can still prove that $A(\lambda_n) = O[\vartheta(\lambda_n)]$.*

(ii) *In the case in which r is a positive integer, we can conclude that $A(\lambda_n) = O[\vartheta(\lambda_n)]$ dropping the condition involving h as well as the condition that $\phi(x)$ should be non-decreasing.* For, the condition mentioned last is required only when changing from λ_n to ω in $A(\lambda_n) = O[\vartheta(\lambda_n)]$ and when applying lemma 1. The application of this lemma becomes superfluous if r is a positive integer, since we can work then with (2·31) instead of (2·32), and with (2·37) instead of (2·38), replacing p by r and x by λ_m. Mordell's theorem (9) thus appears as the particular case $\lambda_n = n$, $r = 1$ of our modification (ii), apart of course from his multiplying factor C in $|A(n)| < C\vartheta(n)$.

3. *Remarks on theorem* 1. (i) We can replace the hypothesis $\theta(x) \leqslant x$ by $\theta(x) = O(x)$ as $x \to \infty$, introducing certain obvious modifications elsewhere.

(ii) If $\theta(x) \leqslant x$, we can replace the relation between x' and x in (2·2) and (2·3), viz. $|x'-x| \leqslant \eta\theta(x)$, by $|(x'-x)/x| \leqslant \eta$.

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* We sketch below the referee's alternative proof mentioned in an earlier footnote. In the proof of $A(\omega) = O_R[\vartheta(\omega)]$ we could use, instead of lemma 6, the identity

$$\Delta^p_{-\varphi} A^p(x) = \Gamma(p+1)\int_{x-\varphi}^{x} dt_1 \int_{t_1-\varphi}^{t_1} dt_2 \int_{t_2-\varphi}^{t_2} \cdots \int_{t_{p-1}-\varphi}^{t_{p-1}} A(t)\,dt,$$

proved easily by induction, employing for example lemma 4. (2·31) would then become

$$\varphi^p A(\omega) = \frac{\Delta^p_{-\varphi} A^p(x)}{\Gamma(p+1)} + \int_{x-\varphi}^{x} dt_1 \int_{t_1-\varphi}^{t_1} \cdots \int_{t_{p-1}-\varphi}^{t_{p-1}} \{A(\omega)-A(t)\}\,dt,$$

and (2·32) would become

$$\begin{aligned}\frac{\varphi^r A(\omega)}{r-p} &= \frac{1}{\Gamma(p+1)}\int_{\lambda_m-\varphi}^{\lambda_m} (\lambda_m - x)^{r-p-1}\Delta^p_{-\varphi} A^p(x)\,dx \\ &\quad + \int_{\lambda_m-\varphi}^{\lambda_m} (\lambda_m-x)^{r-p-1}\,dx \int_{x-\varphi}^{x} dt_1 \int_{t_1-\varphi}^{t_1} \cdots \int_{t_{p-1}-\varphi}^{t_{p-1}} \{A(\omega)-A(t)\}\,dt \\ &= J_1 + J, \quad \text{where} \quad J = J_2 + I.\end{aligned}$$

Hence, in place of (2·34) and (2·35), there would be the relation

$$J < KH(\epsilon+\eta)\frac{\varphi^r}{r-p}\vartheta(\lambda_m),$$

which enables us to prove (2·36), not only dispensing with lemma 5 but also with a better constant factor multiplying $\vartheta(\omega)$ in (2·36).

Our proof of $A(\omega) = O_L[\vartheta(\omega)]$ can of course be modified along similar lines.

(iii) That some such condition as $\theta(x) = O(x)$ is essential to the truth of theorem 1 is brought out by the following example. If $r = 1$, $\lambda_n = n$, $a_n = 1/n \equiv \psi(n)$, then $A(n) = a_1 + a_2 + \ldots + a_n \sim \log n$, whence

$$A^1(n) = A(1) + A(2) + \ldots + A(n) \sim n \log n \equiv \phi(n)$$

and consequently $A(n)/\vartheta(n) \sim \sqrt{(\log n)} \to \infty$. In this example

$$\theta(x)/x = \sqrt{(\log x)} \to \infty$$

which accounts for the fact that $A(n) = O[\vartheta(n)]$ does not hold.

4. *Deductions from theorem* 1. Suppose, in (2·6), δ can be made as small as we like for all sufficiently large ω. Then, in the case of integral r, arguing as in § 2·4 (ii) and without appealing to lemma 1, we can put $\delta = \epsilon^{r+1}$ in (2·36) and (2·39) whenever $\omega \geqslant \omega_0(\epsilon)$. In the case of non-integral r, we can do the same in (2·25), (2·29), (2·36) and (2·39) appealing to the following lemma of Riesz* where we formerly appealed to lemma 1.

LEMMA 1 A. *If* $0 \leqslant \xi \leqslant \omega$ *and* p *denotes the greatest integer less than non-integral* $r > 0$, *then*
$$A^r(\omega) = o[\phi(\omega)], \quad \omega \to \infty,$$
where $\phi(\omega)$ *is a positive non-decreasing function, implies*

$$\int_0^\xi (\omega - t)^{r-p-1} A^p(t)\, dt = o[\phi(\omega)]$$

uniformly with respect to ξ.

Thus, when δ in (2·6) and η in (2·4) can both be made arbitrarily small for all large ω and n, theorem 1 yields

COROLLARY 1·1. *If, in theorem* 1, (2·4) *and* (2·6) *are replaced by*

$$\lambda_n - \lambda_{n-1} = o[\theta(\lambda_n)] \quad \textit{and} \quad A^r(\omega) = o[\phi(\omega)]$$

respectively, (2·7) *will be replaced by* $A(\omega) = o[\vartheta(\omega)]$.

The conclusion in the above corollary can be substantially ensured even without (2·4) provided that, in the argument leading to (2·11 *b*),

$$a_{q+1} = o_R[\vartheta(\lambda_{q+1})]$$

as $q \to \infty$. This is done in

COROLLARY 1·2. *If, in theorem* 1, (2·6) *is replaced by* $A^r(\omega) = o[\phi(\omega)]$, (2·5) *is supplemented by* $\overline{\lim_{n\to\infty}}\, a_n/\vartheta(\lambda_n) \leqslant 0$, (2·4) *and the condition involving* h *in* (2·3) *are left out*, (2·7) *will become* $A(\lambda_n) = o[\vartheta(\lambda_n)]$.†

* Riesz (**10**, 121) states his lemma in the form: *Let functions* $f(x)$, $\phi(x)$ *be defined for positive* x *so that the first is of bounded variation in any finite interval and the second is positive non-decreasing; let* $0 < \alpha < 1$. *Then* $\int_0^x (x-t)^{\alpha-1} f(t)\, dt = o[\phi(x)]$, $x \to \infty$, *implies* $\int_0^\xi (x-t)^{\alpha-1} f(t)\, dt = o[\phi(x)]$ *uniformly with respect to* ξ *in* $(0, x)$.

† Cf. Szász's convergence condition for series summable-$R(\lambda_n, 1)$ (**6**, 36).

As a particular case of corollary 1·1 we have

COROLLARY 1·3. *Let* $a_n/(\lambda_n-\lambda_{n-1})\leqslant K\lambda_n^{\alpha}$, $A^r(\omega)=o(\omega^{\beta})$, *where* $r>0$, $0\leqslant\beta\leqslant\alpha+r+1$. *Then* $A(\omega)=o(\omega^{(\alpha r+\beta)/(r+1)})$ *provided* $\lambda_n-\lambda_{n-1}=o(\lambda_n^{(\beta-\alpha)/(r+1)})$.

This corollary can of course be suitably modified by substituting "O" conditions for the "o" conditions. It is a reformulation of Ananda Rau's theorem referred to in § 1, with a restriction on $\{\lambda_n\}$ and the replacement of the "O" condition on a_n by the corresponding "O_R" condition.*

Corollary 1·1 with $\phi(x)=\{xg(x)\}^r$, $r>0$ and $\psi(x)=1/xg(x)$ gives

COROLLARY 1·4. *Let* $g(x)$ *be a positive function of* $x>0$ *such that* $xg(x)$ *is non-decreasing and* $O(x)$ *as* $x\to\infty$. *Also, let* $0<h\leqslant g(x')/g(x)\leqslant H$ *for* $|(x'-x)/x|\leqslant\eta<1$. *Then* $a_n/(\lambda_n-\lambda_{n-1})\leqslant K/\lambda_n g(\lambda_n)$ *and* $\sigma^r(\omega)=o[\{g(\omega)\}^r]$, $r>0$, *together imply* $A(\omega)=o(1)$, *provided that* $(\lambda_n-\lambda_{n-1})/\lambda_n=o[g(\lambda_n)]$.

The case $r=1$ of the corollary bears to the Hardy-Landau convergence theorem mentioned in § 1 a relation which may be expressed thus. In the result: $a_n/(\lambda_n-\lambda_{n-1})\leqslant K\psi(\lambda_n)$ and $\sigma^1(\lambda_n)=o(1)$ together imply $s_n=o(1)$, when $(\lambda_n-\lambda_{n-1})/\lambda_n=o(1)$, we know that $\psi(x)=1/x$ and cannot be replaced by $\chi(x)/x$ if $\chi(x)\to\infty$. The corollary however shows that when the rapidity of convergence of $\sigma^1(\lambda_n)$ to 0 is taken into account by postulating

$$\sigma^1(\lambda_n)=o[g(\lambda_n)],$$

we can have $\psi(x)=1/xg(x)$, provided the rapidity of convergence of $(\lambda_n-\lambda_{n-1})/\lambda_n$ to 0 is not less, i.e. $(\lambda_n-\lambda_{n-1})/\lambda_n=o[g(\lambda_n)]$. We may take $g(x)$ to be a suitable L-function, $x^{\alpha}(\log x)^{\alpha_1}(\log_2 x)^{\alpha_2}\dots(\log_p x)^{\alpha_p}$—in particular, any one of the functions $1/\sqrt{x}$, $1/\log x$, $\log x/x$, etc.

5. *Further deductions.* It is easy to prove that theorem 1 can be given a form such as the following.

THEOREM 2. *Hypotheses*:

(i) $\phi(x)$, $\vartheta(x)$ *are positive functions of* $x>0$ *such that*

$$\theta(x)=\{\phi(x)/\vartheta(x)\}^{1/r}\leqslant x\quad(r>0). \tag{5·1}$$

(ii) $\phi(x)$ *is non-decreasing*; *for* $0<x'-x\leqslant\eta\theta(x)$, $\eta<1$, *there is an* $H\geqslant 1$ *such that*

$$\phi(x')/\phi(x)\leqslant H. \tag{5·2}$$

* If we omit the trivial part of Ananda Rau's theorem corresponding to $\beta>\alpha+r+1$, as well as Bosanquet's relaxation of the condition $\beta>0$ to $\beta>-1$, we can state the theorem thus. *Let* $a_n/(\lambda_n-\lambda_{n-1})=O(\lambda_n^{\alpha})$ *and* $A^r(\omega)=o(\omega^{\beta})$, *where* $r>0$, $0<\beta\leqslant\alpha+r+1$. *Then* $A(\omega)=o(\omega^{(\alpha r+\beta)/(r+1)})$.

Clearly Bosanquet's arguments for the replacement of $\beta>0$ by $\beta>-1$ in this theorem (**2, 243** f.) justify the replacement in corollary 1·3 as well.

(iii) *For* $\omega-\eta\theta(\omega)\leqslant u<v\leqslant\omega+\eta\theta(\omega)$,

$$A(v)-A(u)<W(\eta)\vartheta(\omega). \tag{5·3}$$

(iv)
$$|A^r(\omega)|\leqslant\delta\phi(\omega). \tag{5·4}$$

Conclusion:
$$A(\omega)=O[\vartheta(\omega)]. \tag{5·5}$$

In particular, if $W(\eta)\to 0$ *as* $\eta\to 0$ *and* (5·4) *is altered to* $A^r(\omega)=o[\phi(\omega)]$, (5·5) *will become* $A(\omega)=o[\vartheta(\omega)]$.

Proof. We have only to employ (5·3) in place of lemmas 2, 3, with the result that λ_m in lemma 5 and §§ 2·2, 2·3 is changed to ω, φ being defined by $(p+1)\varphi=\epsilon\theta(\omega)$.

5·1. From theorem 2 we can deduce

COROLLARY 2·1. *If* $\phi(x)$, $\psi(x)$ *are both positive*, $\phi(x)$ *is non-decreasing*, $\phi(x)$, $\psi(x)$ *are both subject to a condition like* (5·2) *for* $|x'-x|\leqslant\eta\theta(x)$, *and related so that* $\theta(x)=\{\phi(x)/\psi(x)\}^{1/(r+1)}\leqslant x$, $r>0$, *and if* $A^\kappa(\omega)\leqslant K\psi(\omega)$, $K\geqslant 0$ *and* $A^{\kappa+r+1}(\omega)=O[\phi(\omega)]$, *then* $A^{\kappa+1}(\omega)=O[\vartheta(\omega)]$, *where*

$$\vartheta(\omega)=\{\phi(\omega)\}^{1/(r+1)}\{\psi(\omega)\}^{r/(r+1)}.$$

Proof. For $\omega-\eta\theta(\omega)\leqslant u<v\leqslant\omega+\eta\theta(\omega)$,

$$\begin{aligned}A^{\kappa+1}(v)-A^{\kappa+1}(u)&=(\kappa+1)\int_u^v A^\kappa(t)\,dt\\&\leqslant 2(\kappa+1)KH\eta\theta(\omega)\psi(\omega)=2(\kappa+1)KH\eta\vartheta(\omega).\end{aligned} \tag{5·11}$$

Now setting $B(\omega)=A^{\kappa+1}(\omega)$, we find in consequence of a result of Hardy and Riesz (**4**, lemma 6) that

$$A^{\kappa+r+1}(\omega)=\frac{\Gamma(\kappa+r+2)}{\Gamma(r+1)\Gamma(\kappa+2)}B^r(\omega),$$

whence
$$B^r(\omega)=O[\phi(\omega)]. \tag{5·12}$$

Comparing (5·11) with (5·3) and (5·12) with (5·4) we see that the proof can be completed by an appeal to theorem 2 after changing $A(\omega)$ there to $B(\omega)$.

If in corollary 2·1 we alter the hypothesis in respect of $A^{\kappa+r+1}(\omega)$ to $A^{\kappa+r+1}(\omega)=o[\phi(\omega)]$, the conclusion will become $A^{\kappa+1}(\omega)=o[\vartheta(\omega)]$. If further we take $\psi(\omega)=\omega^\kappa$, $\phi(\omega)=\omega^{\kappa+r+1}$, the result is

COROLLARY 2·2. *If, for* $\kappa\geqslant 0$, $r>0$, *a series* Σa_n *has its Riesz mean* $\sigma^\kappa(\omega)$ *bounded on one side as* $\omega\to\infty$ *and if the series is summable*-$R(\lambda_n,\kappa+r+1)$ *to zero, then the series is summable*-$R(\lambda_n,\kappa+1)$ *to zero.*

This corollary is substantially equivalent to a theorem of Doetsch (**3**, 178, Satz 2).

6. This paper would be incomplete if we did not mention the effect, on the conclusion of theorem 1, of varying the Tauberian condition (2·5) in the manner of Karamata (6 *a*). Three such variations are included in the following theorem which supplements corollary 1·2 and its O-version.

THEOREM 1A. *If we postulate hypotheses* (i), (ii), (v) *as in theorem* 1, *without the "h" condition in* (ii), *then* EITHER

$$\text{(iv) } a_n/(\lambda_n-\lambda_{n-1}) \leqslant K\psi(\lambda_n) \quad \text{OR} \quad \text{(iv}'\text{) } a_{n-1}/(\lambda_n-\lambda_{n-1}) \leqslant K\psi(\lambda_{n-1})$$

ensures the conclusion EITHER

$$A(\lambda_n) = O_L[\vartheta(\lambda_n)] \quad \text{OR} \quad A(\lambda_{n-1}) = O_R[\vartheta(\lambda_n)]$$

as $n \to \infty$.

Further, the alteration of (v) to $A^r(\omega) = o[\phi(\omega)]$ changes the "O" in the conclusion to "o".

The first alternative of the theorem follows from the second half of the proof in § 2·3. A proof, like the first half of the one in § 2·3, can be given for the second alternative, using the lemma which follows in place of lemma 2.

LEMMA 2A. *Under the hypotheses* (2·1), (2·3) *without the "h" condition, and* (iv′),

$$A(\lambda_{m-1}) - A(t) < KH\epsilon\vartheta(\lambda_m) \quad (o < \epsilon \leqslant \eta),$$

provided that

$$\lambda_m - \epsilon\theta(\lambda_m) \leqslant t < \lambda_m.$$

Taking $\phi(x) = \{xg(x)\}^r$ and $\psi(x) = 1/xg(x)$ in corollary 1·2 and theorem 1 A we get the following supplement to corollary 1·4.

COROLLARY 1A. *Corollary* 1·4 *remains true when its two assumptions*:

$$a_n\lambda_n g(\lambda_n) \leqslant K(\lambda_n-\lambda_{n-1}), \quad \lambda_n-\lambda_{n-1} = o[\lambda_n g(\lambda_n)],$$

are replaced by one of the following (*without any other change*):

$$a_n\lambda_n g(\lambda_n) \leqslant K(\lambda_n-\lambda_{n-1}), \quad \overline{\lim_{n\to\infty}}\, a_n \leqslant 0;$$

$$|\,a_n\,|\,\lambda_n g(\lambda_n) \leqslant \text{ EITHER } K(\lambda_n-\lambda_{n-1}) \text{ OR } K(\lambda_{n+1}-\lambda_n);$$

$$a_n\lambda_n g(\lambda_n) \leqslant \text{ BOTH } K(\lambda_n-\lambda_{n-1}) \text{ AND } K(\lambda_{n+1}-\lambda_n).$$

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Andhra University
Guntur (S. India)

Madras Christian College
Tambaram (S. India)

UNE INTERPRÉTATION ALGÉBRIQUE DE LA SUITE DES ORDRES DE MULTIPLICITÉ D'UNE BRANCHE ALGÉBRIQUE

By CAHIT ARF

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La suite des multiplicités des points successifs d'une branche algébrique peut se définir au moyen de notions purement algébriques. Dans ce qui suit nous exposons une telle définition qui ne diffère, du reste, de la définition géométrique que par la forme. Nous espérons que cette définition constitue une réponse à la question posée par P. Du Val* sur la relation qui doit exister entre ses résultats et les développements en series entières de la branche considérée.

1. k étant un corps quelconque, nous considérons un anneau H constitué par des series entières d'une variable t avec coefficients dans k. Soient

$$W(H) = \{i_0 = 0, i_1, i_2, \ldots, i_r, i_{r+1}, \ldots\}$$

les ordres (c.à.d. les ordres des premiers termes avec coefficients non nuls) des éléments de H. Les entiers $i_0, i_1, \ldots, i_r, \ldots$ constituent un semi-groupe d'entiers positifs. $S_0, S_{i_1}, S_{i_2}, \ldots, S_{i_r}, \ldots$ étant des éléments d'ordres respectifs $i_0, i_1, \ldots, i_r, \ldots$ dans H, tout élément de cet anneau est de la forme

$$\sum_{l=0}^{\infty} \alpha_l S_{i_l} \quad (\alpha_l \in k).$$

Nous admetterons que H contient toutes les séries de cette forme. Nous désignerons par I_h l'ensemble des éléments de H d'ordres supérieurs ou égaux à h. I_h est visiblement un idéal de H et ses éléments sont de la forme

$$\sum_{i_l \geq h} \alpha_l S_{i_l} \quad (\alpha_l \in k).$$

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* P. Du Val, "The Jacobian algorithm and the multiplicity sequence of an algebraic branch", *Rev. Faculté Sci. Univ. Istanbul* (Série A), 7 (1942), 107–112.

THÉORÈME AUXILIAIRE 1. *ν étant le p.g.c.d. des éléments de $W(H)$, pour r suffisamment grand on a*

$$i_{r+1} = i_r + \nu,\ i_{r+2} = i_r + 2\nu,\ \ldots,\ i_{r+l} = i_r + l\nu,\ \ldots$$

et il existe une série entière d'ordre 1,

$$\tau = t\left(1 + \sum_{l=1}^{\infty} \delta_l t^l\right) \quad (\delta_l \in k)$$

telle que tout élément de H soit de la forme $\sum_{j=0}^{\infty} \alpha_j \tau^{j\nu}$.

Démonstration. Désignons par ν_l le p.g.c.d. des entiers $i_1, i_2, \ldots, i_l$. Chacun des nombres $\nu_1, \nu_2, \ldots, \nu_h, \ldots$ divise alors tous ceux qui le précèdent. Il en résulte que pour ρ suffisamment grand on a $\nu_\rho = \nu_{\rho+1} = \nu_{\rho+2} = \ldots = \nu$. Soit alors

$$\nu = m_1 i_1 + m_2 i_2 + \ldots + m_\rho i_\rho,$$

$m_1, m_2, \ldots, m_\rho$ étant des entiers positifs, nuls ou négatifs. m étant le plus grand des entiers $|m_h(i_1/\nu - 1)|$, les multiples de ν qui sont supérieurs à

$$i = mi_1 + mi_2 + \ldots + mi_\rho$$

sont contenus dans $W(H)$. On a en effet, pour $l = 0, 1, 2, \ldots, i_1/\nu - 1$,

$$\begin{aligned} i + l\nu &= (m + lm_1)\, i_1 + (m + lm_2)\, i_2 + \ldots + (m + lm_\rho)\, i_\rho \\ &= n_1 i_1 + n_2 i_2 + \ldots + n_\rho i_\rho, \end{aligned}$$

avec $n_h \geqslant 0$; puisque $m \geqslant |m_h l|$. Pour $l = i_1/\nu$, on aura $i + l\nu = i + i_1 \in W(H)$. D'une manière générale, les multiples de ν qui sont supérieurs à i peuvent s'écrire sous la forme $i + ji_1 + l\nu$ ($l = 0, 1, 2, \ldots, i_1/\nu - 1$, $j \geqslant 0$) et il est évident que tous ces entiers sont de la forme $\sum_{h=1}^{\rho} n_h i_h$ avec $n_h \geqslant 0$; c.à.d. contenus dans $W(H)$. $S_i = \sum_{l=i_1}^{\infty} \sigma_l t^l$ ($\sigma_l \in k$, $\sigma_{i_1} \neq 0$) étant un élément d'ordre i_1 de H, on peut choisir une série entière de la forme $\tau = t\left(1 + \sum_{l=1}^{\infty} \delta_l t^l\right)$, ($\delta_l \in k$) de manière que l'on ait $S_{i_1} = \sigma_{i_1} \tau^{i_1}$. Dans ces conditions les séries entières en t avec coefficients dans k peuvent s'écrire sous formes de séries entières en τ avec coefficients dans k. *En particulier les éléments de H peuvent s'écrire sous la forme* $\sum_{j=0}^{\infty} \alpha_{j\nu} \tau^{j\nu}$. Nous pouvons nous contenter de démontrer ceci pour les éléments de H qui sont d'ordres supérieurs à i; puisque tout élément de H peut être considéré comme quotient d'un élément d'ordre supérieur à i de H par une puissance convenablement choisie de $S_{i_1} = \sigma_{i_1} \tau^{\nu(i_1/\nu)}$. Les ordres des éléments de H étant des multiples de ν, un élément quelconque de H est de la forme $\sum_{j=N\nu}^{\infty} \alpha_j \tau^j$ ($\alpha_j \in k$, $\alpha_{N\nu} \neq 0$). Pour $N\nu \geqslant i$, l'anneau H contient des

éléments, $S_{N\nu+\nu}$, $S_{N\nu+2\nu}$, $\dots \left(S_{N\nu+l\nu} = \sum_{j=N\nu+l\nu}^{\infty} \alpha_{l,j}\tau^j,\ \alpha_{lj} \in k,\ \alpha_{l,N\nu+l\nu} \neq 0 \right)$ d'ordres respectifs $N\nu+\nu$, $N\nu+2\nu$, ... On peut donc choisir la série $\sum_{l=1}^{\infty} \beta_l S_{N\nu+l\nu}$ de manière que la différence

$$S_{N\nu} = \sum_{j=N\nu}^{\infty} \alpha_j \tau^j - \sum_{l=1}^{\infty} \beta_l S_{N\nu+l\nu} = \alpha_{N\nu}\tau^{N\nu} + \tilde{\alpha}_\mu \tau^\mu + \dots$$

ne contienne aucun terme d'ordre divisible par ν, autre que le premier. Supposons en effect qu'on ait pu choisir β_1, β_2, ..., β_h de manière que les terms d'ordres $N\nu+\nu$, $N\nu+2\nu$, ..., $N\nu+h\nu$ de la différence

$$\sum_{j=N\nu}^{\infty} \alpha_j \tau^j - \sum_{l=1}^{h} \beta_l S_{N\nu+l\nu} = \alpha_{N\nu}\tau^{N\nu} + \alpha_{\mu_h}^{(h)} \tau_{\mu_h} + \dots$$

disparaissent; il suffit alors de poser

$$\beta_{h+1} = \frac{\alpha^{(h)}_{N\nu+h\nu+\nu}}{\alpha_{h+1,\,N\nu+h\nu+\nu}},$$

pour que les termes d'ordres $N\nu+\nu$, $N\nu+2\nu$, ..., $N\nu+h\nu$, $N\nu+h\nu+\nu$ de la différence

$$\sum_{j=N\nu}^{\infty} \alpha_j \tau^j - \sum_{l=1}^{h+1} \beta_l S_{N\nu+l\nu} = \alpha_{N\nu}\tau^{N\nu} + \alpha_{\mu_{h+1}}^{(h+1)} \tau^{\mu_{h+1}} + \dots$$

disparaissent. Dans ces conditions la série $S_{N\nu}$ se réduit à $\alpha_{N\nu}\tau^{N\nu}$. Car sinon, la différence

$$S_{N\nu}^{i_1/\nu} - \alpha_{N\nu}^{i_1/\nu}\left(\frac{S_{i_1}}{\sigma_{i_1}}\right)^N = \frac{i_1}{\nu}\alpha_{N\nu}^{i_1/\nu-1}\tilde{\alpha}_\mu \tau^{N\nu(i_1/\nu-1)+\mu} + \dots$$

dont l'ordre n'est pas divisible par ν serait contenue dans H. Donc tout élément d'ordre supérieur à i de H est une combinaison linéaire à coefficients dans k des éléments de la forme $\alpha_{N\nu}\tau^{N\nu} = S_{N\nu}$.

Remarque. D'après le théorème qui précède, l'anneau H peut être considéré comme un sous anneau de l'anneau des séries entières de la variable $T = \tau^\nu$ avec coefficients dans k. Posons ${}^*i_h = i_h/\nu$. Les ordres des éléments de H par rapport à cette nouvelle variable seront ${}^*i_0 = 0$, ${}^*i_1$, ${}^*i_2$, ..., ${}^*i_r$, ..., et pour r suffisamment grand, on aura

$${}^*i_{r+1} = {}^*i_r + 1,\ {}^*i_{r+2} = {}^*i_r + 2,\ \dots.$$

THÉORÈME AUXILIAIRE 2. *L'inverse de tout élément d'ordre zéro de H est aussi un élément de H.*

Démonstration. Si l'ordre de $a = \sum_{h=0}^{\infty} \alpha_h S_{i_h}$ est zéro, α_0 est différent de zéro. Or les coefficients β_h du produit

$$\alpha_0^{-1} \prod_{h=1}^{\infty} (1 + \beta_h S_{i_h})$$

peuvent être choisis de manière que l'on ait

$$a\alpha_0^{-1}\prod_{h=1}^{n}(1+\beta_h S_{i_h})\equiv 1, \quad \operatorname{mod} t^{i_n+1}.$$

Supposons en effet que ce choix ait pu être fait pour $\beta_1, \beta_2, \ldots, \beta_{n-1}$. On aura

$$a\alpha_0^{-1}\prod_{h=1}^{n-1}(1+\beta_h S_{i_h}) = 1+\gamma_n S_{i_n}+\gamma_{n+1}S_{i_{n+1}}+\cdots$$

et il suffira de poser $\beta_n = -\gamma_n$ pour avoir

$$a\alpha_0^{-1}\prod_{h=1}^{n}(1+\beta_h S_{i_h})\equiv 1, \quad \operatorname{mod} t^{i_n+1}.$$

Pour les coefficients β_h ainsi choisis on aura visiblement

$$a\alpha_0^{-1}\prod_{h=1}^{\infty}(1+\beta_h S_{i_h}) = 1.$$

Remarque. $\sum_{h=0}^{\infty}\alpha_h S_{i_h}$ étant un élément d'ordre zéro de H, à chaque racine $n$ième de α_0 contenue dans k correspond une racine $n$ième de $\sum_{h=0}^{\infty}\alpha_h S_{i_h}$ contenue dans H. La démonstration de ce fait est analogue à celle du théorème auxiliaire 2.

2. THÉORÈME AUXILIAIRE 3. *Si l'on désigne par I_h/S_h l'ensemble des quotients des éléments de I_h par S_h, et par $[I_h/S_h]$ l'anneau engendré par I_h/S_h, l'anneau $[I_h/S_h]$ ne dépend pas du choix de S_h parmis les éléments d'ordre h de H.*

Démonstration. Remarquons d'abord que l'ensemble I_h/S_h contient l'anneau H et par conséquent $[I_h/S_h]\supseteq H$.

Soit $S_h' = \epsilon S_h$ un autre élément d'ordre h de H. ϵ est alors un élément de $[I_h/S_h]$. Il en résulte d'après le théorème auxiliaire 2, que ϵ^{-1} est aussi un élément de $[I_h/S_h]$. On a donc

$$I_h/S_h' = I_h/\epsilon S_h = \epsilon^{-1}(I_h/S_h)\subseteq [I_h/S_h]$$

et par conséquent $$[I_h/S_h']\subseteq [I_h/S_h].$$

On peut évidemment montrer exactement de la même manière que l'on a aussi $$[I_h/S_h]\subseteq [I_h/S_h'].$$

On a donc $[I_h/S_h'] = [I_h/S_h]$.

L'anneau $[I_h/S_h]$ ne dépendant pas du choix S_h, nous pouvons le désigner par $[I_h]$.

Remarque. Le semi-groupe $W([I_{i_h}])$ contient visiblement le semi-groupe engendré par les entiers

$$i_h-i_h = 0,\ i_{h+1}-i_h,\ i_{h+2}-i_h,\ \ldots$$

qui sont les ordres des éléments de I_{i_h}/S_{i_h}. Mais $W([I_{i_h}])$ n'est pas nécéssairement identique à ce semi-groupe comme le montre l'exemple suivant: Considérons l'anneau H formé de toutes les séries de la forme

$$\sum_{i,j\geqslant 0} \alpha_{ij} X^i Y^j \quad (\alpha_{ij} \in k),$$

avec $X = t^4$, $Y = t^{10}+t^{15}$. On montre facilement que $W(H)$ est formé des entiers

$$0, 4, 8, 10, 12, 14, 16, 18, 20, 22, 24, 25, 26,$$
$$28, 29, 30, 32, 33, 34, 35, 36, 37, 38, \ldots.$$

Les ordres des éléments de I_4/X sont alors les entiers

$$0, 4, 6, 8, 10, 12, 14, 16, 18, 20, 21, 22,$$
$$24, 25, 26, 28, 29, 30, 31, 32, 33, 34, \ldots$$

qui engendrent le semi-groupe

$$0, 4, 6, 8, 10, 12, 14, 16, 18, 20, 21, 22, 24,$$
$$25, 26, 27, 28, 29, 30, 31, 32, 33, 34, \ldots$$

tandis que $[I_4]$ contient l'élément $(Y/X)^2 - X^3 = 2t^{17}+t^{22}$ dont l'ordre est 17.

Remarque. Si pour un choix particulier de S_h, l'anneau $[I_h]$ est identique à I_h/S_h, il en est de même pour tous les choix de S_h. En effet, $S'_h = \epsilon S_h$ étant un autre élément d'ordre h de H, on aura

$$I_h/S'_h = \epsilon^{-1} I_h/S_h = \epsilon^{-1}[I_h] = [I_h];$$

puisque tout élément S de $[I_h]$ est égal à l'élément ϵS de $[I_h]$ multiplié par ϵ^{-1}.

Définition. Nous disons qu'un anneau H est canonique si l'on a $[I_h] = I_h/S_h$ quelque soit $h \in W(H)$.

Remarque. Si H est un anneau canonique, les entiers

$$i_h - i_h = 0,\ i_{h+1} - i_h,\ i_{h+2} - i_h,\ \ldots$$

forment un semi-groupe quelque soit h. *Un semi-groupe d'entiers non négatifs*

$$i_0 = 0,\ i_1,\ i_2,\ \ldots,\ i_h,\ \ldots$$

est appelé canonique si la suite

$$i_h - i_h = 0,\ i_{h+1} - i_h,\ i_{h+2} - i_h,\ \ldots$$

est un semi-groupe pour chaque h. Si la suite des entiers croissants

$$i_0 = 0,\ i_1,\ i_2,\ \ldots,\ i_h,\ \ldots$$

est un semi-groupe canonique, les séries entières

$$\sum_{h=0}^{\infty} \alpha_h t^{i_h} \quad (\alpha_h \in k),$$

forment visiblement un anneau canonique. *$W(H)$ peut être canonique sans qu'il en soit de même pour H*: L'anneau H formé des séries de la forme $\sum_{i,j,l\geqq 0} \alpha_{ijl} X^i Y^j Z^l (\alpha_{ijl} \in k)$, avec $X = t^4$, $Y = t^{10}+t^{15}$, $Z = t^{27}$, est tel que les ordres

$$0,\ 4,\ 8,\ 10,\ 12,\ 14,\ 16,\ 18,\ 20,\ 22,\ 24,\ 25,\ 26,\ 27,\ 28,\ 29,\ 30,\ \ldots$$

de ses éléments forment, comme on le vérifie facilement, un semi-groupe canonique, alors que H n'est pas un anneau canonique, puisque $[I_4]$ contient l'élément d'ordre 17, $(Y/X)^2 - X^3 = 2t^{17}+t^{22}$, qui n'est pas contenu dans I_4/X.

THÉORÈME AUXILIAIRE 4. *La partie commune de plusieurs anneaux canoniques est un anneau canonique.*

Démonstration. Il suffit évidemment de démontrer le théorème pour la partie commune de deux anneaux canoniques seulement. H et H' étant deux anneaux canonique, soit S un élément commun de ces deux anneaux. Soit h l'ordre de S et soient I_h et I'_h les ensembles d'éléments de H et de H' dont les ordres ne sont pas inférieurs à h. Il suffit de montrer que

$$(I_h \cap I'_h)/S = I_h/S \cap I'_h/S$$

est un anneau. Or I_h/S et I'_h/S étant des anneaux, il en est de même de leur partie commune.

Remarque. Si H est un anneau canonique, il en est de même de $[I_{i_h}]$. Considérons en effet l'ensemble des éléments de I_{i_h}. Ces éléments sont de la forme

$$\sum_{\nu=h}^{\infty} \alpha_\nu S_{i_\nu} \quad (\alpha_\nu \in k).$$

H étant un anneau canonique l'anneau $[I_h]$ est constitué par l'ensemble des séries de la forme $\sum_{\nu=h}^{\infty} \alpha_\nu \dfrac{S_{i_\nu}}{S_{i_h}}$ dont les ordres sont les nombres

$$0,\ j_1 = i_{h+1} - i_h,\ j_2 = i_{h+2} - i_h,\ \ldots.$$

L'ensemble des éléments de $[I_{i_h}]$ d'ordres supérieurs ou égaux à j_l est alors l'ensemble des séries de la forme

$$\sum_{\nu=h+l}^{\infty} \alpha_\nu \frac{S_{i_\nu}}{S_{i_h}} \quad (\alpha_\nu \in k).$$

$S_{i_{h+l}}/S_{i_h}$ étant un élément d'ordre $j_l = i_{h+l} - i_h$ de cet ensemble, l'ensemble des éléments

$$\left(\sum_{\nu=h+l}^{\infty} \alpha_\nu \frac{S_{i_\nu}}{S_{i_h}}\right) \Big/ \frac{S_{i_{h+l}}}{S_{i_h}} = \sum_{\nu=h+l}^{\infty} \alpha_\nu \frac{S_{i_\nu}}{S_{i_{h+l}}}$$

constituent l'anneau $[I_{i_{h+l}}]$.

$\mathfrak{G}$ étant l'ensemble de tous les entiers non négatifs,* on montre d'une manière analogue que si
$$\{0, i_1, i_2, \ldots, i_r + \mathfrak{G}\nu\}$$
est une semi-groupe canonique, il en est de même de
$$\{0, i_{h+1} - i_h, \ldots, i_r - i_h + \mathfrak{G}\nu\}.$$

Remarque. Si les entiers
$$i_0 = 0, i_1, i_2, \ldots, i_h, \ldots$$
forment une semi-groupe canonique on a $i_{h+1} - i_h \leqslant i_h - i_{h-1}$. En effet, les entiers $i_{h-1} - i_{h-1} = 0, i_h - i_{h-1}, i_{h+1} - i_{h-1}, \ldots, i_r - i_{h-1}, \ldots$ devant former un semi-groupe, on doit avoir $i_{h+1} - i_{h-1} \leqslant 2(i_h - i_{h-1})$; d'où résulte immédiatement l'inégalité $i_{h+1} - i_h \leqslant i_h - i_{h-1}$.

3. D'après la remarque qui suit immédiatement le théorème auxiliaire 1, $I_{(N-1)\nu}$ contient toutes les séries entières dont les ordres en $T = \tau^\nu$ sont supérieurs ou égaux à $N-1$ pourvu que N soit suffisamment grand. $[I_{(N-1)\nu}]$ est alors l'anneau $k[T]$ de toutes les séries entières en T avec coefficients dans k. Cette remarque nous conduit à la construction suivante qui nous permet d'obtenir tous les anneaux canoniques ainsi que tous les semi-groupes canoniques:

On commence par considérer l'anneau $[I_{(N-1)\nu}] = k[T]$ de toutes les séries entières en T de même que le semi-groupe $\mathfrak{G}\nu$ des multiples entiers non négatifs de ν. On choisit un élément d'ordre non nul, T_{r-1} de $[I_{(N-1)\nu}]$ de même qu'un élément non nul ν_{r-1} $(= w(T_{r-1})$†) de $\mathfrak{G}\nu$ et l'on pose
$$[I_{i_{r-1}}] = k + T_{r-1}[I_{(N-1)\nu}] \quad (i_r = (N-1)\nu).$$
L'anneau $[I_{i_{r-1}}]$ et le semi-groupe $\{0, \nu_{r-1} + \mathfrak{G}\nu\}$ $(= W([I_{i_{r-1}}]))$ sont canoniques. On choisit de même un élément T_{r-2} d'ordre non nul dans $[I_{i_{r-1}}]$ ainsi qu'un entier positif ν_{r-2} $(= w(T_{r-2}))$ dans $\{0, \nu_{r-1} + \mathfrak{G}\nu\}$, et l'on pose
$$\begin{aligned}[I_{i_{r-2}}] &= k + T_{r-2}[I_{i_{r-1}}] \\ &= k + kT_{r-2} + T_{r-2}T_{r-1}k[T], \\ W([I_{i_{r-2}}]) &= \{0, \nu_{r-2}, \nu_{r-2} + \nu_{r-1} + \mathfrak{G}\nu\}.\end{aligned}$$
On obtient ainsi un nouvel anneau canonique ainsi qu'un semi-groupe canonique. En continuant de cette manière on obtient finalement l'anneau canonique
$$k + kT_1 + kT_1T_2 + \ldots + kT_1T_2 \ldots T_{r-2} + k[T]\, T_1T_2 \ldots T_{r-2}T_{r-1}$$

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* Dans ce qui suit $\mathfrak{G}$ désignera toujours l'ensemble de tous les entiers non négatifs.

† Dans ce qui suit $w\left(\sum_{i=\mu}^{\infty} \alpha_i t^i\right)$ désigne l'ordre en t de la série entière $\sum_{i=\mu}^{\infty} \alpha_i t^i$.

et le semi-groupe canonique

$$\{0, \nu_1, \nu_1+\nu_2, \ldots, \nu_1+\nu_2+\ldots+\nu_{r-2}, \nu_1+\nu_2+\ldots+\nu_{r-1}+\mathfrak{G}\nu\}$$

avec $$T_h \in kT_{h+1}+kT_{h+1}T_{h+2}+\ldots+k[T]\,T_{h+1}T_{h+2}\ldots T_{r-1},$$

$$(w(T_h)=)\,\nu_h \in \{\nu_{h+1}, \nu_{h+1}+\nu_{h+2}, \ldots, \nu_{h+1}+\nu_{h+2}+\ldots+\nu_{r-1}+\mathfrak{G}\nu\}.$$

4. Etant donné un anneau H la partie commune de tous les anneaux canoniques contenant H est un anneau canonique $^*H$ que nous appelerons la *fermeture canonique* de H. De même $G=\{0, i_1, i_2, \ldots, i_{r-1}+\mathfrak{G}\nu\}$ étant un semi-groupe d'entiers non négatifs ($\nu=(i_1, i_2, \ldots, i_{r-1}, i_{r-1}+\nu)$), la partie commune de tous les semi-groupes canoniques contenant G est un semi-groupe canonique $^*G$; nous l'appelerons la *fermeture canonique* de G.

D'après cette définition il est clair que $W(^*H)$ contient le semi-groupe canonique $^*W(H)$; mais ces deux groupes ne sont pas nécéssairement égaux, puisque $W(H)$ peut être canonique sans que H le soit.

5. Etant donné le semi-groupe

$$G=\{0, i_1, i_2, \ldots, i_{r-1}+\mathfrak{G}\nu\} \quad (\nu=(i_1, i_2, \ldots, i_{r-1}, i_{r-1}+\nu)),$$

la fermeture canonique $^*G$ de G peut s'obtenir de la manière suivante: On considère le semi-groupe $\{0, i_1+G_1\}$ ou G_1 est le semi-groupe des entiers de la forme

$$\alpha_2(i_2-i_1)+\alpha_3(i_3-i_1)+\ldots+\alpha_n(i_n-i_1),$$

les coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ étant des entiers non négatifs. Le semi-groupe $\{0, i_1+G_1\}$ qui contient alors G est visiblement contenu dans $^*G$. Remarquons que les éléments de G_1 qui sont inférieur à $i_{h+1}-i_1$ sont de la forme

$$\alpha_2(i_2-i_1)+\alpha_3(i_3-i_1)+\ldots+\alpha_h(i_h-i_1);$$

les nombres $$\alpha_2(i_2-i_1)+\alpha_3(i_3-i_1)+\ldots+\alpha_n(i_n-i_1)$$

avec $n \geqslant h+1$, $\alpha_n \neq 0$ sont en effet supérieur ou égaux à $i_{h+1}-i_1$. En particulier l'élément le plus petit de G_1 est i_2-i_1. Il en résulte en outre que les éléments de $\{0, i_1+G_1\}$ qui sont inférieurs à i_{h+1} ne dépendent que des nombres $i_1, i_2, \ldots, i_h$, dont ils sont des combinaisons linéaires à coefficients entiers. Le semi-groupe $\{0, i_1+G_1\}$ étant contenu dans $^*G$, il en serait de même de $\{0, i_1+{}^*G_1\}$ qui contient $\{0, i_1+G_1\} \supseteq G$, et qui est canonique. On a donc $^*G=\{0, i_1+{}^*G_1\}$. La construction de $^*G$ est ainsi ramenée à la construction de la fermeture canonique d'un semi-groupe de la forme

$$G_1=\{0, i'_1, i'_2, \ldots, i'_{r'-1}+\mathfrak{G}\nu\};$$

pour lequel on a $i'_{r'-1} \leqslant i_{r-1}-i_1$. La répétition de cette construction nous permet donc de ramener la construction proposée à celle de la fermeture canonique d'un semi-groupe G_N qui se réduit, pour N suffisamment grand,

au semi-groupe $\mathfrak{G}\nu$ de tous les multiples non négatifs de ν. $\mathfrak{G}\nu$ étant sa propre fermeture canonique, la construction proposée se trouve ainsi effectuée. Remarquons que les éléments de $^*G$ ainsi construits ne dépendent que des éléments de G qui ne les dépassent pas; et qu'ils en sont des combinaisons linéaires à coefficients entiers. Supposons en effet que ceci ait été démontré pour la fermeture $^*G_1$ de G_1. Les éléments de $^*G_1$, qui sont inférieurs à $i_{h+1}-i_1$, ne dépendent alors que des éléments de G_1 qui sont inférieurs à $i_{h+1}-i_1$, et ils en sont des combinaisons linéaires à coefficients entiers; or ces derniers ne dépendent à leurs tours que des nombres $i_1, i_2, \ldots, i_h$ dont ils sont des combinaisons linéaires à coefficients entiers. Il en résulte que les éléments de $\{0, i_1+{}^*G_1\} = {}^*G$ qui sont inférieurs à i_{h+1} ne dépendent que des $i_1, i_2, \ldots, i_h$, dont ils sont des combinaisons linéaires à coefficients entiers.

Etant donné le semi-groupe canonique

$$^*G = \{0, i_1, i_2, \ldots, i_{r-1}+\mathfrak{G}\nu\} \quad (\nu = (i_1, i_2, \ldots, i_{r-1}, i_{r-1}+\nu)),$$

il n'existe qu'un nombre fini de semi-groupes g tels que $^*g = {}^*G$. Soit en effet

$$g = \{0, j_1, j_2, \ldots, j_s, j_{s+1}, \ldots\}$$

un tel semi-groupe. Soient $j_1, j_2, \ldots, j_n$ ceux des entiers $j_1, j_2, \ldots, j_s, \ldots$ qui sont inférieur à $i_{r+1} = i_{r-1}+2\nu$. i_{r-1} et $i_{r-1}+\nu$ étant des combinaisons linéaires à coefficients entiers des $j_1, j_2, \ldots, j_n$, le p.g.c.d. de ces nombres est ν. Or à chaque système d'entiers positifs inférieurs à $i_{r-1}+2\nu$ dont le p.g.c.d. est ν, on peut associer un multiple $j\nu$ de ν tel que tout semi-groupe d'entiers non négatifs contenant le système, contienne tous les multiples de ν supérieurs à $j\nu$. Soit $L\nu$ le plus grand des entiers $j\nu$ qui sont ainsi associés aux systèmes de multiples positifs de ν inférieurs à $i_{r-1}+2\nu$. Les semi-groupes g pour lesquels on a $^*g = {}^*G$ contiennent alors tous les multiples de ν qui dépassent $L\nu$ et ne diffèrent entre eux que par leurs éléments qui sont inférieurs à $L\nu$.

THÉORÈME 1. *La partie commune de tous les semi-groupes g tels que $^*g = {}^*G$, est un semi-groupe g_χ tel que $^*g_\chi = {}^*G$.*

Démonstration. Soit g un semi-groupe tel que l'on ait $^*g = {}^*G$ et tel qu'aucun sous semi-groupe de g ne possède cette propriété; nous allons montrer que $g = g_\chi$. Soit i le plus petit élément de g qui n'est pas contenu dans g_χ. Soient $i_0 = 0, i_1, i_2, \ldots, i_h$ les éléments de g et de g_χ qui sont inférieurs à i. i n'étant pas contenu dans g_χ, le nombre i n'est pas de la forme

$$\alpha_1 i_1 + \alpha_2 i_2 + \ldots + \alpha_h i_h,$$

avec $\alpha_1, \alpha_2, \ldots, \alpha_h$ entiers non négatifs. D'autre part g_χ étant la partie commune des semi-groupes dont la fermeture canonique est $^*G$, il existe un semi-groupe g' tel que $^*g' = {}^*G$ qui ne contient pas le nombre i. Les éléments

de $^*G = {}^*g$ qui sont inférieurs à i ne dépendant que des entiers $i_1, i_2, \ldots, i_h$, le semi-groupe g'' obtenu en suprimant dans g' tous les entiers positifs inférieurs à i autres que $i_1, i_2, \ldots, i_h$ est encore tel que $^*g'' = {}^*G$. Il en résulte que les éléments de $^*G$ qui sont inférieurs ou *ègaux* à i ne dépendent que des nombres $i_1, i_2, \ldots, i_h$; puisque g'' ne contient pas le nombre i. Donc la fermeture du sous semi-groupe de g obtenu en y suprimant le nombre i est encore égal à $^*G$. Contrairement au choix de g. On a donc $g_\chi = g$ et par conséquent $^*g_\chi = {}^*G$.

Le sous semi-groupe g_χ défini dans l'énconcé du théorème 1 s'appelle le *sous semi-groupe caractéristique* de tous les g tels que $^*g = {}^*G$. Il est clair que le semi-groupe g_χ est tel que tout sous semi-groupe propre de g_χ ait une fermeture canonique distinct de $^*g_\chi = {}^*G$. Inversement si g_χ est tel que pour tout sous semi-groupe propre g' de g_χ on ait $^*g' \neq {}^*g_\chi$, g_χ est son propre sous semi-groupe caractéristique.

$g_\chi = \{0, i_1, i_2, \ldots, i_{r-1}, i_r, \ldots\}$ étant le sous semi-groupe caractéristique de g, considérons les entiers $\chi_1, \chi_2, \ldots, \chi_h$ définis de la manière suivante: $\chi_1 = i_1$; χ_2 est le plus petit de ceux des entiers $i_1, i_2, \ldots, i_r, \ldots$ qui ne soient pas de la forme $\alpha_1\chi_1$ avec α_1, entier non négatif; χ_3 est le plus petit de ceux des entiers $i_1, i_2, \ldots, i_r, \ldots$ qui ne soient pas de la forme $\alpha_1\chi_1 + \alpha_2\chi_2$ avec α_1, α_2 entiers non négatifs; enfin $\chi_1, \chi_2, \ldots, \chi_n$ étant définis χ_{n+1} est le plus petit de ceux des entiers $i_1, i_2, \ldots, i_r, \ldots$ qui ne soient pas de la forme

$$\alpha_1\chi_1 + \alpha_2\chi_2 + \ldots + \alpha_n\chi_n$$

avec $\alpha_1, \alpha_2, \ldots, \alpha_n$ entiers non negatifs. Les nombres $\chi_1, \chi_2, \ldots, \chi_h$ ainsi définis s'appellent *les caractères* de g.

THÉORÈME 2. *$\gamma_1 < \gamma_2 < \ldots < \gamma_l$ étant un ensemble d'entiers positifs, l'ensemble des caractères du semi-groupe g des entiers de la forme*

$$\alpha_1\gamma_1 + \alpha_2\gamma_2 + \ldots + \alpha_l\gamma_l,$$

avec $\alpha_1, \alpha_2, \ldots, \alpha_l$ entiers non négatifs, est contenu dans l'ensemble $\gamma_1, \gamma_2, \ldots, \gamma_l$.

Démonstration. Soit χ_i le plus petit des caractères $\chi_1, \chi_2, \ldots, \chi_h$ de g qui ne soient pas contenus dans l'ensemble $\gamma_1, \gamma_2, \ldots, \gamma_l$. D'après la définition de g_χ, χ_i est de la forme $\alpha_1\gamma_1 + \alpha_2\gamma_2 + \ldots + \alpha_{l'}\gamma_{l'}$ avec $\alpha_1, \alpha_2, \ldots, \alpha_{l'}$ entiers non négatifs; $\gamma_1, \gamma_2, \ldots, \gamma_{l'}$ étant ceux des entiers $\gamma_1, \gamma_2, \ldots, \gamma_l$ qui sont inférieurs à χ_i. $\gamma_1, \gamma_2, \ldots, \gamma_l$ étant des éléments de la fermeture canonique de g_χ, tout semi-groupe canonique contenant $\chi_1, \chi_2, \ldots, \chi_{i-1}$ contient aussi $\gamma_1, \gamma_2, \ldots, \gamma_{l'}$. Ceci impliquerait que la fermeture canonique du semi-groupe des combinaisons linéaires à coefficients entiers non négatifs de $\chi_1, \chi_2, \ldots, \chi_{i-1}, \chi_{i+1}, \ldots, \chi_h$ contient χ_i, et il en résulterait que g_χ n'est pas un semi groupe caractéristique. Donc l'ensemble $\gamma_1, \gamma_2, \ldots, \gamma_l$ contient nécéssairement l'ensemble $\chi_1, \chi_2, \ldots, \chi_h$.

THÉORÈME 3. *g étant le semi-groupe des combinaisons linéaires à coefficients entiers non négatifs de $0 < \gamma_1 < \gamma_2 < \ldots < \gamma_l$, les entiers*

$$\nu_1, \nu_2, \ldots, \nu_{N-2}, \nu_{N-1}, \nu$$

tels que l'on ait

$$^*g = \{0, \nu_1, \nu_1 + \nu_2, \ldots, \nu_1 + \nu_2 + \ldots + \nu_{N-1} + \mathfrak{G}\nu\}$$

se déduisent de $\gamma_1, \gamma_2, \ldots, \gamma_l$ par l'algorithme quasi-jacobien de Du Val. Les entiers $\nu_1, \nu_2, \ldots, \nu_{N-1}, \nu$ y figurent comme diviseurs, tandis que les quotients partiels représentent les nombres de fois que chacun des diviseurs est répété dans la suite $\nu_1, \nu_2, \ldots, \nu_{N-1}, \nu$. Réciproquement si les nombres*

$$\nu_1, \nu_2, \ldots, \nu_{N-2}, \nu_{N-1}, \nu$$

se déduisent de $\gamma_1, \gamma_2, \ldots, \gamma_l$ par l'algorithme quasi-jacobien de Du Val, les quotients partiels étant les nombres de fois que chacun des diviseurs figure dans la suite $\nu_1, \nu_2, \ldots, \nu_{N-1}, \nu$, la fermeture canonique du semi groupe des entiers de la forme

$$\alpha_1\gamma_1 + \alpha_2\gamma_2 + \ldots + \alpha_l\gamma_l,$$

avec $\alpha_1, \alpha_2, \ldots, \alpha_l$ entiers non négatifs, est $^*g$.

Démonstration. ν étant le plus grand commun diviseur des éléments de g, on a $\gamma_1 \geqslant \nu$. Si $\gamma_1 = \nu$ le semi-groupe g est constitué par l'ensemble de tous les multiples de $\gamma_1 = \nu$, et l'on a $g = {}^*g = \{\mathfrak{G}\nu\}$. Dans ces conditions l'algorithme se termine dès le commencement. Admettons que la proposition ait été établie pour tout ensemble $\gamma'_1 < \gamma'_2 < \ldots < \gamma'_{l'}$ pour lequel $\gamma'_1 < \gamma_1$, et établissons le pour l'ensemble $\gamma_1, \gamma_2, \ldots, \gamma_l$. Soit γ_i le plus petit des entiers $\gamma_1, \gamma_2, \ldots, \gamma_l$ qui ne soit pas divisible par γ_1. Soit q le quotient de γ_i par γ_1 et considérons le semi-groupe Γ des combinaisons linéaires à coefficients entiers non négatifs des nombres $\gamma_i - q\gamma_1\, \gamma_{i+1} - q\gamma_1, \ldots, \gamma_l - q\gamma_1, \gamma_1$. Le semi-groupe $^*g$ contient visiblement le semi-groupe $\{0, \gamma_1, 2\gamma_1, \ldots, q\gamma_1 + \Gamma\}$ qui contient g. On a donc

$$^*g = \{0, \gamma_1, 2\gamma_1, \ldots, q\gamma_1 + {}^*\Gamma\},$$

c.à.d.

$$\nu_1 = \gamma_1, \nu_2 = \gamma_1, \ldots, \nu_q = \gamma_1,$$

$$^*\Gamma = \{0, \nu_{q+1}, \nu_{q+1} + \nu_{q+2}, \ldots, \nu_{q+1} + \ldots + \nu_{N-1} + \mathfrak{G}\nu\}.$$

Les entiers $\gamma_i - q\gamma_1, \gamma_{i+1} - q\gamma_1, \ldots, \gamma_l - q\gamma_i, \gamma_1$ étant les restes de la $(i-1)$-ième division de l'algorithme appliqué aux nombres $\gamma_1, \gamma_2, \ldots, \gamma_l$, il suffit de montrer que les entiers $\nu_{q+1}, \nu_{q+2}, \ldots, \nu_{N-1}, \nu$ s'obtiennent en appliquant l'algorithme aux entiers $\gamma_i - q\gamma_1, \gamma_{i+1} - q\gamma_1, \ldots, \gamma_l - q\gamma_1, \gamma_1$. Or $\gamma_i - q\gamma_1$ étant inférieur à γ_1, ceci a été admis comme établi. Réciproquement, si les

* Du Val, loc. cit.

nombres $\nu_1, \nu_2, \ldots, \nu_{N-1}, \nu$ se déduisent de $\gamma_1, \gamma_2, \ldots, \gamma_l$ par l'algorithme quasi-jacobien de Du Val, la fermeture canonique du semi-groupe des combinaisons linéaires à coefficients entiers non négatifs de $\gamma_1, \gamma_2, \ldots, \gamma_l$ est

$$\{0, \nu_1, \nu_1+\nu_2, \ldots, \nu_1+\nu_2+\ldots+\nu_{N-1}+\mathfrak{G}\nu\}$$

d'après la proposition directe que l'on vient d'établir.

$\chi_1, \chi_2, \ldots, \chi_h$ étant les caractères de g, le semi groupe des combinaisons linéaires à coefficients entiers non négatif de $\chi_1, \chi_2, \ldots, \chi_h$ est le sous semi-groupe caractéristique g_χ de g. Il en résulte, d'après les théorèmes 3 et 2, que les entiers $\nu_1, \nu_2, \ldots, \nu_{N-1}, \nu$ se déduisent des caractères de g par l'algorithme quasi-jacobien de Du Val, et que tous les systèmes d'entiers non négatifs $\gamma_1, \gamma_2, \ldots, \gamma_l$, pour lesquels l'algorithme fournit le même résultat, se déduisent du système des caractères de g en lui adjoignant des nombres arbitrairement choisis dans $^*g$.

6. Considérons maintenant un anneau H et sa fermeture canonique $^*H$. L'anneau H étant de la forme

$$H = k+kS_{i_1}+kS_{i_2}+\ldots+k[T]\,S_{i_h}$$

sa fermeture canonique $^*H$ peut se construire de la manière suivante:

Désignons par H_1 l'anneau

$$[I_{i_1}] = \Sigma k\left(\frac{S_{i_2}}{S_{i_1}}\right)^{\alpha_2}\left(\frac{S_{i_3}}{S_{i_1}}\right)^{\alpha_3}\ldots\left(\frac{S_{i_{h-1}}}{S_{i_1}}\right)^{\alpha_{h-1}}+k[T]\frac{S_{i_h}}{S_{i_1}},$$

où la sommation est étendue à tous les systèmes d'exposants entiers non négatifs $\alpha_2, \alpha_3, \ldots, \alpha_{h-1}$ tels que $\alpha_2(i_2-i_1)+\alpha_3(i_3-i_1)+\ldots+\alpha_{h-1}(i_{h-1}-i_1)$ soit inférieur à i_h-i_1. La fermeture canonique $^*H$ de H contient visiblement

$$k+H_1S_{i_1}$$

qui contient H et l'on a $^*H = k+{}^*H_1S_{i_1}$, en désignant par $^*H_1$, la fermeture canonique de H_1. D'une manière générale H_i étant défini, désignons par H_{i+1} l'anneau déduit de H_i de la même manière que H_1 est déduit de H. Il est clair que pour N suffisamment grand H_N est identique à $k[T]$. Soit T_{i+1} un élément d'ordre positif le plus petit de H_i. On a alors visiblement

$$\begin{aligned} {}^*H &= k+kT_1+{}^*H_2T_1T_2 \qquad\qquad (\text{avec } T = S_i)\\ &= k+kT_1+kT_1T_2+{}^*H_3T_1T_2T_3\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &= k+kT_1+kT_1T_2+\ldots+kT_1T_2\ldots T_{N-1}+{}^*H_NT_1T_2\ldots T_{N-1}T_N\\ &= k+kT_1+kT_1T_2+\ldots+kT_1T_2\ldots T_{N-1}+k[T]\,T_1T_2\ldots T_{N-1}T_N. \end{aligned}$$

Remarque. n étant un entier quelconque, l'anneau $k+H_1S_{i_1} \bmod t^n$ ne dépend que de $H \bmod t^n$. Pour le montrer il suffit d'observer que $H_1 \bmod t^{n-i_1}$

ne dépend que de $H \bmod t^n$. De même l'anneau $k+H_2T_2 \bmod t^{n-i_1}$ ne dépend que de $H_1 \bmod t^{n-i_1}$. L'anneau $k+kT_1+H_2T_1T_2 \bmod t^n$ ne dépend donc que de $H \bmod t^n$. En continuant de cette manière on observe finalement que

$$^*H = k+kT_1+kT_1T_2+\ldots+kT_1T_2\ldots T_{N-1}+H_NT_1T_2\ldots T_N \quad \bmod t^n$$

ne dépend que de $H \bmod t^n$.

THÉORÈME AUXILIAIRE 5. *Si H mod t^n est identique à $^*H$ mod t^n, l'ensemble $^*H$ mod t^{n+1} est identique à l'un des ensembles*

$$k+kS_{i_1}+kS_{i_2}+\ldots+kS_{i_{l-1}}+[I_{i_l}]S_{i_l} \bmod t^{n+1} \quad (i_l<n+1).$$

Démonstration. L'ensemble $^*H \bmod t^n$ étant identique à $H \bmod t^n$, l'ensemble $^*H \bmod t^{n+1}$, qui contient l'ensemble $H \bmod t^{n+1}$, est constitué par les éléments de la forme

$$S+\alpha{}^*S_n \bmod t^{n+1}$$

où S est un élément de H, $^*S_n$ un élément fixe d'ordre n de $^*H$, et α un élément de k. Donc tout anneau $H' \bmod t^{n+1}$, contenant l'anneau $H \bmod t^{n+1}$, est identique à $H \bmod t^{n+1}$, s'il est contenu dans $^*H \bmod t^{n+1}$ sans lui être identique. Considérons maintenant l'anneau

$$k+S_{i_1}[I_{i_1}] \bmod t^{n+1}$$

qui contient $H \bmod t^{n+1}$ et qui est contenu dans $^*H \bmod t^{n+1}$. D'après ce que nous venons de remarquer l'anneau $k+S_{i_1}[I_{i_1}] \bmod t^{n+1}$ est identique à l'un des deux anneaux

$$^*H \bmod t^{n+1}, \quad H \bmod t^{n+1}.$$

S'il n'est pas identique au premier on a $[I_{i_1}] = I_{i_1}/S_{i_1} \bmod t^{n+1-i_1}$. Comme $^*[I_{i_1}] \bmod t^{n+1-i_1}$ ne dépend que de $[I_{i_1}] \bmod t^{n+1-i_1}$, les ensembles

$$^*[I_{i_1}] \bmod t^{n+1-i_1}, \quad k+\frac{S_{i_2}}{S_{i_1}}{}^*[I_{i_2}] \bmod t^{n+1-i_1}$$

seront alors identiques, puisque I_{i_2}/S_{i_1} est l'ensemble des éléments d'ordre positifs de I_{i_1}/S_{i_1}. Il en résulte que $^*H \bmod t^{n+1}$ est identique à l'un des deux anneaux

$$k+S_{i_1}[I_{i_1}] \bmod t^{n+1}, \quad k+kS_{i_1}+{}^*[I_{i_2}]S_{i_2} \bmod t^{n+1}.$$

Si $^*H \bmod t^{n+1}$ n'est identique ni à $k+S_{i_1}[I_{i_1}] \bmod t^{n+1}$ ni à

$$k+kS_{i_1}+S_{i_2}[I_{i_2}] \bmod t^{n+1},$$

ces deux anneaux sont identiques à $H \bmod t^{n+1}$. Dans ces conditions on aura $[I_{i_2}] \equiv I_{i_2}/S_{i_2} \bmod t^{n+1-i_2}$, dont on peut déduire l'identité des deux ensembles

$$^*[I_{i_2}] \bmod t^{n+1-i_2}, \quad k+{}^*[I_{i_3}]\frac{S_{i_3}}{S_{i_2}} \bmod t^{n+1-i_2}.$$

$^*H \bmod t^{n+1}$ est donc identique à l'un des ensembles

$$k+[I_{i_1}]S_{i_1} \bmod t^{n+1}, \quad k+kS_{i_1}+[I_{i_2}]S_{i_2} \bmod t^{n+1},$$
$$k+kS_{i_1}+kS_{i_2}+{}^*[I_{i_3}]S_{i_3} \bmod t^{n+1}.$$

En continuant de cette manière on montre que $^*H \bmod t^{n+1}$ est identique à l'un des ensembles

$$\begin{array}{ll} k+[I_{i_1}]S_{i_1} & \bmod t^{n+1}, \\ k+kS_{i_1}+[I_{i_2}]S_{i_2} & \bmod t^{n+1}, \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ k+kS_{i_1}+kS_{i_2}+\dots+[I_{i_l}]S_{i_l} & \bmod t^{n+1}, \\ k+kS_{i_1}+kS_{i_2}+\dots+kS_{i_l}+{}^*[I_{i_{l+1}}]S_{i_{l+1}} & \bmod t^{n+1}. \end{array}$$

Or pour $i_{l+1} \geqslant n+1$, le dernier de ces ensembles est $H \bmod t^{n+1}$. $^*H \bmod t^{n+1}$ est donc identique à l'un des ensembles

$$k+kS_{i_1}+kS_{i_2}+\dots+[I_{i_l}]S_{i_l} \quad \bmod t^{n+1}$$

avec $\leqslant i_l n$.

$X_1, X_2, \dots, X_n$ étant des séries entières d'ordres positifs en t, nous désignerons par $k[X_1, X_2, \dots, X_n]$ l'anneau formé des séries de la forme

$$\Sigma \alpha_{j_1 j_2 \dots j_n} X_1^{j_1} X_2^{j_2} \dots X_n^{j_n}$$

avec $\alpha_{j_1 j_2 \dots j_n} \in k$, la sommation étant étendue à tous les systèmes d'entiers $j_1, j_2, \dots, j_n$ non négatifs.

THÉORÈME AUXILIAIRE 6. *Les éléments $Y_1, Y_2, \dots, Y_\nu, \dots$ de $^*H$ étant choisis de manière que $w(Y_j)$ soit le plus petit élément de $W(^*H)$ non contenu dans $W(k[Y_1, Y_2, \dots, Y_{j-1}])$, si les éléments $Y_1', Y_2', \dots, Y_{\nu-1}'$, sont respectivement congrus à $Y_1, Y_2, \dots, Y_{\nu-1} \bmod t^{w(Y_\nu)}$, le plus petit élément de $W(^*H)$ non contenu dans $W(k[Y_1', Y_2', \dots, Y_{\nu-1}'])$ est $w(Y_\nu)$.*

Démonstration. Les anneaux

$$^*H \bmod t^{w(Y_\nu)}, \quad k[Y_1, Y_2, \dots, Y_{\nu-1}] \bmod t^{w(Y_\nu)}, \quad k[Y_1', \dots, Y_{\nu-1}'] \bmod t^{w(Y_\nu)}$$

étant visiblement identiques il suffit de montrer que $k[Y_1', Y_2', \dots, Y_{\nu-1}']$ ne contient pas d'élément d'ordre $w(Y_\nu)$. Tout élément de $k[Y_1', Y_2', \dots, Y_{\nu-1}']$ $\bmod t^{w(Y_\nu)+1}$ étant de la forme

$$P(Y_1', Y_2', \dots, Y_{\nu-1}') \bmod t^{w(Y_\nu)+1}$$

où $P(Y_1', Y_2', \dots, Y_{\nu-1}')$ est un polynôme à coefficients dans k, il suffit de montrer que $w(P(Y_1', Y_2', \dots, Y_{\nu-1}'))$ ne peut être égal à $w(Y_\nu)$. Si le polynôme $P(Y_1', Y_2', \dots, Y_{\nu-1}')$ contient un terme constant, $Y_1', Y_2', \dots, Y_{\nu-1}'$ étant tous d'ordres positifs, on aura $w(P(Y_1', Y_2', \dots, Y_{\nu-1}')) = 0 \neq w(Y_\nu)$. Si

$$P(Y_1', Y_2', \dots, Y_{\nu-1}')$$

contient des termes de degrés 1 sans contenir de terme constant, on peut l'écrire sous la forme

$$P_1(Y_1', Y_2', \dots, Y_{j-1}') + \beta Y_j' + P_2(Y_1', Y_2', \dots, Y_{\nu-1}')$$

avec $\beta \neq 0$; $P_2(Y_1', Y_2', \dots, Y_{\nu-1}')$ étant la somme des termes de degrés non nuls par rapport à l'ensemble Y_j', Y_{j+1}', $\dots$, $Y_{\nu-1}'$ sauf le terme $\beta Y_j'$. $w(P_2(Y_1', Y_2', \dots, Y_{\nu-1}'))$ est alors supérieur à $w(Y_j')$ qui est par définition différent de l'ordre de

$$P_1(Y_1', Y_2', \dots, Y_{j-1}') \equiv P_1(Y_1, Y_2, \dots, Y_{j-1}) \bmod t^{w(Y_\nu)}.$$

On a donc

$$w(P(Y_1', Y_2', \dots, Y_{\nu-1}')) = \min(w(Y_j')), \quad w(P_1(Y_1', Y_2', \dots, Y_{j-1}')) < w(Y_\nu).$$

Si enfin $P(Y_1', Y_2', \dots, Y_{\nu-1}')$ ne contient aucun terme de degré 1 ni de degré zéro, on peut écrire

$$P(Y_1', Y_2', \dots, Y_{\nu-1}') \equiv P(Y_1, Y_2, \dots, Y_{\nu-1}) \bmod t^{w(Y_\nu)+1}$$

$w(P(Y_1, Y_2, \dots, Y_{\nu-1}))$ étant différent de $w(Y_\nu)$ il en est de même de

$$w(P(Y_1', Y_2', \dots, Y_{\nu-1}')).$$

THÉORÈME AUXILIAIRE 7. $Y_1, Y_2, \dots, Y_{\nu-1}, Y_\nu$ et $Y_1', Y_2', \dots, Y_{\nu-1}'$ *ayant les mêmes significations que dans l'énoncé du théorème auxiliaire* 6, *si la fermeture canonique de* $k[Y_1, Y_2, \dots, Y_{\nu-1}]$ *ne contient pas d'élément d'ordre* $w(Y_\nu)$, *il en est de même de fermeture canonique de* $k[Y_1', Y_2', \dots, Y_{\nu-1}']$.

Démonstration. Soient $i_0 = 0$, i_1, i_2, $\dots$, i_μ, $\dots$ les orders des éléments de $k[Y_1', Y_2', \dots, Y_{\nu-1}']$ écrits dans l'ordre croissant et soit I_{i_μ}' l'ensemble des éléments de $k[Y_1', Y_2', \dots, Y_{\nu-1}']$ dont les ordres ne sont pas inférieurs à i_μ. Désignons par S_{i_l}' un élément d'ordre i_l de $k[Y_1', Y_2', \dots, Y_{\nu-1}']$, et par $\mathfrak{H}'$ la fermeture canonique de $k[Y_1', Y_2', \dots, Y_{\nu-1}']$. Les anneaux

$$^*H \bmod t^{w(Y_\nu)}, \quad \mathfrak{H}' \bmod t^{w(Y_\nu)}, \quad k[Y_1', Y_2', \dots, Y_{\nu-1}'] \bmod t^{w(Y_\nu)}$$

étant identiques, d'après le théorème auxiliaire 5 l'anneau $\mathfrak{H}' \bmod t^{w(Y_\nu)+1}$ est identique à l'un des anneaux

$$k + kS_{i_1}' + kS_{i_2}' + \dots + [I_{i_l}']\, S_{i_l}' \quad \bmod t^{w(Y_\nu)+1}$$

avec $i_l < w(Y_\nu)$. Soit μ le plus petit des entiers l pour lesquels cette identité a lieu. Si $\mu = 0$, $\mathfrak{H}' \bmod t^{w(Y_\nu)+1}$ est identique à $k[Y_1', Y_2', \dots, Y_{\nu-1}'] \bmod t^{w(Y_\nu)+1}$ qui ne contient pas d'élément d'ordre $w(Y_\nu)$. Supposons donc que μ est positif. Pour montrer que $\mathfrak{H}'$ ne contient pas d'élément d'ordre $w(Y_\nu)$, il suffit de montrer que $[I_{i_\mu}']$ ne contient pas d'élément d'ordre $w(Y_\nu) - i_\mu$. Soient I_{i_μ} et $^*I_{i_\mu}$ les ensembles des éléments d'ordres non inférieurs à i_μ de $k[Y_1, \dots, Y_{\nu-1}]$ et de $^*H$. Les anneaux

$$^*H \bmod t^{w(Y_\nu)}, \quad k[Y_1, Y_2, \dots, Y_{\nu-1}] \bmod t^{w(Y_\nu)}, \quad k[Y_1', Y_2', \dots, Y_{\nu-1}'] \bmod t^{w(Y_\nu)}$$

étant identiques, il en est de même des ensembles

$$[{}^*I_{i_\mu}] \quad \bmod t^{w(Y_\nu)-i_\mu}$$

$$I_{i_\mu}/S_{i_\mu} \bmod t^{w(Y_\nu)-i_\mu}, \quad I'_{i_\mu}/S'_{i_\mu} \bmod t^{w(Y_\nu)-i_\mu},$$

où S_{i_μ} est une élément de $k[Y_1, Y_2, \ldots, Y_{\nu-1}]$, tel que l'on ait

$$S_{i_\mu} \equiv S'_{i_\mu} \quad \bmod t^{w(Y_\nu)}.$$

Il en résulte qu'on peut associer à tout élément Z' de I'_{i_μ}/S'_{i_μ} un élément Z de I_{i_μ}/S_{i_μ} de manière que l'on ait

$$Z = Z' \quad \bmod t^{w(Y_\nu)-i_\mu}.$$

Considérons en particulier un ensemble d'éléments $Z'_1, Z'_2, \ldots, Z'_\rho$ de I'_{i_μ}/S'_{i_μ} choisis de la manière suivante:

(1) Z'_1 est un élément d'ordre positif le plus petit de I'_{i_μ}/S'_{i_μ},

(2) $Z'_1, Z'_2, \ldots, Z'_{j-1}$ étant choisis, on choisit Z'_j de manière que $w(Z'_j)$ soit le plus petit élément positif de $W(I'_{i_\mu}/S'_{i_\mu})$ qui ne soit pas contenu dans $W(k[Z'_1, Z'_2, \ldots, Z'_{j-1}])$,

(3) $w(Z'_\rho) < w(Y_\nu) - i_\mu + 1$ et tout élément de $W(I'_{i_\mu}/S'_{i_\mu})$ inférieur à $w(Y_\nu) - i_\mu + 1$ soit contenu dans $W(k[Z'_1, Z'_2, \ldots, Z'_\rho])$.

$k[Y'_1, Y'_2, \ldots, Y'_{\nu-1}] \bmod t^{w(Y_\nu)+1}$ étant distinct de $\mathfrak{H}' \bmod t^{w(Y_\nu)+1}$ tandis que $k[Y'_1, Y'_2, \ldots, Y'_{\nu-1}] \bmod t^{w(Y_\nu)}$ est identique à $\mathfrak{H}' \bmod t^{w(Y_\nu)}$, l'anneau $k[Y'_1, Y'_2, \ldots, Y'_{\nu-1}]$ ne peut contenir d'éléments d'ordres $w(Y_\nu)$. Il en résulte que les nombres $w(Z'_1), w(Z'_2), \ldots, w(Z'_\rho)$ sont inférieurs à $w(Y_\nu) - i_\mu$. Des conditions imposées au choix des $Z'_1, Z'_2, \ldots, Z'_\rho$ résulte en outre l'identité des anneaux

$$[I'_{i_\mu}] \bmod t^{w(Y_\nu)-i_\mu+1}, \quad k[Z'_1, Z'_2, \ldots, Z'_\rho] \bmod t^{w(Y_\nu)-i_\mu+1};$$

il suffit donc de montrer que $k[Z'_1, Z'_2, \ldots, Z'_\rho]$ ne contient pas d'élément d'ordre $w(Y_\nu) - i_\mu$. Or soient $Z_1, Z_2, \ldots, Z_\rho$ les éléments de I_{i_μ}/S_{i_μ} tels que l'on ait

$$Z_j \equiv Z'_j \bmod t^{w(Y_\nu)-i_\mu} \quad (j = 1, 2, \ldots, \rho).$$

La fermeture canonique de $k[Y_1, Y_2, \ldots, Y_{\nu-1}]$ ne contenant aucun élément d'ordre $w(Y_\nu)$, l'anneau $k[Z_1, Z_2, \ldots, Z_\rho]$ ne contient aucun élément d'ordre $w(Y_\nu) - i_\mu$. Les éléments $Z_1, Z_2, \ldots, Z_\rho, Z_{\rho+1} = Y_\nu/S_{i_\mu}$ de $[{}^*I_{i_\mu}]$ et les $Z'_1, Z'_2, \ldots, Z'_\rho$ remplissent donc les conditions de l'énoncé du théorème auxiliaire 6 vis à vis de l'anneau canonique $[{}^*I_{i_\mu}]$. L'anneau $k[Z'_1, Z'_2, \ldots, Z'_\rho]$ ne peut donc contenir d'éléments d'ordres $w(Z_{\rho+1}) = w(Y_\nu) - i_\mu$.

Considérons maintenant un ensemble d'éléments $X_1, X_2, \ldots, X_m$ de ${}^*H$ choisis de la manière suivante: X_1 est un élément d'ordre positif le plus petit de ${}^*H$; $X_1, X_2, \ldots, X_{l-1}$ étant choisis, X_l est un élément de ${}^*H$ tel que $w(X_l)$ soit le plus petit élément de $W({}^*H)$ qui ne soit pas contenu dans

$W(\mathfrak{H}_{l-1})$, où $\mathfrak{H}_{l-1}$ désigne la fermeture canonique de $k[X_1, X_2, \ldots, X_{l-1}]$. Les éléments de $W(^*H)$ étant des combinaisons linéaires à coefficients entiers non negatifs d'un nombre fini d'entre eux, les éléments $X_1, X_2, \ldots, X_l, \ldots$ ainsi choisis ne peuvent être qu'en nombre fini. Un tel ensemble d'éléments $(X_1, X_2, \ldots, X_m)$ sera appelé dans ce qui suit une *base* de $^*H$.

THÉORÈME 4. *$(X_1, X_2, \ldots, X_m)$ étant une base de $^*H$, les entiers*

$$w(X_1),\ w(X_2),\ \ldots,\ w(X_m)$$

ne dépendent que de $^*H$ et ils constituent une partie des caractères de $^*H$.

Démontrons d'abord la proposition suivante qui facilitera la démonstration de ce théorème.

THÉORÈME AUXILIAIRE 8. *$\mathfrak{H}_l$ étant la fermeture canonique de $k[X_1, X_2, \ldots, X_l]$ où $X_1, X_2, \ldots, X_m$ est une base de $^*H$, on peut choisir les éléments $Y_1, Y_2, \ldots, Y_\nu, \ldots$ de $\mathfrak{H}_l$ remplissant les conditions de l'énoncé du théorème auxiliaire* 6 *envisagé pour l'anneau canonique $\mathfrak{H}_l$* (c.à.d. *$w(Y_j)$ étant le plus petit élément de $W(\mathfrak{H}_l)$ non contenu dans $W(k[Y_1, Y_2, \ldots, Y_{j-1}])$*) *de manière que la suite $Y_1, Y_2, \ldots, Y_\nu, \ldots$ contienne l'ensemble $X_1, X_2, \ldots, X_l$.*

Démonstration. Pour $l = 1$, on a visiblement $\mathfrak{H}_1 = k[X_1]$ et l'on peut poser $Y_1 = X_1$. Supposons que la proposition ait été démontrée pour l et démontrons la pour $l+1$. Soient $Y_1, Y_2, \ldots, Y_\nu$ ceux des éléments choisis de $\mathfrak{H}_l$ dont les ordres sont plus petits que $w(X_{l+1})$. Les éléments de $W(\mathfrak{H}_l)$ qui sont inférieurs à $w(X_{l+1})$ sont alors les mêmes que ceux de $W(k[Y_1, Y_2, \ldots, Y_\nu])$. Le plus petit élément de $W(\mathfrak{H}_{l+1})$ non contenu dans $W(\mathfrak{H}_l)$ étant $w(X_{l+1})$, posons $Y_{\nu+1} = X_{l+1}$, et choisissons $Y_{\nu+2}, Y_{\nu+3}, \ldots$ dans $\mathfrak{H}_{l+1}$ conformément à l'énoncé du théorème auxiliaire 6 envisagé pour $\mathfrak{H}_{l+1}$. La suite

$$Y_1,\ Y_2,\ \ldots,\ Y_\nu,\ Y_{\nu+1},\ \ldots$$

remplit alors les conditions de l'énoncé de la proposition que nous voulons démontrer pour $\mathfrak{H}_{l+1}$.

Démonstration du théorème 4. Soient $X_1, X_2, \ldots, X_m$ et $X'_1, X'_2, \ldots, X'_{m'}$ deux bases de $^*H$. Si les entiers $w(X_1), w(X_2), \ldots, w(X_m)$ et les entiers $w(X'_1), w(X'_2), \ldots, w(X'_{m'})$ n'étaient pas les mêmes, l'un au moins des entiers $(w(X_1), w(X_2), \ldots, w(X_m), w(X'_1), w(X'_2), \ldots, w(X'_{m'}))$ n'appartient qu'à l'un des ensembles $(w(X_1), w(X_2), \ldots, w(X_m))$, $(w(X'_1), w(X'_2), \ldots, w(X'_{m'}))$. Soit $w(X'_{l+1})$ le plus petit de ces entiers qui n'appartient qu'à l'un de ces ensembles, et considérons les fermetures canoniques $\mathfrak{H}_l$, $\mathfrak{H}'_l$ des anneaux $k[X_1, X_2, \ldots, X_l]$, $k[X'_1, X'_2, \ldots, X'_l]$. D'après les choix des X'_j, X_j les anneaux $\mathfrak{H}_l \bmod t^{w(X_{l+1})}$ $\mathfrak{H}'_l \bmod t^{w(X'_{l+1})}$ sont respectivement identiques à $^*H \bmod t^{w(X_{l+1})}$, $^*H \bmod t^{w(X'_{l+1})}$. $w(X_{l+1})$ étant par définition supérieur à $w(X'_{l+1})$, l'anneau $\mathfrak{H}_l$ devrait contenir un élément d'ordre $w(X'_{l+1})$. Or soit $(Y_1, Y_2, \ldots, Y_\nu, \ldots)$ un ensemble d'éléments

de $\mathfrak{H}_l$ choisis conformément à l'énoncé du théorème auxiliaire 8 et soient $Y_1, Y_2, \ldots, Y_\nu$, ceux des éléments de cet ensemble dont les ordres sont inférieurs à $w(X'_{l+1})$. Les anneaux

$$^*H \bmod t^{w(X'_{l+1})}, \quad \mathfrak{H}_l \bmod t^{w(X_{l+1})}, \quad \mathfrak{H}'_l \bmod t^{w(X'_{l+1})},$$
$$k[Y_1, Y_2, \ldots, Y_\nu] \bmod t^{w(X'_{l+1})}$$

étant identiques, il existe des éléments $Y'_1, Y'_2, \ldots, Y'_\nu$ de $\mathfrak{H}'_l$ tels que

$$Y'_j = Y_j \bmod t^{w(X'_{l+1})} \quad (j = 1, 2, \ldots, \nu).$$

La fermeture canonique de $k[Y'_1, Y'_2, \ldots, Y'_\nu]$ qui est contenu dans $\mathfrak{H}'_l$ ne peut contenir aucun élément d'ordre $w(X'_{l+1})$. Donc la fermeture canonique de $k[Y_1, Y_2, \ldots, Y_\nu]$ qui n'est autre que $\mathfrak{H}_l$ (puisque l'ensemble $(Y_1, Y_2, \ldots, Y_\nu)$ contient l'ensemble $(X_1, X_2, \ldots, X_l)$) ne contient pas d'éléments d'ordre $w(X'_{l+1})$ (théorème auxiliaire 7). Donc $w(X_{l+1})$ est égal à $w(X'_{l+1})$ contrairement à l'hypothèse.

Le fait que les nombres $w(X_1), w(X_2), \ldots, w(X_n)$ constituent une partie des caractères de $^*H$ se démontre comme suit: $w(X_1)$ étant le plus petit élément de $W(^*H)$ on a $w(X_1) = \chi_1$. Supposons que $w(X_l)$ soit le plus petit de ceux des nombres $w(X_1), w(X_2), \ldots, w(X_m)$ qui ne sont pas des caractères de $^*H$. $w(X_l)$ serait alors contenu dans la fermeture canonique du semi-groupe engendré par les éléments de $W(^*H)$ qui sont inférieurs à $w(X_l)$. Or les éléments de $W(^*H)$ qui sont inférieurs à $w(X_l)$ sont contenus dans $W(\mathfrak{H}_{l-1})$. On a donc $w(X_l) \in W(\mathfrak{H}_{l-1})$ contrairement au choix des X_j.

Dans ce qui suit nous allons appeler les nombres

$$w(X_1) = {}^*\chi_1, \; w(X_2) = {}^*\chi_2, \; \ldots, \; w(X_m) = {}^*\chi_m$$

les caractères de base de $^*H$. De la définition d'une base de $^*H$ et du théorème 4 résulte immédiatement que tout système d'éléments $^*X_1, {}^*X_2, \ldots, {}^*X_m$ de $^*H$ tels que $w(^*X_1) = {}^*\chi_1, w(^*X_2) = {}^*\chi_2, \ldots, w(^*X_m) = {}^*\chi_m$ constitue une base de $^*H$.

Un ensemble d'éléments $Y_1, Y_2, \ldots, Y_\nu$ de H s'appelle *un système de générateurs*, si $^*H$ est la fermeture canonique de $k[Y_1 - \eta_1, Y_2 - \eta_2, \ldots, Y_\nu - \eta_\nu]$ où $\eta_1, \eta_2, \ldots, \eta_\nu$ désignent les termes constants de $Y_1, Y_2, \ldots, Y_\nu$.

$X_1, X_2, \ldots, X_m$ étant une base de $^*H$, considérons un ensemble d'éléments $Y_1, Y_2, \ldots, Y_m$ choisis de la manière suivante:

$$\begin{array}{ll} Y_1 = X_1 + X'_1 & X'_1 \in k \\ Y_2 = X_2 + X'_2 & X'_2 \in \mathfrak{H}_1 \\ \cdots\cdots\cdots\cdots & \cdots\cdots \\ Y_m = X_m + X'_m & X'_m \in \mathfrak{H}_{m-1} \end{array}$$

où $\mathfrak{H}_i$ désigne la fermeture canonique de $k[X_1, X_2, \ldots, X_i]$; les éléments $Y_1, Y_2, \ldots, Y_m$ constituent visiblement un système de générateurs de $^*H$. Inversement tout système de générateurs contient un ensemble partiel choisi de cette manière. En effet $Y_1, Y_2, \ldots, Y_\nu$ étant un système de générateurs de $^*H$, désignons par $\eta_1, \eta_2, \ldots, \eta_\nu$ les termes constant de $Y_1, Y_2, \ldots, Y_\nu$. L'un au moins des entiers $w(Y_1-\eta_1)$, $w(Y_2-\eta_2)$, $\ldots$, $w(Y_\nu-\eta_\nu)$ est alors égal à $^*\chi_1$, soit $w(Y_1-\eta_1) = {}^*\chi_1$. On peut donc poser $X_1 = Y_1-\eta_1$. $W(\mathfrak{H}_1)$ contenant tous les éléments de $W(^*H)$ qui sont inférieurs à $^*\chi_2$, on peut choisir $X'_i \in \mathfrak{H}_1$, de manière que l'on ait

$$w(Y_i - X'_i) \geqslant {}^*\chi_2 \quad (i = 2, 3, \ldots, \nu).$$

L'un au moins des entiers $w(Y_i - X'_i)$ est égal à $^*\chi_2$; car sinon la fermeture canonique de $k[X_1, Y_2 - X'_2, \ldots, Y_\nu - X'_\nu]$ qui est par définition identique à $^*H$ ne contiendrait pas d'éléments d'ordre $^*\chi_2$. Soit $w(Y_2 - X'_2) = {}^*\chi_2$. On peut alors poser $X_2 = Y_2 - X'_2$ et ainsi de suite. Il résulte de ces considérations que tout système de générateurs de $^*H$ contient au moins m éléments, m étant le nombre des caractères de base de $^*H$; nous l'appellerons le nombre *de dimension de* $^*H$.

7. $^*H = k + kT_1 + kT_1T_2 + \ldots + k[T]\,T_1T_2\ldots T_{N-1}$ étant un anneau canonique, les caractères ainsi que les caractères de base des anneaux

$$[I_{i_h}] = {}^*H_h = k + kT_{h+1} + \ldots + k[T]\,T_{h+1}T_{h+2}\ldots T_{N-1}$$

sont des invariants de $^*H$. Les caractères de $^*H_h$ sont visiblement détérminés par ceux de $^*H$. Mais il n'en est plus de même des caractères de base de $^*H_h$.

Considérons par exemple les anneaux

$$\left.\begin{aligned} {}^*H &= k + kt^{4\nu}(1+t) + kt^{6\nu}(1+t) + kt^{7\nu}(1+t) + k[t]\,t^{8\nu}, \\ {}^*H' &= k + kt^{4\nu} + kt^{6\nu}(1+t) + kt^{7\nu}(1+t) + k[t]\,t^{8\nu}. \end{aligned}\right\} \quad (\nu > 1)$$

On vérifie facilement que ces anneaux sont tous les deux canoniques et que leurs caractères qui sont ceux du semi-groupe

$$W(^*H) = W(^*H') = \{0, 4\nu, 6\nu, 7\nu, 8\nu+1, 8\nu+2, 8\nu+3, \ldots\}$$

sont les mêmes. Ces caractères sont visiblement 4ν, 6ν, 7ν, $8\nu+1$. Construisons maintenant une base de $^*H$: On peut évidemment poser $X_1 = t^{4\nu}(1+t)$; $k[X_1]$ est un anneau canonique et le plus petit élément de $W(^*H)$ non contenu dans $W(k[X_1])$ est 6ν; on peut donc poser $X_2 = t^{6\nu}(1+t)$. La fermeture canonique de $k[X_1, X_2]$ est

$$k[\overline{X_1, X_2}] = k + kt^{4\nu}(1+t) + kt^{6\nu}(1+t) + k[t]\,t^{8\nu}.$$

On peut donc choisir $X_3 = t^{7\nu}(1+t)$ comme troisième élément de base de $^*H$. La fermeture canonique de $k[X_1, X_2, X_3]$ étant alors $^*H$, les caractères de

base de $^*H$ sont 4ν, 6ν, 7ν. On observe de la même manière que les éléments $X_1' = t^{4\nu}$, $X_2' = t^{6\nu}(1+t)$, $X_3' = t^{7\nu}(1+t)$ constituent une base de $^*H'$. Les caractères de base de $^*H$ et de $^*H'$ sont donc les mêmes. Calculons maintenant les caractères de base des anneaux

$$^*H_1 = k + kt^{2\nu} + kt^{3\nu} + k[t]\,t^{4\nu},$$
$$^*H_1' = k + kt^{2\nu}(1+t) + kt^{3\nu}(1+t) + k[t]\,t^{4\nu}.$$

Une base de $^*H_1'$, est constitué par $t^{2\nu}$, $t^{3\nu}$, $t^{4\nu+1}$ tandis que les éléments $t^{2\nu}(1+t), t^{3\nu}(1+t)$ constituent une base de $^*H_1'$, puisque la fermeture canonique de $k[t^{2\nu}(1+t), t^{3\nu}(1+t)]$ contient l'élément

$$t^{4\nu}(1+t)^2 - t^{2\nu}(1+t)\left(\frac{t^{3\nu}(1+t)}{t^{\cdot\nu}(1+t)}\right)^2 = t^{4\nu+1}(1+t)$$

dont l'ordre est $4\nu+1$. Les caractères de base de $^*H_1$, sont donc 2ν, 3ν, $4\nu+1$ tandis que ceux de $^*H_1'$ sont 2ν, 3ν.

Les caractères de base des anneaux $[I_{i_h}] = {}^*H_h$ *constituent donc des éléments invariants nouveaux pour* $^*H$.

Les considérations qui suivent, permettent de détérminer successivement les caractères de bases des $^*H_h$. *Considérons un élément quelconque d'ordre positif de* $^*H$. *Soit* T *cet élément et soit* $(X_1, X_2, \ldots, X_m)$ *une base de* $^*H$. *Désignons par* $^*\chi_i$ *le plus petit des nombres*

$$^*\chi_1 = w(X_1),\ {}^*\chi_2 = w(X_2),\ \ldots,\ {}^*\chi_m = w(X_m),\ \chi_{m+1} = \infty$$

tel que la fermeture canonique de $k[X_1, X_2, \ldots, X_{i-1}, T]$ *contienne un élément d'ordre* $^*\chi_i$. *Les éléments* T, TX_1, TX_2, $\ldots$, TX_{i-1}, TX_{i+1}, $\ldots$, TX_m *constituent alors une base de* $k + {}^*HT$ *qui est canonique.* En effet

$$\overline{k[X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m, T]}$$

étant la fermeture canonique de $k[X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m, T]$, la fermeture canonique de $k[T, TX_1, \ldots, TX_{i-1}, TX_{i+1}, \ldots, TX_m]$ contient visiblement l'anneau

$$k + T\overline{k[X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m, T]}.$$

Comme $\overline{k[X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m, T]}$ contient un élément d'ordre $^*\chi_i$, on a

$$\overline{k[X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m, T]} = {}^*H.$$

La fermeture canonique de $k[T, TX_1, \ldots, TX_{i-1}, TX_{i+1}, \ldots, TX_m]$ est donc identique à

$$k + T^*H$$

qu'elle contient; puisque l'anneau $k[T, TX_1, \ldots, TX_{i-1}, TX_{i+1}, \ldots, TX_m]$ est lui même contenu dans $k + T^*H$. Donc pour montrer que les

$$T,\ TX_1,\ \ldots,\ TX_{i-1},\ TX_{i+1},\ \ldots,\ TX_m$$

constituent une base de $k+T^*H$ il suffit de montrer que les fermetures canoniques des anneaux

$$\begin{array}{ll} k[T, TX_1, \dots, TX_j] & (1 \leqslant j < i-1) \\ k[T, TX_1, \dots, TX_{i-1}] & \\ k[T, TX_1, \dots, TX_{i-1}, TX_{i+1}, \dots, TX_h] & (n > h \geqslant i+1) \end{array}$$

sont dépourvus d'éléments d'ordres respectifs

$$w(TX_{j+1}), \quad w(TX_{i+1}), \quad w(TX_{h+1}).$$

Or ces fermetures sont respectivement identiques à

$$k+T\overline{k[X_1, X_2, \dots, X_j, T]},$$
$$k+T\overline{k[X_1, X_2, \dots, X_{i-1}, T]},$$
$$k+T\overline{k[X_1, X_2, \dots, X_{i-1}, X_{i+1}, \dots, X_h, T]},$$

où les symboles surlignés désignent toujours les fermetures canoniques des anneaux correspondants. Il suffit donc de montrer que les fermetures canoniques des anneaux $k[X_1, X_2, \dots, X_j, T]$, $k[X_1, X_2, \dots, X_{i-1}, T]$, $k[X_1, X_2, \dots, X_{i-1}, X_{i+1}, \dots, X_h, T]$ ne contienent pas d'éléments d'ordres respectifs $w(X_{j+1})$, $w(X_{i+1})$, $w(X_{h+1})$. Or le fait que la fermeture canonique de $k[X_1, X_2, \dots, X_j, T]$ pour $j < i-1$ ne contient pas d'éléments d'ordre $w(X_{j+1})$ est impliqué par la définition de i. Si l'anneau

$$k[X_1, X_2, \dots, X_{i-1}, T]$$

contenait un élément d'ordre $w(X_{i+1})$ ou l'anneau

$$k[X_1, X_2, \dots, X_{i-1}, X_{i+1}, \dots, X_h, T]$$

un élément d'ordre $w(X_{h+1})$, la fermeture canonique de l'un des anneaux

$$\begin{array}{ll} k[X_1, X_2, \dots, X_{i-1}, X_{i+2}, \dots, X_m, T], & \\ k[X_1, X_2, \dots, X_{i-1}, X_{i+1}, \dots, X_h, X_{h+2}, \dots, X_m, T], & \text{pour } h < m-1, \\ k[X_1, X_2, \dots, X_{i-1}, X_{i+1}, \dots, X_{m-1}, T], & \text{pour } h = m-1, \end{array}$$

contiendrait un système d'éléments d'ordre respectifs ${}^*\chi_1$, ${}^*\chi_2$, …, ${}^*\chi_m$ et par conséquent une base de ${}^*H$. Ce qui impliquerait l'existence d'un système de générateurs de ${}^*H$ contenant $m-1$ éléments seulement, contrairement à ce qui a été établi plus haut (voir § 6).

Les caractères de base de $k+T\,{}^*H$ sont donc

$$w(T),\ w(T)+{}^*\chi_1,\ w(T)+{}^*\chi_2,\ \dots,\ w(T)+{}^*\chi_{i-1},\ w(T)+{}^*\chi_{i+1},\ \dots,\ w(T)+{}^*\chi_m.$$

Comme les caractères de base de $k+T\,{}^*H$ ne dépendent pas du choix des éléments $X_1, X_2, \dots, X_m$, le nombre ${}^*\chi_i$ ne dépend que de T et de ${}^*H$ seulement. Nous allons le désigner par ${}^*\chi_i = {}^*\chi(T, {}^*H)$.

D'une manière analogue les caractères $k+T{}^*H$ se déduisent de ceux de ${}^*H$ par les expressions

$$w(T),\ \chi_1+w(T),\ \ldots,\ \chi_l+w(T),\quad \text{pour}\quad w(T)\neq\chi_1,\ \chi_2,\ \ldots,\ \chi_l,$$

et par

$$w(T),\ \chi_1+w(T),\ \ldots,\ \chi_{j-1}+w(T),\ \chi_{j+1}+w(T),\ \ldots,\ \chi_l+w(T),\ \text{pour } w(T)=\chi_j,$$

en désignant par $\chi_1, \chi_2, \ldots, \chi_l$ les caractères de ${}^*H$.

En particulier dans le cas où tous les caractères de ${}^*H$ sont aussi ses caractères de base, tous les caractères de $k+T{}^*H$ seront aussi ses caractères de base si $w(T)$ est un caractère de ${}^*H$ ou si $\chi(T, {}^*H)$ est infini.

Remarque. ρ étant un élément quelconque de $W({}^*H)$, on peut toujours choisir l'élément T d'ordre $w(T)=\rho$ de ${}^*H$, de manière que $\chi(T, {}^*H)$ soit égal à l'un quelconque de ceux des nombres ${}^*\chi_1, {}^*\chi_2, \ldots, {}^*\chi_m, {}^*\chi_{m+1}=\infty$ qui dépassent ρ, pourvu que ρ soit différent des nombres ${}^*\chi_i$. Supposons en effet que ρ soit distinct des nombres ${}^*\chi_1<{}^*\chi_2<\ldots<{}^*\chi_m$ et soit ${}^*\chi_l$ celui de ces derniers pour lequel on a ${}^*\chi_l<\rho<{}^*\chi_{l+1}$. Si $X_1, X_2, \ldots, X_m$ est une base de ${}^*H$, la fermeture canonique de $k[X_1, X_2, \ldots, X_l]$ contient, par définition des éléments d'ordre ρ. Soit T' l'un de ces éléments, et posons $T=T'+X_h$ (avec $h>l$, $X_{m+1}=0$). Pour $l\leqslant j<h-1$ les ensembles

$$\overline{k[X_1, X_2, \ldots, X_j, T]} \bmod t^{*\chi_h},\quad \overline{k[X_1, X_2, \ldots, X_j, T']} \bmod t^{*\chi_h},$$
$$\overline{k[X_1, X_2, \ldots, X_j]} \bmod t^{*\chi_h}$$

étant identique, l'anneau $\overline{k[X_1, X_2, \ldots, X_j, T]}$ n'a pas d'éléments d'ordre $w(X_{j+1})={}^*\chi_{j+1}$. Pour $j<l$, $\rho=w(T)$ étant plus grand que ${}^*\chi_{j+1}$, les ensembles

$$\overline{k[X_1, X_2, \ldots, X_j, T]} \bmod t^{*\chi_{j+1}+1},\quad \overline{k[X_1, X_2, \ldots, X_j]} \bmod t^{*\chi_{j+1}+1}$$

sont identiques et par conséquent $\overline{k[X_1, X_2, \ldots, X_j, T]}$ ne contient pas d'éléments d'ordre $w(X_{j+1})$. Par contre l'anneau

$$\overline{k[X_1, X_2, \ldots, X_{h-1}, T]},$$

qui contient l'élément T', contient aussi l'élément $T-T'=X_h$. On a donc $\chi(T, {}^*H)={}^*\chi_h$.

Considérons maintenant un semi-groupe canonique quelconque

$${}^*G={}^*G_0=\{0, \nu_1, \nu_1+\nu_2, \ldots, \nu_1+\nu_2+\ldots+\nu_{N-1}+\mathfrak{G}\nu\}\quad (\nu_{N-1}\neq\nu).$$

Le semi groupe

$${}^*G_{N-1}=\{\mathfrak{G}\nu\}$$

a visiblement un seul caractère qui est $\chi_1^{(N-1)}=\nu$. Les caractères de

$${}^*G_{N-2}=\{0, \nu_{N-1}+\mathfrak{G}\nu\}$$

sont alors, d'après la règle indiquée plus haut,

$$\chi_1^{(N-2)}=\nu_{N-1},\quad \chi_2^{(N-2)}=\nu_{N-1}+\nu.$$

Les caractères de ${}^*G_{N-3}$ se déduisent des précédents, d'après la même règle:

$$\left.\begin{array}{l}\chi_1^{(N-3)} = \nu_{N-2}, \quad \chi_2^{(N-3)} = \nu_{N-2}+\nu_{N-1}, \\ \qquad\qquad\qquad \chi_3^{(N-3)} = \nu_{N-2}+\nu_{N-1}+\nu,\end{array}\right\} \quad \text{pour} \quad \nu_{N-2} > \nu_{N-1}+\nu,$$

$$\chi_1^{(N-3)} = \nu_{N-2}, \quad \chi_2^{(N-3)} = \nu_{N-2}+\nu_{N-1}, \qquad \text{pour} \quad \nu_{N-2} = \nu_{N-1}+\nu,$$

$$\chi_1^{(N-3)} = \nu_{N-2}, \quad \chi_2^{(N-3)} = \nu_{N-2}+\nu_{N-1}+\nu, \qquad \text{pour} \quad \nu_{N-2} = \nu_{N-1}.$$

On en déduit successivement, en appliquant toujours la même règle, les caractères

$$\chi_1^{(i)}, \chi_2^{(i)}, \ldots, \chi_{l_i}^{(i)}$$

de tous les semi-groupes ${}^*G_i = \{0, \nu_{i+1}+G_{i+1}\}$.

Posons maintenant

$${}^*l_{N-1} = 1, \quad {}^*\chi_1^{(N-1)} = \nu;$$

$${}^*l_{N-2} = 2, \quad {}^*\chi_1^{(N-2)} = \nu_{N-1}, \quad {}^*\chi_2^{(N-2)} = \nu_{N-1}+\nu;$$

et d'une manière générale

$$\begin{array}{l}{}^*l_{i-1} = {}^*l_i, \quad {}^*\chi_1^{(i-1)} = \nu_i, \ {}^*\chi_2^{(i-1)} = \nu_i + {}^*\chi_1^{(i)}, \ \ldots, \ {}^*\chi_{h_i}^{(i-1)} = \nu_i + {}^*\chi_{{}^*h_i-1}^{(i)}, \\ \qquad {}^*\chi_{{}^*h_i+1}^{(i-1)} = \nu_i + {}^*\chi_{{}^*h_i+1}^{(i)}, \ \ldots, \ {}^*\chi_{{}^*l_{i-1}}^{(i-1)} = \nu_i + {}^*\chi_{{}^*l_i}^{(i)}, \quad \text{pour} \quad {}^*h_i \leqslant {}^*l_i,\end{array}$$

$$\begin{array}{l}{}^*l_{i-1} = {}^*l_i+1, \ {}^*\chi_1^{(i-1)} = \nu_i, \ {}^*\chi_2^{(i-1)} = \nu_i + {}^*\chi_1^{(i)}, \ \ldots, \ {}^*\chi_h^{(i-1)} = \nu_i + {}^*\chi_{h-1}^{(i)}, \\ \qquad {}^*\chi_{h+1}^{(i-1)} = \nu_i + {}^*\chi_h^{(i)}, \ \ldots, \ {}^*\chi_{{}^*l_{i-1}}^{(i-1)} = \nu_i + {}^*\chi_{{}^*l_i}^{(i)}, \quad \text{pour} \quad {}^*h_i = {}^*l_i+1,\end{array}$$

où ${}^*h_i$ est l'un quelconque des entiers positifs $h \leqslant {}^*l_i+1$ pour lesquels on a $\nu_i < {}^*\chi_h^{(i)}$ avec ${}^*\chi_{{}^*l_i+1}^{(i)} = \infty$, si $\nu_i \neq {}^*\chi_1^{(i)}, \ldots, {}^*\chi_{{}^*l_i}^{(i)}$; sinon ${}^*\chi_{{}^*h_i}^{(i)}$ est celui des nombres ${}^*\chi_1^{(i)}, {}^*\chi_2^{(i)}, \ldots, {}^*\chi_{{}^*l_i}^{(i)}$ qui est égal à ν_i.

De la remarque précédente et des considérations qui la précèdent résulte immédiatement qu'on peut toujours choisir les éléments $T_i \in {}^*H_i$ de manière que les caractères et les caractères de base des anneaux

$$\begin{array}{ll}{}^*H_{N-1} = k[T], & w(T) = \nu, \\ {}^*H_{N-2} = k + {}^*H_{N-1}T_{N-1}, & w(T_{N-1}) = \nu_{N-1}, \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots \\ {}^*H_{i-1} = k + {}^*H_i T_i, & w(T_i) = \nu_i, \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots \\ {}^*H = {}^*H_0 = k + {}^*H_1 T_1, & w(T_1) = \nu_1\end{array}$$

soient respectivement

(Les caractères)	(Les caractères de bases)
$\chi_1^{(N-1)}$;	${}^*\chi_1^{(N-1)}$;
$\chi_1^{(N-2)}, \chi_2^{(N-2)}$;	${}^*\chi_1^{(N-2)}, {}^*\chi_2^{(N-2)}$;
$\ldots\ldots\ldots$	$\ldots\ldots\ldots$
$\chi_1^{(i-1)}, \chi_2^{(i-1)}, \ldots, \chi_{l_{i-1}}^{(i-1)}$;	${}^*\chi_1^{(i-1)}, {}^*\chi_2^{(i-1)}, \ldots, {}^*\chi_{{}^*l_{i-1}}^{(i-1)}$;
$\ldots\ldots\ldots$	$\ldots\ldots\ldots$
$\chi_1^{(0)}, \chi_2^{(0)}, \ldots, \ldots, \chi_{l_0}^{(0)}$;	${}^*\chi_1^{(0)}, {}^*\chi_2^{(0)}, \ldots, \ldots, {}^*\chi_{{}^*l_0}^{(0)}$.

En particulier les caractères de base de ${}^*H = {}^*H_0$ coincident avec ses caractères si et seulement si l'on a choisi ${}^*h_i = {}^*l_i + 1$, chaque fois qu'on a eu à en faire le choix; le nombre de dimensions de ${}^*H$ serait alors le plus grand des dimensions des anneaux canoniques ayant les mêmes caractères.

THÉORÈME 5. *Si les caractères be base*

$$ {}^*\chi_1^{(N-1)};\ {}^*\chi_1^{(N-2)},\ {}^*\chi_2^{(N-2)};\ \dots;\ {}^*\chi_1^{(i-1)},\ {}^*\chi_2^{(i-1)},\ \dots,\ {}^*\chi_{{}^*l_{i-1}}^{(i-1)};\ \dots;\ {}^*\chi_1^{(0)},\ \dots,\ {}^*\chi_{{}^*l_0}^{(0)} $$

ont été construits en posant

$$ {}^*\chi_{{}^*h_j}^{(j)} = \textit{le plus petit de ceux des nombres } {}^*\chi_1^{(j)}, {}^*\chi_2^{(j)}, \dots, {}^*\chi_{l_j+1}^{(j)} $$
$$ \textit{qui ne sont pas inférieurs à } \nu_j, $$

le nombre de dimensions de l'anneau correspondant ${}^*H$ est le plus petit possible parmis les dimensions des anneaux canoniques ayant les mêmes caractères.

Démonstration. Soient

$$ {}^\dagger\chi_1^{(N-1)};\ {}^\dagger\chi_1^{(N-2)},\ {}^\dagger\chi_2^{(N-2)};\ \dots;\ {}^\dagger\chi_1^{(i-1)},\ {}^\dagger\chi_2^{(i-1)},\ \dots,\ {}^\dagger\chi_{{}^\dagger l_{i-1}}^{(i-1)};\ \dots $$

un autre système de caractères de bases, déduits des mêmes nombres ν_j. Nous avons à montrer que l'on a ${}^\dagger l_i \geqslant {}^*l_i$ $(i = N-1, N-2, \dots, 0)$. ν étant un entier quelconque, désignons par ${}^*l_i(\nu)$ le nombre de ceux des

$$ {}^*\chi_1^{(i)},\ {}^*\chi_2^{(i)},\ \dots,\ {}^*\chi_{{}^*l_i}^{(i)} $$

qui ne sont pas inférieurs à ν. Soit de même ${}^\dagger l_i(\nu)$ le nombre de ceux des ${}^\dagger\chi_1^{(i)}, {}^\dagger\chi_2^{(i)}, \dots, {}^\dagger\chi_{{}^\dagger l_i}^{(i)}$ qui ne sont pas inférieurs à ν. Nous allons démontrer, en même temps, que l'on a

$$ {}^\dagger l_i(\nu) - {}^*l_i(\nu) \leqslant {}^\dagger l_i - {}^*l_i. $$

Les égalités $\quad {}^\dagger l_{N-1} = {}^*l_{N-1} = 1, \quad {}^\dagger l_{N-2} = {}^*l_{N-2} = 2,$

$$ {}^\dagger l_{N-1} - {}^*l_{N-1} = {}^\dagger l_{N-1}(\nu) - {}^*l_{N-1}(\nu) = 0, \quad {}^\dagger l_{N-2} - {}^*l_{N-2} = {}^\dagger l_{N-2}(\nu) - {}^*l_{N-2}(\nu) = 0 $$

étant évidentes, il suffit de déduire de

$$ {}^\dagger l_i \geqslant {}^*l_i, \quad {}^\dagger l_i(\nu) - {}^*l_i(\nu) \leqslant {}^\dagger l_i - {}^*l_i $$

les inégalités $\quad {}^\dagger l_{i-1} \geqslant {}^*l_{i-1}, \quad {}^\dagger l_{i-1}(\nu) - {}^*l_{i-1}(\nu) \leqslant {}^\dagger l_{i-1} - {}^*l_{i-1}.$

Nous distinguons les cas suivants:

(1) ${}^\dagger l_i = {}^*l_i$, ${}^\dagger\chi_{{}^\dagger h_i}^{(i)}$ fini;

(2) ${}^\dagger l_i \geqslant {}^*l_i$, ${}^\dagger\chi_{{}^\dagger h_i}^{(i)}$ infini, ${}^*\chi_{{}^*h_i}^{(i)}$ fini;

(3) ${}^\dagger l_i \geqslant {}^*l_i$, ${}^\dagger\chi_{{}^\dagger h_i}^{(i)}$ infini, ${}^*\chi_{{}^*h_i}^{(i)}$ infini;

(4) ${}^\dagger l_i > {}^*l_i$, ${}^\dagger\chi_{{}^\dagger h_i}^{(i)}$ fini, ${}^*\chi_{{}^*h_i}^{(i)}$ infini;

(5) ${}^\dagger l > {}^*l_i$, ${}^\dagger\chi_{{}^\dagger h_i}^{(i)}$ fini, ${}^*\chi_{{}^*h_i}^{(i)}$ fini.

(1) ${}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$ étant fini, ${}^{\dagger}l_i(\nu_i)$ n'est pas nul. ${}^{\dagger}l_i(\nu_i)-{}^{*}l_i(\nu_i)$ étant inférieur ou égal à ${}^{\dagger}l_i-{}^{*}l_i=0$ le nombre ${}^{*}l_i(\nu_i)$ est non nul. Donc ${}^{*}\chi^{(i)}_{{}^{*}h_i}$ est fini. Il en résulte que l'on a

$$ {}^{\dagger}l_{i-1}={}^{\dagger}l_i={}^{*}l_i={}^{*}l_{i-1}. $$

Montrons qu'on a encore,

$$ {}^{\dagger}l_{i-1}(\nu)-{}^{*}l_{i-1}(\nu)\leqslant {}^{\dagger}l_{i-1}-{}^{*}l_{i-1}\ (=0) $$

pour tous les ν. D'après les formules de récurrence

$$ \begin{aligned} &{}^{\dagger}\chi^{(i-1)}_1=\nu_i,\ {}^{\dagger}\chi^{(i-1)}_2=\nu_i+{}^{\dagger}\chi^{(i)}_1,\ \ldots,\ {}^{\dagger}\chi^{(i-1)}_{{}^{\dagger}h_i}=\nu_i+{}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i-1},\\ &\qquad {}^{\dagger}\chi^{(i-1)}_{{}^{\dagger}h_i+1}=\nu_i+{}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i+1},\ \ldots,\ {}^{\dagger}\chi^{(i-1)}_{{}^{\dagger}l_{i-1}}=\nu_i+{}^{\dagger}\chi^{(i)}_{{}^{\dagger}l_i};\\ &{}^{*}\chi^{(i-1)}_1=\nu_i,\ {}^{*}\chi^{(i-1)}_2=\nu_i+{}^{*}\chi^{(i)}_1,\ \ldots,\ {}^{*}\chi^{(i-1)}_{{}^{*}h_i}=\nu_i+{}^{*}\chi^{(i)}_{{}^{*}h_i-1},\\ &\qquad {}^{*}\chi^{(i-1)}_{{}^{*}h_i+1}=\nu_i+{}^{*}\chi^{(i)}_{{}^{*}h_i+1},\ \ldots,\ {}^{*}\chi^{(i-1)}_{{}^{*}l_{i-1}}=\nu_i+{}^{*}\chi^{(i)}_{{}^{*}l_i}; \end{aligned} $$

il est clair que l'on a

$$ \begin{aligned} {}^{\dagger}l_{i-1}(\nu)&={}^{\dagger}l_i, &&\text{pour}\quad \nu\leqslant\nu_i,\\ {}^{\dagger}l_{i-1}(\nu)&={}^{\dagger}l_i(\nu-\nu_i)-1, &&\text{pour}\quad \nu_i<\nu\leqslant\nu_i+{}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i},\\ {}^{\dagger}l_{i-1}(\nu)&={}^{\dagger}l_i(\nu-\nu_i), &&\text{pour}\quad \nu_i+{}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}<\nu,\\ {}^{*}l_{i-1}(\nu)&={}^{*}l_i, &&\text{pour}\quad \nu\leqslant\nu_i,\\ {}^{*}l_{i-1}(\nu)&={}^{*}l_i(\nu-\nu_i)-1, &&\text{pour}\quad \nu_i<\nu\leqslant\nu_i+{}^{*}\chi^{(i)}_{{}^{*}h_i},\\ {}^{*}l_{i-1}(\nu)&={}^{*}l_i(\nu-\nu_i), &&\text{pour}\quad \nu_i+{}^{*}\chi^{(i)}_{{}^{*}h_i}<\nu. \end{aligned} $$

Il en résulte que, pour

$$ \nu\leqslant\nu_i+\min({}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i},{}^{*}\chi^{(i)}_{{}^{*}h_i})\quad\text{et pour}\quad \nu>\nu_1+\max({}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i},{}^{*}\chi^{(i)}_{{}^{*}h_i}), $$

on a

$$ {}^{\dagger}l_{i-1}(\nu)-{}^{*}l_{i-1}(\nu)={}^{\dagger}l_i(\nu-\nu_i)-{}^{*}l_i(\nu-\nu_i)\leqslant 0. $$

Si ${}^{*}\chi^{(i)}_{{}^{*}h_i}<{}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$, on aura $\min({}^{*}\chi^{(i)}_{{}^{*}h_i},{}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i})={}^{*}\chi^{(i)}_{{}^{*}h_i}$, $\max({}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i},{}^{*}\chi^{(i)}_{{}^{*}h_i})={}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$ et

$$ {}^{\dagger}l_{i-1}(\nu)-{}^{*}l_{i-1}(\nu)={}^{\dagger}l_i(\nu-\nu_i)-{}^{*}l_i(\nu-\nu_i)-1<0 $$
$$ (\text{pour } \nu_i+{}^{*}\chi^{(i)}_{{}^{*}h_i}<\nu\leqslant\nu_i+{}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}). $$

Si ${}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}<{}^{*}\chi^{(i)}_{{}^{*}h_i}$, ν_i étant inférieur ou égal à ${}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$, il n'existe aucun nombre ${}^{*}\chi^{(i)}_j$ compris entre ${}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$ et ${}^{*}\chi^{(i)}_{{}^{*}h_i}$. On a donc pour $\nu_i+{}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}<\nu\leqslant\nu_i+{}^{*}\chi^{(i)}_{{}^{*}h_i}$

$$ \begin{aligned} {}^{\dagger}l_{i-1}(\nu)-{}^{*}l_{i-1}(\nu)&={}^{\dagger}l_i(\nu-\nu_i)-{}^{*}l_i(\nu-\nu_i)+1\\ &={}^{\dagger}l_i(\nu-\nu_i)-{}^{*}l_i({}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i})+1\\ &<{}^{\dagger}l_i({}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i})-{}^{*}l_i({}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i})+1\leqslant 1. \end{aligned} $$

(2) ${}^{\dagger}l_i\geqslant{}^{*}l_i$, ${}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$ infini, ${}^{*}\chi^{(i)}_{{}^{*}h_i}$ fini. Dans ce cas on aura visiblement ${}^{\dagger}l_{i-1}={}^{\dagger}l_i+1$, ${}^{*}l_{i-1}={}^{*}l_i$, et par conséquent ${}^{\dagger}l_{i-1}>{}^{*}l_{i-1}$. Des formules de

récurrence qui fournissent les nombres ${}^{\dagger}\chi_j^{(i-1)}$ et ${}^{*}\chi_j^{(i-1)}$ résulte en outre que l'on a

$$\begin{aligned}
{}^{\dagger}l_{i-1}(\nu) &= {}^{\dagger}l_i+1, && \text{pour} \quad \nu\leqslant\nu_i,\\
{}^{\dagger}l_{i-1}(\nu) &= {}^{\dagger}l_i(\nu-\nu_i), && \text{pour} \quad \nu_i<\nu,\\
{}^{*}l_{i-1}(\nu) &= {}^{*}l_i, && \text{pour} \quad \nu\leqslant\nu_i,\\
{}^{*}l_{i-1}(\nu) &= {}^{*}l_i(\nu-\nu_i)-1, && \text{pour} \quad \nu_i<\nu\leqslant\nu_i+{}^{*}\chi_{*h_i}^{(i)},\\
{}^{*}l_{i-1}(\nu) &= {}^{*}l_i(\nu-\nu_i), && \text{pour} \quad \nu_i+{}^{*}\chi_{*h_i}^{(i)}<\nu,
\end{aligned}$$

dont on déduit facilement l'inégalité

$${}^{\dagger}l_{i-1}(\nu)-{}^{*}l_{i-1}(\nu)\leqslant {}^{\dagger}l_{i-1}-{}^{*}l_{i-1}\leqslant 1.$$

(3) Pour ${}^{\dagger}l_i\geqslant {}^{*}l_i$, ${}^{\dagger}\chi_{\dagger h_i}^{(i)}$ infini, ${}^{*}\chi_{*h_i}^{(i)}$ infini, il est clair que l'on a ${}^{\dagger}l_{i-1}={}^{\dagger}l_i+1$, ${}^{*}l_{i-1}={}^{*}l_i+1$ et par conséquent ${}^{\dagger}l_{i-1}\geqslant {}^{*}l_{i-1}$. Des formules de récurrence qui fournissent les nombres ${}^{\dagger}\chi_j^{(i-1)}$, ${}^{*}\chi_j^{(i-1)}$ résultent en outre

$$\begin{aligned}
{}^{\dagger}l_{i-1}(\nu) &= {}^{\dagger}l_i+1, & {}^{*}l_{i-1}(\nu) &= {}^{*}l_i+1, && \text{pour} \quad \nu\leqslant\nu_i,\\
{}^{\dagger}l_{i-1}(\nu) &= {}^{\dagger}l_i(\nu-\nu_i), & {}^{*}l_{i-1}(\nu) &= {}^{*}l_i(\nu-\nu_i), && \text{pour} \quad \nu_i<\nu,
\end{aligned}$$

d'où l'on tire $${}^{\dagger}l_{i-1}(\nu)-{}^{*}l_{i-1}(\nu)\leqslant {}^{\dagger}l_{i-1}-{}^{*}l_{i-1}.$$

(4) ${}^{\dagger}l_i>{}^{*}l_i$, ${}^{\dagger}\chi_{\dagger h_i}^{(i)}$ fini, ${}^{*}\chi_{*h_i}^{(i)}$ infini. On aura alors

$${}^{\dagger}l_{i-1}={}^{\dagger}l_i, \quad {}^{*}l_{i-1}=l_i+1,$$

$$\begin{aligned}
{}^{\dagger}l_{i-1}(\nu) &= {}^{\dagger}l_i, & {}^{*}l_{i-1}(\nu) &= {}^{*}l_i+1, && \text{pour} \quad \nu\leqslant\nu_i,\\
{}^{\dagger}l_{i-1}(\nu) &= {}^{\dagger}l_i(\nu-\nu_i)-1, & {}^{*}l_{i-1}(\nu) &= {}^{*}l_i(\nu-\nu_i), && \text{pour} \quad \nu_i<\nu\leqslant\nu_1+{}^{\dagger}\chi_{\dagger h_i}^{(i)},\\
{}^{\dagger}l_{i-1}(\nu) &= {}^{\dagger}l_i(\nu-\nu_i), & {}^{*}l_{i-1}(\nu) &= {}^{*}l_i(\nu-\nu_i), && \text{pour} \quad \nu_i+{}^{\dagger}\chi_{\dagger h_i}^{(i)}<\nu,
\end{aligned}$$

et par conséquent

$$\begin{aligned}
&{}^{\dagger}l_{i-1}\geqslant {}^{*}l_{i-1},\\
&{}^{\dagger}l_{i-1}(\nu)-{}^{*}l_{i-1}(\nu) = {}^{\dagger}l_{i-1}-{}^{*}l_{i-1}, && \text{pour} \quad \nu\leqslant\nu_i,\\
&{}^{\dagger}l_{i-1}(\nu)-{}^{*}l_{i-1}(\nu) = {}^{\dagger}l_i(\nu-\nu_i)-{}^{*}l_i(\nu-\nu_i)-1, && \text{pour} \quad \nu_i<\nu\leqslant\nu_i+{}^{\dagger}\chi_{\dagger h_i}^{(i)}.\\
&\qquad\leqslant {}^{\dagger}l_{i-1}-{}^{*}l_{i-1},
\end{aligned}$$

${}^{\dagger}\chi_{\dagger h_i}^{(i)}$ étant fini mais supérieur ou égal ν_i tandis que ${}^{*}\chi_{*h_i}^{(i)}$ est infini, on aura

$${}^{*}l_{i-1}(\nu)={}^{*}l_i(\nu-\nu_i)=0, \quad \text{pour} \quad \nu\geqslant\nu_i+{}^{\dagger}\chi_{\dagger h_i}^{(i)},$$

et par conséquent

$$\begin{aligned}
{}^{\dagger}l_{i-1}(\nu)-{}^{*}l_{i-1}(\nu) = {}^{\dagger}l_{i-1}(\nu) &\leqslant {}^{\dagger}l_{i-1}(\nu_i+{}^{\dagger}\chi_{\dagger h_i}^{(i)}), \quad \text{pour} \quad \nu_i+{}^{\dagger}\chi_{\dagger h_i}^{(i)}<\nu,\\
&\leqslant {}^{\dagger}l_{i-1}(\nu_i+{}^{\dagger}\chi_{\dagger h_i}^{(i)})-{}^{*}l_{i-1}(\nu_i+{}^{\dagger}\chi_{\dagger h_i}^{(i)})\\
&\leqslant {}^{\dagger}l_{i-1}-{}^{*}l_{i-1}.
\end{aligned}$$

(5) ${}^{\dagger}l_i > {}^{*}l_i$, ${}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$ fini, ${}^{*}\chi^{(i)}_{{}^{*}h_i}$ fini. Dans ce cas les inégalités

$$ {}^{\dagger}l_{i-1} \geqslant {}^{*}l_i, \quad {}^{\dagger}l_{i-1}(\nu) - {}^{*}l_{i-1}(\nu) \leqslant {}^{\dagger}l_{i-1} - {}^{*}l_{i-1} $$

se déduisent de ${}^{\dagger}l_i \geqslant {}^{*}l_{i-1}$, ${}^{\dagger}l_i(\nu) - {}^{*}l_i(\nu) < {}^{\dagger}l_i - {}^{*}l_i$ exactement de la même manière que dans le cas (1).

l_0 étant le nombre des caractères de

$$ {}^{*}G = \{0, \nu_1, \nu_1 + \nu_2, \ldots, \nu_1 + \nu_2 + \ldots + \nu_{N-1} + \mathfrak{G}\nu\}, $$

${}^{*}l_0$ le nombre des caractères de base ${}^{*}\chi_1^{(0)}, {}^{*}\chi_2^{(0)}, \ldots$ déduits de ${}^{*}G$ conformément à l'énoncé du théorème 5, nous venons de voir que le nombre des caractères de base d'un anneau canonique ${}^{\dagger}H$, tel que $W({}^{\dagger}H) = {}^{*}G$, est compris entre ${}^{*}l_0$ et l_0. Inversement on a

THÉORÈME 6. *n étant un entier quelconque compris entre ${}^{*}l_0$ et l_0 il existe un anneau canonique de dimension n dont les caractères sont ceux de ${}^{*}G$.*

Démonstration. Il suffit de déduire de l'existence d'un anneau canonique de dimension $n-1$ celle d'un anneau canonique de dimension n. Supposons donc qu'il existe une système de caractère de base

$$ {}^{\dagger}\chi_1^{(N-1)};\ {}^{\dagger}\chi_1^{(N-2)}, {}^{\dagger}\chi_2^{(N-2)};\ \ldots;\ {}^{\dagger}\chi_1^{(0)}, {}^{\dagger}\chi_2^{(0)}, \ldots, {}^{\dagger}\chi^{(0)}_{{}^{\dagger}l_0} $$

déduits de ${}^{*}G$ suivant les règles indiquées plus haut et que l'on a ${}^{\dagger}l_0 = n-1$. Le nombre ${}^{\dagger}l_0$ étant plus petit que l_0, il existe des entiers i pour lesquels ${}^{\dagger}\chi^{(i)}_{{}^{\dagger}h_i}$ est fini sans être égal à ν_i; soit μ le plus petit de ces entiers. Nous pouvons supposer que le système de caractères de base

$$ {}^{\dagger}\chi_1^{(N-1)};\ {}^{\dagger}\chi_1^{(N-2)}, {}^{\dagger}\chi_2^{(N-2)};\ \ldots;\ {}^{\dagger}\chi_1^{(0)}, \ldots, {}^{\dagger}\chi^{(0)}_{{}^{\dagger}l_0} $$

ait été choisi parmis les systèmes qui remplissent les mêmes conditions, de telle manière que *μ soit le plus grand possible.* Cela étant, posons

$$
\begin{array}{ll}
{}^{\dagger}l'_{N-1} = {}^{\dagger}l_{N-1} = 1, & {}^{\dagger}\chi_1'^{(N-1)} = {}^{\dagger}\chi_1^{(N-1)}, \\
{}^{\dagger}l'_{N-2} = {}^{\dagger}l_{N-2} = 2, & {}^{\dagger}\chi_1'^{(N-2)} = {}^{\dagger}\chi_1^{(N-2)},\ {}^{\dagger}\chi_2'^{(N-2)} = {}^{\dagger}\chi_2^{(N-2)}, \\
\cdots\cdots & \cdots\cdots\cdots\cdots \\
{}^{\dagger}l'_{\mu} = {}^{\dagger}l_{\mu}, & {}^{\dagger}\chi_1'^{(\mu)} = {}^{\dagger}\chi_1^{(\mu)},\ {}^{\dagger}\chi_2'^{(\mu)} = {}^{\dagger}\chi_2^{(\mu)}, \ldots, {}^{\dagger}\chi'^{(\mu)}_{{}^{\dagger}l'_\mu} = {}^{\dagger}\chi^{(\mu)}_{{}^{\dagger}l_\mu}, \\
{}^{\dagger}l'_{\mu-1} = {}^{\dagger}l_{\mu-1} + 1, & {}^{\dagger}\chi_1'^{(\mu-1)} = {}^{\dagger}\chi_1^{(\mu-1)},\ {}^{\dagger}\chi_2'^{(\mu-1)} = {}^{\dagger}\chi_2^{(\mu-1)}, \ldots, {}^{\dagger}\chi'^{(\mu-1)}_{{}^{\dagger}h_\mu} = {}^{\dagger}\chi^{(\mu-1)}_{{}^{\dagger}h_\mu}, \\
& {}^{\dagger}\chi'^{(\mu-1)}_{{}^{\dagger}h_\mu+1} = \nu_\mu + {}^{\dagger}\chi^{(\mu)}_{{}^{\dagger}h_\mu},\ {}^{\dagger}\chi'^{(\mu-1)}_{{}^{\dagger}h_\mu+2} = {}^{\dagger}\chi^{(\mu-1)}_{{}^{\dagger}h_\mu+1}, \ldots
\end{array}
$$

avec ${}^{\dagger}\chi'^{(\mu)}_{{}^{\dagger}h'_\mu} = \infty$. L'ensemble ${}^{\dagger}\chi_1'^{(\mu-1)}, {}^{\dagger}\chi_2'^{(\mu-1)}, \ldots, {}^{\dagger}\chi'^{(\mu-1)}_{{}^{\dagger}l'_{\mu-1}}$ est visiblement constitué par l'ensemble ${}^{\dagger}\chi_1^{(\mu-1)}, {}^{\dagger}\chi_2^{(\mu-1)}, \ldots, {}^{\dagger}\chi^{(\mu-1)}_{{}^{\dagger}l_{\mu-1}}$ et par le nombre ${}^{\dagger}\chi'^{(\mu-1)}_{{}^{\dagger}h_\mu+1} = \nu_\mu + {}^{\dagger}\chi^{(\mu)}_{{}^{\dagger}h_\mu}$. Le nombre $\nu_{\mu-1}$ ne peut pas être égal à ${}^{\dagger}\chi'^{(\mu-1)}_{{}^{\dagger}h_\mu+1}$. Car sinon on aurait ${}^{\dagger}\chi^{(\mu-1)}_{{}^{\dagger}h_{\mu-1}} = \infty$, ${}^{\dagger}\chi'^{(\mu-1)}_{{}^{\dagger}h'_{\mu-1}} = {}^{\dagger}\chi'^{(\mu-1)}_{{}^{\dagger}h_\mu+1}$ et le système correspondant

$$ {}^{\dagger}\chi_1'^{(\mu-2)} = \nu_{\mu-1},\ {}^{\dagger}\chi_2'^{(\mu-2)} = {}^{\dagger}\chi_1'^{(\mu-1)} + \nu_{\mu-1}, \ldots, $$
$$ {}^{\dagger}\chi'^{(\mu-2)}_{{}^{\dagger}h_\mu+1} = {}^{\dagger}\chi'^{(\mu-1)}_{{}^{\dagger}h_\mu} + \nu_{\mu-1},\ {}^{\dagger}\chi'^{(\mu-2)}_{{}^{\dagger}h_\mu+2} = {}^{\dagger}\chi'^{(\mu-1)}_{{}^{\dagger}h_\mu+2} + \nu_{\mu-1}, \ldots $$

serait composé des mêmes nombres que le système

$$^{\dagger}\chi_1^{(\mu-2)} = \nu_{\mu-1},\ ^{\dagger}\chi_2^{(\mu-2)} = {}^{\dagger}\chi_1^{(\mu-1)} + \nu_{\mu-1},\ \ldots,$$
$$^{\dagger}\chi_{^{\dagger}h_\mu+1}^{(\mu-2)} = {}^{\dagger}\chi_{^{\dagger}h_\mu}^{(\mu-1)} + \nu_{\mu-1},\ ^{\dagger}\chi_{^{\dagger}h_\mu+2}^{(\mu-2)} = {}^{\dagger}\chi_{^{\dagger}h_\mu+1}^{(\mu-1)} + \nu_{\mu-1},\ \ldots.$$

Ceci nous permetterait alors de construire, en posant

$$^{\dagger}\chi_1'^{(\mu-3)} = {}^{\dagger}\chi_1^{(\mu-3)},\ \ldots,\ ^{\dagger}\chi_{^{\dagger}l_{\mu-3}}'^{(\mu-3)} = {}^{\dagger}\chi_{^{\dagger}l_{\mu-3}}^{(\mu-3)};$$
$$^{\dagger}\chi_1'^{(\mu-4)} = {}^{\dagger}\chi_1^{(\mu-4)},\ \ldots,\ ^{\dagger}\chi_{^{\dagger}l_{\mu-4}}'^{(\mu-4)} = {}^{\dagger}\chi_{^{\dagger}l_{\mu-4}}^{(\mu-4)};$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$^{\dagger}\chi_1'^{(0)} = {}^{\dagger}\chi_1^{(0)},\quad \ldots,\quad ^{\dagger}\chi_{^{\dagger}l_0}'^{(0)} = {}^{\dagger}\chi_{^{\dagger}l_0}^{(0)};$$

un système de caractère de base $^{\dagger}\chi_1'^{(N-1)}; \ldots; {}^{\dagger}\chi_1'^{(0)}, {}^{\dagger}\chi_2'^{(0)}, \ldots, {}^{\dagger}\chi_{^{\dagger}l_0}'^{(0)}$ remplissant les mêmes conditions que le système $^{\dagger}\chi_1^{(N-1)}; \ldots; {}^{\dagger}\chi_1^{(0)}, {}^{\dagger}\chi_2^{(0)}, \ldots, {}^{\dagger}\chi_{^{\dagger}l_0}^{(0)}$ sauf que les $^{\dagger}\chi_{^{\dagger}h_i}'^{(i)}$ sont infinis ou égaux à ν_i pour $i = 1, 2, \ldots, \mu-1$ et μ. Donc si $^{\dagger}\chi_{^{\dagger}h_{\mu'}}^{(\mu')}$ est le premier des nombres $^{\dagger}\chi_{^{\dagger}h_i}'^{(i)}$ qui n'est ni infini ni égal à $\nu_{\mu'}$, on aurait $\mu' > \mu$, contrairement au choix du système

$$^{\dagger}\chi_1^{(N-1)};\ ^{\dagger}\chi_1^{(N-2)},\ ^{\dagger}\chi_2^{(N-2)};\ \ldots;\ ^{\dagger}\chi_1^{(0)},\ ^{\dagger}\chi_2^{(0)},\ \ldots,\ ^{\dagger}\chi_{^{\dagger}l_0}^{(0)}.$$

$\nu_{\mu-1}$ étant ainsi différent de $^{\dagger}\chi_{^{\dagger}h_\mu+1}'^{(\mu-1)}$ qui est le seul des nombres $^{\dagger}\chi_i'^{(\mu-1)}$ qui ne soit pas égal à un nombre $^{\dagger}\chi_i^{(\mu-1)}$ on peut poser $^{\dagger}\chi_{^{\dagger}h_{\mu-1}}'^{(\mu-1)} = {}^{\dagger}\chi_{^{\dagger}h_{\mu-1}}^{(\mu-1)}$ et considérer l'ensemble

$$^{\dagger}\chi_1'^{(\mu-2)} = \nu_{\mu-1},\ ^{\dagger}\chi_2'^{(\mu-2)} = \nu_{\mu-1} + {}^{\dagger}\chi_1'^{(\mu-1)},\ \ldots$$

qui est alors composé des nombres

$$^{\dagger}\chi_1^{(\mu-2)},\ ^{\dagger}\chi_2^{(\mu-2)},\ \ldots,\ ^{\dagger}\chi_{^{\dagger}l_{\mu-2}}^{(\mu-2)}$$

et de $^{\dagger}\chi_{^{\dagger}h_\mu+1}'^{(\mu-1)} + \nu_{\mu-1} = {}^{\dagger}\chi_{^{\dagger}h_\mu}^{(\mu)} + \nu_\mu + \nu_{\mu-1}$. On montre de même que $\nu_{\mu-2}$ est distinct de $^{\dagger}\chi_{^{\dagger}h_\mu+1}'^{(\mu-1)} + \nu_{\mu-1}$; ce qui nous permet de poser $^{\dagger}\chi_{^{\dagger}h_{\mu-2}}'^{(\mu-2)} = {}^{\dagger}\chi_{^{\dagger}h_{\mu-2}}^{(\mu-2)}$. En continuant de cette manière on construit finalement le système

$$^{\dagger}\chi_1'^{(0)},\ ^{\dagger}\chi_2'^{(0)},\ \ldots,\ ^{\dagger}\chi_{^{\dagger}l_0}'^{(0)}$$

qui se compose de $\quad ^{\dagger}\chi_1^{(0)},\ ^{\dagger}\chi_2^{(0)},\ \ldots,\ ^{\dagger}\chi_{^{\dagger}l_0}^{(0)}$

et du nombre $^{\dagger}\chi_{^{\dagger}h_\mu}^{(\mu)} + \nu_\mu + \nu_{\mu-1} + \ldots + \nu_1$. On a donc

$$^{\dagger}l_0' = {}^{\dagger}l_0 + 1 = n - 1 + 1 = n.$$

Le tableau ci-contre montre les systèmes de caractères de base qui correspondent au semi-groupe

$$^{*}G = \{0,\ 702,\ 1404,\ 1620,\ 1836,\ 2052,\ 2106,\ 2160,\ 2214,\ 2268,$$
$$2322,\ 2340,\ 2358,\ 2376,\ 2383,\ 2390,\ 2394,\ 2397 + \mathfrak{G}\};$$

la première colonne du tableau étant en même temps le système des caractères de $^{*}G$.

	1ère colonne				2ème colonne			3ème colonne			4ème colonne			5ème colonne	
H_{17}	1				1			1			1			1	
H_{16}	3	4			3	4		3	4		3	4		3	4
H_{15}	4	7			4	7		4	7		4	7		4	7
H_{14}	7	11			7	11		7	11		7	11		7	11
H_{13}	7	18			7	18		7	18		7	18		7	18
H_{12}	18	25			18	25		18	25		18	25		18	25
H_{11}	18	43			18	43		18	43		18	43		18	43
H_{10}	18	61			18	61		18	61		18	61		18	61
H_9	54	72	115		54	72	115	54	72	115	54	72		54	72
H_8	54	126	169		54	126	169	54	126	169	54	126		54	126
H_7	54	180	223		54	180	223	54	180	223	54	180		54	180
H_6	54	234	277		54	234	277	54	234	277	54	234		54	234
H_5	54	288	331		54	288	331	54	288	331	54	288		54	288
H_4	216	270	504	547	216	270	547	216	270	504	216	270	504	216	270
H_3	216	486	720	763	216	486	763	216	486	720	216	486	720	216	486
H_2	216	702	936	979	216	702	979	216	702	936	216	702	936	216	702
H_1	702	918	1638	1681	702	918	1681	702	918	1638	702	918	1638	702	918
H	702	1620	2340	2383	702	1620	2383	702	1620	2340	702	1620	2340	702	1620

Comme exemples d'anneaux H dont les caractères sont 702, 1620, 2340, 2383 on peut citer les suivants:

$$\overline{k[t^{702}, t^{1620}, t^{2340}, t^{2383}]}$$

$$\overline{k[t^{702}(1+t^{72})^3, t^{1620}(1+t^{72})^7, t^{2383}(1+t^{72})^9]}$$

$$\overline{k[t^{702}(1+t^{115})^3, t^{1620}(1+t^{115})^7, t^{2340}(1+t^{115})^9]}$$

$$\overline{k[t^{702}(1+t^7)^{13}, t^{1620}(1+t^7)^{30}, t^{2340}(1+t^7)^{44}]}$$

$$\overline{k[t^{702}(1+t^7)^{13}\,(1+t^{79})^3, t^{1620}(1+t^7)^3\,(1+t^{79})^7]}$$

dont les suites des caractères de bases sont respectivement donnés par les cinq colonnes du tableau ci-dessus.

Remarquons enfin que les caractères de $^*H$ et les caractères de base de $^*H$, $^*H_1$, ..., $^*H_{N-1}$ qui sont, comme nous l'avons vu, des invariants de $^*H$, ne constituent pas un système complet d'invariants. C'est à dire on peut construire des anneaux canoniques $^*H$ et $^*H'$ non transformable l'une dans l'autre par une substitution de la forme

$$t \to t(\alpha_0 + \alpha_1 t + \ldots + \alpha_n t^n + \ldots), \quad (\alpha_0 \neq 0)$$

de manière que les caractères de $^*H$ et de $^*H'$ ainsi que les caractères de base de $^*H$, $^*H_1$, ..., $^*H_{N-1}$ et de $^*H'$, $^*H'_1$, ..., $^*H'_{N-1}$ soient respectivement les mêmes. Soient par exemple

$$^*H = k + kt^{4\nu}(1+t) + kt^{6\nu}(1+t) + kt^{7\nu}(1+t) + k[t]\,t^{8\nu},$$

$$^*H' = k + kt^{4\nu}(1+t+t^2) + kt^{6\nu}(1+t+t^2) + kt^{7\nu}(1+t+t^2) + k[t]\,t^{8\nu}$$

avec $\nu > 2$. Ces anneaux ont les mêmes caractères qui sont

$$4\nu,\ 6\nu,\ 7\nu,\ 8\nu+1.$$

Leurs caractères de base sont également les mêmes:

$$4\nu,\ 6\nu,\ 7\nu.$$

Les anneaux $^*H_1$, $^*H'_1$ étant tous les deux identiques à

$$k + kt^{2\nu} + kt^{3\nu} + k[t]\, t^{4\nu},$$

les caractères de base de $^*H'_1$, $^*H'_2$, $^*H'_3$, $^*H'_4$ sont respectivement les mêmes que ceux de $^*H_1$, $^*H_2$, $^*H_3$, $^*H_4$. Par contre il n'existe aucune substitution de la forme

$$(\alpha) \qquad t \to t(\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \dots)$$

qui transforme $^*H$ en $^*H'$. En effet une telle substitution qui doit transformer $^*H$ en $^*H'$ devrait transformer $^*H_1$ en $^*H'_1$, c.à.d. en lui même. Or, en supposant que 2ν est non divisible par le caractéristique de k, les substitutions de la forme (α), qui transforment l'anneau

$$^*H_1 = k + kt^{2\nu} + kt^{3\nu} + k[t]\, t^{4\nu}$$

en lui même, sont de la forme

$$t \to t(\alpha_0 + \alpha_\nu t^\nu + \alpha_{2\nu} t^{2\nu} + \alpha_{2\nu+1} t^{2\nu+1} + \dots)$$

dont aucune ne transforme l'élément

$$t^{4\nu} + t^{4\nu+1}$$

de $^*H$ en un élément du même ordre de $^*H'$ qui est de la forme

$$\xi_0(t^{4\nu} + t^{4\nu+1} + t^{4\nu+2}) + \xi_1(t^{6\nu} + t^{6\nu+1} + t^{6\nu+2}) + \dots.$$

8. Considérons maintenant une branche algébrique passant par l'origine et défini par

$$Y_1 = Y_1(t),\ Y_2 = Y_2(t),\ \dots,\ Y_n = Y_n(t),$$

où $Y_1(t), Y_2(t), \dots, Y_n(t)$ sont des séries entières en t, dont les termes constants sont nuls. Considérons l'anneau $k[Y_1(t), Y_2(t), \dots, Y_n(t)]$. Nous pouvons admettre que cet anneau contient des éléments de tous les ordres qui dépassent un nombre suffisamment grand (théorème auxiliaire 2).

THÉORÈME 7. *$^*H$ étant la fermeture canonique de $k[Y_1(t), Y_2(t), \dots, Y_n(t)]$ soit $W(^*H) = \{0, \nu_1, \nu_1 + \nu_2, \dots, \nu_1 + \nu_2 + \dots + \nu_{N-1} + \mathfrak{G}\}$. Les ordres de multiplicités des points successif de la branche $Y_1(t), Y_2(t), \dots, Y_n(t)$ sont*

$$\nu_1,\ \nu_2,\ \dots,\ \nu_{N-2},\ \nu_{N-1},\ 1,\ 1,\ \dots.$$

Démonstration. Soit $w(Y_1(t))$ le plus petit des nombres

$$w(Y_1(t)),\ w(Y_2(t)),\ \dots,\ w(Y_n(t)).$$

Le point $O = (0, 0, \ldots, 0)$ est alors un point multiple d'ordre $w(Y_1(t))$. D'autre part il est clair que $w(Y_1(t)) = \nu_1$. Il suffit donc de montrer que les ordres de multiplicité des points successifs $(t = 0)$ de la branche*

$$Y'_1(t) = Y_1(t),\ Y'_2(t) = \frac{Y_2(t)}{Y_1(t)},\ \ldots,\ Y'_n(t) = \frac{Y_n(t)}{Y_1(t)}$$

qu'on déduit de $Y_1(t), Y_2(t), \ldots, Y_n(t)$ en résolvent le point O, sont

$$\nu_2,\ \nu_3,\ \ldots,\ \nu_{N-1},\ 1,\ 1,\ \ldots.$$

Nous transportons l'origine des coordonnés au point $t = 0$ de la branche $Y'_1(t), Y'_2(t), \ldots, Y'_n(t)$, qui devient alors

$$Y'_1(t) - \eta_1,\ Y'_2(t) - \eta_2,\ \ldots,\ Y'_n(t) - \eta_n$$

où $\eta_1, \eta_2, \ldots, \eta_n$ désignent les termes constants des séries $Y'_1(t), Y'_2(t), \ldots, Y'_n(t)$. I_{ν_1} étant l'idéal de $k[Y_1(t), Y_2(t), \ldots, Y_n(t)]$ formé par ses éléments d'ordres supérieurs ou égaux à ν_1, il est évident que

$$[I_{\nu_1}] = k[Y'_1(t) - \eta_1, Y'_2(t) - \eta_2, \ldots, Y'_n(t) - \eta_n].$$

Or nous savons que $\quad {}^*H = k + Y_1(t)\,\overline{[I_{\nu_1}]}$

et que $\quad W(\overline{[I_{\nu_1}]}) = \{0, \nu_2, \nu_2 + \nu_3, \ldots, \nu_2 + \nu_3 + \ldots + \nu_{N-1} + \mathfrak{G}\}.$

Donc l'origine est un point multiple d'ordre ν_2, pour la branche

$$Y'_1(t) - \eta_1,\ Y'_2(t) - \eta_2,\ \ldots,\ Y'_n(t) - \eta_n;$$

autrement dit le plus petit des nombres

$$w(Y'_1(t) - \eta_1),\ w(Y'_2(t) - \eta_2),\ \ldots,\ w(Y'_n(t) - \eta_n)$$

est ν_2. On achève la démonstration du théorème 5 en répétant un nombre quelconque de fois ce raisonnement.

D'après le théorème 3 les nombres $\nu_1, \nu_2, \ldots, \nu_{N-1}, \ldots$ se déduisent des caractères de ${}^*H$ exactement de la même manière que ces nombres, considérés comme ordres de multiplicités de la branche, se déduisent des caractères de Du Val associés à la branche $Y_1(t), Y_2(t), Y_3(t), \ldots, Y_n(t)$. Donc les caractères de Du Val de cette branche sont les mêmes que ceux de $k[Y_1(t), Y_2(t), \ldots, Y_n(t)]$.

Il est évident que si deux branches

$$Y_1(t),\ Y_2(t),\ \ldots,\ Y_n(t); \quad Z_1(t),\ Z_2(t),\ \ldots,\ Z_m(t)$$

passant par l'origine peuvent être transformées l'une dans l'autre par une transformation birationnelle régulière à l'origine, les anneaux

$$k[Y_1(t), Y_2(t), \ldots, Y_n(t)], \quad k[Z_1(t), Z_2(t), \ldots, Z_m(t)]$$

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* Voir P. Du Val, loc. cit. et J. G. Semple, "Singularities of space algebraic curves", *Proc. London Math. Soc.* (2), 44 (1938), 149–174.

sont identiques ou, d'une manière plus précise, transformables l'une dans l'autre par une substitution de la forme $t \to t(\alpha_0+\alpha_1 t+\dots)$, $(\alpha_0 \neq 0)$ et réciproquement. Nous dirons alors que ces deux branches sont régulièrement équivalentes entre elles. Pour deux branches régulièrements équivalentes, les anneaux

$$^*H = \overline{k[Y_1(t), Y_2(t), \dots, Y_n(t)]}, \quad ^*H' = \overline{k[Z_1(t), Z_2(t), \dots, Z_m(t)]}$$

peuvent évidemment être transformés l'un dans l'autre par une substitution de la forme $t \to t(\alpha_0+\alpha_1 t+\dots)$, $(\alpha_0 \neq 0)$; mais de l'identité $^*H = {}^*H'$ on ne peut pas déduire l'identité des

$$k[Y_1(t), Y_2(t), \dots, Y_n(t)], \quad k[Z_1(t), Z_2(t), \dots, Z_m(t)].$$

Nous dirons que les deux branches donnés sont canoniquement équivalentes si l'on a $^*H = {}^*H'$. Deux branches régulièrement équivalentes sont aussi canoniquement équivalentes sans que l'inverse soit nécéssairement vrai. Les caractères de $^*H$ ainsi que les caractères de base de $^*H_1, {}^*H_2, \dots, {}^*H_{N-1}$ sont donc des invariants de la branche $Y_1(t), Y_2(t), \dots, Y_n(t)$ pour l'équivalence canonique et par conséquent pour l'équivalence régulière. Remarquons toutefois que les caractères ainsi que les caractères de base de $^*H$, $^*H_1$, $^*H_2$, $\dots$, $^*H_{N-1}$ ne constituent pas un système complet d'invariants ni pour l'équivalence canonique ni pour l'équivalence régulières; puisque comme nous l'avons vu plus haut ces caractères et ces caractères de base ne suffisent pas à déterminer $^*H$.

Les séries $Y_1(t)$, $Y_2(t)$, $\dots$, $Y_n(t)$ constituent visiblement un système de générateurs de $^*H = \overline{k[Y_1(t), Y_2(t), \dots, Y_n(t)]}$.

A la fin du § 6 nous avons vu comment on pouvait construire tous les systèmes de générateurs de $^*H$ en partant de ses bases. Nous avons vu en particulier que, m étant le nombre de dimension de $^*H$, c.à.d. le nombre de ses caractères de base, tout système de générateurs de $^*H$ contient m éléments qui constituent à eux seuls un système de générateurs de $^*H$. Cela s'exprime géométriquement en disant que si m est le nombre des caractères de base de $k[Y_1(t), Y_2(t), \dots, Y_n(t)]$ l'une des projections de dimensions m de la branche $Y_1(t)$, $Y_2(t)$, $\dots$, $Y_n(t)$ est son équivalent canonique tandis qu' aucune projection de dimensions inférieures à m n'est canoniquement équivalentes à $Y_1(t)$, $Y_2(t)$, $\dots$, $Y_n(t)$.

The University
Istanbul

NOTE ON CAHIT ARF'S "UNE INTERPRÉTATION ALGÉBRIQUE DE LA SUITE DES ORDRES DE MULTIPLICITÉ D'UNE BRANCHE ALGÉBRIQUE"*

By PATRICK DU VAL

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Cahit Arf's results being severely algebraic in form, their geometrical meaning may not, at first sight, be evident to all geometers; it is felt therefore that a word of explanation may not be out of place.

A branch being parametrized as in § 8 of C.A., the orders of the elements of the ring

$$H = k[Y_1(t), \ldots, Y_n(t)]$$

are the possible intersection numbers of all hypersurfaces with the branch; in fact the intersection number of the hypersurface

$$\Sigma a_{i_1 \ldots i_n} Y_1^{i_1} \ldots Y_n^{i_n} = 0$$

is clearly the order of the element

$$\Sigma a_{i_1 \ldots i_n} (Y_1(t))^{i_1} \ldots (Y_n(t))^{i_n}$$

of the ring. If the ring H is canonical, theorem 7 shows that these intersection numbers are all the multiplicity sums of the branch. This accordingly is the characteristic property of a canonical branch. It does not, as one might suppose, follow from this that there exist hypersurfaces passing simply through every number of consecutive points of the branch; for the ring

$$H = k + k[t]\,t^2$$

corresponding to an ordinary cusp

$$Y_1 = t^2, \quad Y_2 = t^3$$

is clearly canonical, whereas there are no curves passing simply through more than the first two points of the branch.

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* *Supra*, 256–287. Referred to as C.A.

Thus the uniqueness of the canonical closure of a ring means that every branch is a projection of sequence, unique save for a transformation regular at the origin.

The number of base characters of the branch clearly gives the minimum space in which there can be a branch canonically equivalent to the given one.

To fix our ideas, let us consider the ring generated by

$$X_1 = t^4, \quad X_2 = t^{10}(1+t^5)$$

considered by C.A. The classical theory of Enriques shows that the points $P_1, P_2, \ldots$ consecutive along this plane branch are of multiplicities

$$4,\ 4,\ 2,\ 2,\ 2,\ 2,\ 1,\ 1,\ \ldots,$$

P_4 and P_8 being satellite and the rest free. Since typical elements of orders

$$0,\ 4,\ 8,\ 10,\ 12,\ 14,\ 16,\ 18,\ 20,\ 22,\ 24,\ 25,\ 26,\ 28,\ 29,\ 30,\ 32,\ 33,\ 34,\ 35$$

are

$$1,\ X_1,\ X_1^2,\ X_2,\ X_1^3,\ X_1X_2,\ X_1^4,\ X_1^2X_2,\ X_1^5,\ X_1^3X_2,\ X_1^6,\ X_2^2-X_1^5,\ X_1^4X_2,$$
$$X_1^7,\ X_1X_2^2-X_1^6,\ X_1^5X_2,\ X_1^8,\ X_1^2X_2^2-X_1^7,\ X_1^6X_2,\ X_2^3-X_1^5X_2,\ \ldots$$

we can write

$$\begin{aligned} H &= k[X_1X_2] \\ &= k+kt^4+kt^8+kt^{10}(1+t^5)+kt^{12}+kt^{14}(1+t^5)+kt^{16}+kt^{18}(1+t^5) \\ &\quad +kt^{20}+kt^{22}(1+t^5)+kt^{24}+kt^{25}(2+t^5)+kt^{26}(1+t^5)+kt^{28}+kt^{29}(2+t^5) \\ &\quad +kt^{30}(1+t^5)+k[t]\,t^{32}, \end{aligned}$$

which is the same thing as

$$\begin{aligned} H = {} & k+kt^4+kt^8+kt^{10}(1+t^5)+kt^{12}+kt^{14}(1+t^5)+kt^{16}+kt^{18}(1+t^5) \\ & +kt^{20}+kt^{22}(1+t^5)+kt^{24}+kt^{25}+kt^{26}(1+t^5)+kt^{28}+kt^{29}+kt^{30}+k[t]\,t^{32}; \end{aligned}$$

we have thus

$$\begin{aligned} \frac{I_4}{X_1} = {} & k+kt^4+kt^6(1+t^5)+kt^8+kt^{10}(1+t^5)+kt^{12}+kt^{14}(1+t^5) \\ & +kt^{16}+kt^{18}(1+t^5)+kt^{20}+kt^{21}+kt^{22}(1+t^5)+kt^{24}+kt^{25}+kt^{26}+k[t]\,t^{28}, \end{aligned}$$

and, as the ring generated by this contains the elements

$$\left(\frac{X_2}{X_1}\right)^2-X_1^3, \quad \left(\frac{X_2}{X_1}\right)^2X_2-X_1^3X_2$$

of order 17, 23 respectively, we have

$$\begin{aligned} H_1 = [I_4] = {} & k+kt^4+kt^6(1+t^5)+kt^8+kt^{10}(1+t^5) \\ & +kt^{12}+kt^{14}(1+t^5)+kt^{16}+kt^{17}+kt^{18}+k[t]\,t^{20}. \end{aligned}$$

By exactly the same method we see that

$$H_2 = k + kt^2(1+t^5) + kt^4 + kt^6 + k[t]\,t^8,$$

and since this is clearly canonical we find that the canonical closure of H_1 is

$$^*H_1 = k + t^4 H_2 = k + kt^4 + kt^6(1+t^5) + kt^8 + kt^{10} + k[t]\,t^{12},$$

and that of H is

$$^*H = k + t^{4*}H_1 = k + kt^4 + kt^8 + kt^{10}(1+t^5) + kt^{12} + kt^{14} + k[t]\,t^{16},$$

of which, as we expect, the characters are, 4, 10, 17, the first, third, and seventh multiplicity sums of the branch. Since

$$^*H = k[X_1, X_2, X_3, X_4],$$

where $X_3 = t^{17}$, $X_4 = t^{19}$, we see that the canonical branch of which the given branch is a projection is in four dimensions. Any projection of this into two dimensions is represented by a ring of the form $k[Y_1, Y_2]$, where Y_1, Y_2 belong to H, and can clearly be chosen to be of orders 4, 10 respectively, i.e. we may take

$$Y_1 = t^4 + a_1 t^{17} + a_2 t^{18} + \ldots, \quad Y_2 = t^{10} + t^{15} + b_1 t^{17} + b_2 t^{18} + \ldots;$$

for different values of the coefficients $a_1, a_2, \ldots, b_1, b_2, \ldots$, these branches are not regularly but only canonically equivalent.

An interesting feature is the apparent unimportance of the term t^{15} in the canonical ring, whereas of course this is of fundamental significance in determining the characters of the plane branch. In fact the canonical ring

$$^*H = k + kt^4 + kt^8 + kt^{10}(1+t^5) + kt^{12} + kt^{14} + k[t]\,t^{16}$$

clearly has the same characters as

$$^*H' = k + kt^4 + kt^8 + kt^{10} + kt^{12} + kt^{14} + k[t]\,t^{16},$$

which is also canonical. This latter, however, cannot be generated by two elements; in fact a base for it must be in some such form as

$$X_1' = t^4, \quad X_2' = t^{10}, \quad X_3' = t^{17};$$

thus of the two canonical branches

$$X_1 = t^4, \quad X_2 = t^{10}(1+t^5), \quad X_3 = t^{17}, \quad X_4 = t^{19}$$

and $$X_1' = t^4, \quad X_2' = t^{10}, \quad X_3' = t^{17}, \quad X_4' = t^{19},$$

both of which have two fourfold followed by four twofold and a succession of simple points, the former can and the latter cannot be projected into

a plane branch with the same multiplicity sequence. In fact a general plane projection of the latter is of the form

$$Y_1' = \frac{t^4+a_1t^{10}+a_2t^{17}}{1+c_1t^4+c_2t^{10}+c_3t^{17}} = t^4+\alpha_1t^8+\alpha_2t^{10}+\alpha_3t^{12}+\alpha_4t^{14}+\alpha_5t^{16}+\alpha_6t^{17}+\ldots,$$

$$Y_2' = \frac{t^{10}+bt^{17}}{1+c_1t^4+c_2t^{10}+c_3t^{17}} = t^{10}+\beta_1t^{12}+\beta_2t^{14}+\beta_3t^{16}+\beta_4t^{17}+\ldots,$$

or in terms of

$$\tau = (Y_1')^{\frac{1}{4}} = t(1+p_1t^4+p_2t^6+p_3t^8+p_4t^{10}+p_5t^{12}+p_6t^{13}+\ldots),$$
$$t = \tau(1+q_1\tau^4+q_2\tau^6+q_3\tau^8+q_4\tau^{10}+q_5\tau^{12}+q_6\tau^{13}+\ldots)$$

and $\quad Y_1' = \tau^4, \quad Y_2' = \tau^{10}+B_1\tau^{12}+B_2\tau^{14}+B_3\tau^{16}+B_4\tau^{17}+\ldots;$

which by Enriques's theory clearly represents a branch with two fourfold followed by not four but five twofold points; the canonical closure of $k[Y_1', Y_2']$ is in fact not $^*H'$ but its canonical subring

$$k+kt^4+kt^8+kt^{10}(1+at^7)+kt^{12}+kt^{14}+kt^{16}+k[t]t^{18},$$

where a is a fixed constant depending on those in the expansions of Y_1', Y_2'. $^*H'$ is, however, the canonical closure of

$$k[X_1', X_2', X_3'],$$

or of the ring representing a general projection of the branch corresponding to $^*H'$ into three dimensions. This canonical branch can accordingly be projected into three but not into two dimensions without changing its characters; in short, whereas the two canonical branches considered both have the characters 4, 10, 17, the base characters of the former are 4, 10, and those of the latter 4, 10, 17.

Projection of any branch from a general point clearly gives one represented by a subring of the ring representing the given branch; if the projection alters the characters (i.e. the multiplicity sequence) of the branch, this means that the canonical closure of the subring is not that of the given ring, but is a subring of the latter; i.e. the multiplicity sums

$$\nu_1', \quad \nu_1'+\nu_2', \quad \nu_1'+\nu_2'+\nu_3', \quad \ldots$$

for the projected branch are a certain selection, not the whole, of the multiplicity sums

$$\nu_1, \quad \nu_1+\nu_2, \quad \nu_1+\nu_2+\nu_3, \quad \ldots$$

of the given branch. Thus we must have for some $i_1, i_2, \ldots$

$$\nu_1' = \sum_{j=1}^{i_1}\nu_j, \quad \nu_2' = \sum_{j=i_1+1}^{i_2}\nu_j, \quad \nu_3' = \sum_{j=i_2+1}^{i_3}\nu_j, \quad \ldots.$$

It seems natural to regard the first point of the projected branch as corresponding to the first i_1 points of the given branch, the next point of the former to the next i_2 points of the latter, and so on. Thus in the projection of the branch

$$k+kt^4+kt^8+kt^{10}+kt^{12}+kt^{14}+k[t]\,t^{16}$$

into a plane we may hold that the seventh and eighth points of the canonical branch (which are its first two simple points) are projected into the same point of the plane, which is accordingly a fifth double point of the plane branch. This is seen to agree with the genesis of a cusp by projection from a point on the tangent, where the double point on the projected branch certainly arises from two consecutive simple points on the original, the second point on the projected branch from the third simple point on the original, and so on.

Thus where a branch, such as

$$X_1' = t^4, \quad X_2' = t^{10}, \quad X_3' = t^{17},$$

cannot be projected into a lower space without altering its characters, this means that every cone (of whatever vertex) which passes through certain of its points inevitably passes through certain others; in the present case every cone which passes through the seventh point passes also (and with the same multiplicity) through the eighth; whereas the branch

$$X_1 = t^4, \quad X_2 = t^{10}(1+t^5), \quad X_3 = t^{17}$$

has not this property.

The canonical rings

$$H_1 = [I_{i_1}], \quad H_2, \quad H_3, \quad \ldots,$$

obtained from a canonical ring H represent the branches obtained by resolving the points of the branch in succession, as is seen in the course of the proof of C.A., theorem 7. Thus the base characters of these rings show the dimensions of the least spaces into which these resolved branches can be projected without altering their characters. For instance, of the branches

$$\left.\begin{aligned} {}^*H &= k+kt^{4\nu}(1+t)+kt^{6\nu}(1+t)+kt^{7\nu}(1+t)+k[t]\,t^{8\nu}, \\ {}^*H' &= k+kt^{4\nu} \qquad\qquad\; +kt^{6\nu}(1+t)+kt^{7\nu}(1+t)+k[t]\,t^{8\nu}, \end{aligned}\right\} \quad (\nu>1)$$

considered in C.A. §7, both are capable of being projected into three dimensions without altering their characters, and each has a 4ν-ple, a 2ν-ple, and two ν-ple followed by simple points. The branches

$${}^*H_1 = k+kt^{2\nu}+kt^{3\nu}+k[t]\,t^{4\nu},$$

$${}^*H_2 = k+kt^{2\nu}(1+t)+kt^{3\nu}(1+t)+k[t]\,t^{4\nu},$$

obtained by resolving the first points of each, have of course the same multiplicity sequence, namely a 2ν-ple, and two ν-ple, followed by simple points; but whereas the former of these cannot be projected into a plane without altering its characters, the latter can. In fact the general plane projection of the former is represented by a ring of the form $k[Y_1, Y_2]$ where

$$Y_1 = t^{2\nu} + at^{4\nu+1} + \ldots, \quad Y_2 = t^{3\nu} + bt^{4\nu+1} + \ldots.$$

By the same method as in the first example the canonical closure of this is found to be

$$k + kt^{2\nu}(1 + at^{2\nu+1} + \ldots) + kt^{3\nu}(1 + bt^{\nu+1} + \ldots) + kt^{4\nu} + k[t]\,t^{5\nu}$$

the characters of which are 2ν, 3ν, $5\nu + 1$, so that the projected branch has a 2ν-fold followed by not two but three ν-ple and a succession of simple points, i.e. the first ν simple points of the branch $^*H_1$ are projected into a single ν-ple point; the same result is obtained by expressing the ring in terms of $\tau = Y_1^{1/2\nu}$, when it takes the form

$$Y_1 = \tau^{2\nu}, \quad Y_2 = \tau^{3\nu} + b\tau^{4\nu+1} + \ldots.$$

By similar methods it can be seen that the projection of either of the original branches $^*H$, $^*H'$ into a plane has one 4ν-ple and two 2ν-ple followed by simple points, i.e. the two ν-ple points are projected into a single 2ν-ple point.

In the same way it can be seen that the branches

(i) $X_1 = t^{702}, \quad X_2 = t^{1620}, \quad X_3 = t^{2340}, \quad X_4 = t^{2383}$,

(ii) $X_1 = t^{702}(1 + t^{72})^3, \quad X_2 = t^{1620}(1 + t^{72})^7, \quad X_3 = t^{2383}(1 + t^{72})^9$,

(iii) $X_1 = t^{702}(1 + t^{115})^3, \quad X_2 = t^{1620}(1 + t^{115})^7, \quad X_3 = t^{2340}(1 + t^{115})^9$,

(iv) $X_1 = t^{702}(1 + t^7)^{13}, \quad X_2 = t^{1620}(1 + t^7)^{30}, \quad X_3 = t^{2340}(1 + t^7)^{44}$,

(v) $X_1 = t^{702}(1 + t^7)^{13}(1 + t^{79})^3, \quad X_2 = t^{1620}(1 + t^7)^3(1 + t^{79})^7$,

considered in C.A. § 7, all of which have the characters

702, 1620, 2340, 2383

and the multiplicity sequence

702 (twice), 216 (three times), 54 (five times),

18 (three times), 7 (twice), 4, 3,

followed by simple points, differ in the minimum space into which their canonical equivalents can be projected without altering their characters. (ii), (iii), (iv), indeed all exist in three dimensions and cannot be projected

into a plane; but whereas the branch obtained by resolving the first five points of (iv) can be projected into a plane without altering its characters, those obtained from (ii) and (iii) in the same way cannot. The difference between these latter is not quite of the same kind, since the number of base characters is the same at every stage; only their values differ; thus whereas in (iii) the ring $k[X_1, X_2, X_3]$ contains an element of order 2340, in (ii) it does not; this means that the branch (iii), or any other in three dimensions canonically equivalent to it, can be cut by a surface (in the given form the plane $X_3 = 0$) so that the intersection number is 2340, i.e. a surface can be drawn passing simply through the first eleven points of the branch (as far as the first 18-ple point), while in the case of (ii) this is not possible—every surface passing through the eleventh point either passes through some further points after it, or has higher multiplicity at some of the earlier points.

The branch (v) is a plane branch, and can therefore by the classical theory be expressed in the form

$$X_1 = \tau^{702}, \quad X_2 = \tau^{1620} + a\tau^{1674} + b\tau^{1692} + c\tau^{1699} + \dots \quad (b \neq 0,\ c \neq 0).$$

The transformation from the variable t to $\tau = X_1^{1/702}$ is lengthy but straightforward.

It is hoped that enough has been said to make clear the geometrical significance of the canonical branch of which a given branch is a projection, and of the number and values of its base characters.

The University
Istanbul

NOTE ON CONVERGENCE AND SUMMABILITY FACTORS. II

By L. S. BOSANQUET

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1. Necessary and sufficient conditions have been stated by Schur* for a series $\Sigma a_n \epsilon_n$† to be summable (C, ρ) whenever Σa_n is bounded or summable (C, κ), where ρ and κ are integers. I have given a proof of Schur's theorem in a recent paper.‡ Here I extend the result by considering also series which are "more than" or "less than" summable (C, κ).

We suppose throughout that ρ and κ are integers. We write§

$$S_n^\alpha = S_n^\alpha(s_\nu) = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu, \tag{1}$$

where A_μ^α is given by the identity

$$(1-x)^{-\alpha-1} = \Sigma A_\mu^\alpha x^\mu \quad (|x|<1) \tag{2}$$

and

$$s_n = a_0 + a_1 + \ldots + a_n. \tag{3}$$

Thus

$$S_n^1 = s_0 + s_1 + \ldots + s_n, \quad S_n^0 = s_n, \quad S_n^{-1} = a_n, \tag{4}$$

and

$$A_n^\alpha = S_n^\alpha(1). \tag{5}$$

We shall use the identities

$$\text{(i)}\ S_n^{\alpha+1} = S_n^1(S_\nu^\alpha), \quad \text{(ii)}\ S_n^{\alpha-1} = S_n^{-1}(S_\nu^\alpha), \tag{6}$$

which are included in

$$S_n^{\alpha+\beta} = S_n^\beta(S_\nu^\alpha). \tag{7}$$

We also note that

$$(n+\alpha)\, S_n^\alpha(s_\nu) - \alpha S_n^{\alpha+1}(s_\nu) = S_n^\alpha(\nu s_\nu), \tag{8}$$

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* I. Schur, *Journal für Math.* 151 (1921), 79–111.

† Here and elsewhere Σ denotes $\sum_0^\infty$.

‡ L. S. Bosanquet, *Journal London Math. Soc.* 20 (1945), 39–48. This will be referred to as I.

§ We shall only require the case where α is an integer (positive, negative or zero).

and, if κ is a positive integer,

$$S_n^{-\kappa}(s_\nu) = \sum_{\nu=n-\kappa}^{n} A_{n-\nu}^{-\kappa-1} s_\nu = \sum_{\mu=0}^{\kappa} (-1)^\mu \binom{\kappa}{\mu} s_{n-\mu} \quad (n \geqslant \kappa), \tag{9}$$

since $A_\mu^{-\kappa-1} = 0$ for $\mu \geqslant \kappa + 1$.

Also, if κ is a positive integer,

$$A_n^\kappa = \frac{(n+1)(n+2)\dots(n+\kappa)}{1.2.\dots.\kappa} \sim \frac{n^\kappa}{\kappa!}. \tag{10}$$

Finally, we write

$$\Delta s_n = \Delta^1 s_n = s_n - s_{n+1}, \qquad \Delta^\kappa = \Delta\Delta^{\kappa-1} \quad (\kappa \geqslant 2), \qquad \Delta^0 s_n = s_n, \tag{11}$$

so that, for $\kappa = 0, 1, \dots,$

$$\Delta^\kappa s_n = (-1)^\kappa S_{n+\kappa}^{-\kappa}(s_\nu). \tag{12}$$

We obtain the following result, where the case $p = 0, \rho \geqslant 0, \kappa \geqslant 0$ is Schur's theorem, which includes the Bohr-Hardy theorem (when $\rho = \kappa$) and a theorem of Bromwich (when $\rho = 0$).*

THEOREM A. *If $-1 \leqslant \rho \leqslant \kappa$ (ρ, κ integers) and p is real, then necessary and sufficient conditions for $\Sigma a_n \epsilon_n$ to be summable (C, ρ) whenever $S_n^\kappa = O(n^{\kappa+p})$ are*

$$\text{(i) } \epsilon_n = o(n^{\rho-\kappa-p}), \quad \text{(ii) } \Sigma(n+1)^{\kappa+p} \,|\, \Delta^{\kappa+1}\epsilon_n \,| < \infty. \tag{13}$$

If $\rho > \kappa \geqslant 0$, the conditions are the same as when $\rho = \kappa$, but, if $\rho > \kappa = -1$, the first condition must be omitted.

The same holds with O and o interchanged.

2. We require a number of lemmas. In all proofs we consider the *first of the alternative statements*. Only slight modifications are required in order to prove the companion results.

LEMMA 1. *If $\kappa \geqslant 0$ and p is real, and if $S_n^\kappa = O(n^{\kappa+p})$, then $S_n^\sigma = O(n^{\kappa+p})$ for $\sigma = -1, 0, \dots, \kappa - 1$.*

The same holds with o in place of O.

This is proved by repetition of (6) ii, since

$$S_n^\sigma = S_n^{-1}(S_\nu^{\sigma+1}) = S_n^{\sigma+1} - S_{n-1}^{\sigma+1} \quad (n \geqslant 1).$$

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* For references see I.

In my first draft of theorem A, I assumed that $p > -\kappa - 1$, but Prof. Hardy kindly pointed out that this condition was not used in the proof. There is, however, a distinction between the cases $p > -\kappa - 1$ and $p \leqslant -\kappa - 1$. For, if $p > -\kappa - 1$, it is reasonable to write $s_n = O(n^p)$ (C, κ), since by lemma 2 this implies $s_n = O(n^p)$ $(C, \kappa + 1)$. But, if $p \leqslant -\kappa - 1$, it is not reasonable to do so, as is seen by taking $S_n^\kappa = (n+1)^{\kappa+p}$, when $S_n^{\kappa+1} = A + O(n^{\kappa+1+p})$ if $p < -\kappa - 1$, and $S_n^{\kappa+1} \sim \log n$ if $p = -\kappa - 1$.

LEMMA 2. *If* $p > -\kappa - 1$* *and* $S_n^\kappa = O(n^{\kappa+p})$, *then* $S_n^{\kappa+1} = O(n^{\kappa+1+p})$.

The same holds with o in place of O.

By (6) i we have

$$S_n^{\kappa+1} = S_n^1(S_\nu^\kappa) = \sum_{\nu=0}^{n} O\{(\nu+1)^{\kappa+p}\} = O(n^{\kappa+1+p}).$$

LEMMA 3.† *If* $\kappa \geqslant 2$ *and q is real, and if*

$$\text{(i) } \epsilon_n = o(n^{-q}), \quad \text{(ii) } \Delta^\kappa \epsilon_n = o(n^{-\kappa-q}), \tag{14}$$

then $\Delta^\sigma \epsilon_n = o(n^{-\sigma-q})$ *for* $\sigma = 1, 2, \ldots, \kappa-1$.

The same holds with O in place of o.

Suppose that $q \neq 0, -1, \ldots, -(\kappa-1)$. Then since, by (14) ii and (12), $S_n^{-\kappa}(\epsilon_\nu) = o(n^{-\kappa-q})$ it follows, by repeated applications of (6) i and (5), that there are constants α_μ $(\mu = 1, 2, \ldots, \kappa)$ such that

$$S_n^{-\sigma}(\epsilon_\nu) = S_n^{\kappa-\sigma}\{S_\nu^{-\kappa}(\epsilon_\mu)\} = \sum_{\mu=1}^{\kappa-\sigma} \alpha_\mu A_n^{\kappa-\sigma-\mu} + o(n^{-\sigma-q}) \tag{15}$$

for $\sigma = \kappa-1, \kappa-2, \ldots, 0$. On taking $\sigma = 0$ in (15) we find from (14) i, by (10), that the numbers α_μ are either zero, if $\mu \leqslant \kappa+q$, or arbitrary, if $\mu > \kappa+q$ (and so may be taken as zero), and hence $S_n^{-\sigma}(\epsilon_\nu) = o(n^{-\sigma-q})$ for $\sigma = 1, 2, \ldots, \kappa-1$, i.e. by (12), $\Delta^\sigma \epsilon_n = o(n^{-\sigma-q})$.

Suppose that $q = -\lambda$ $(\lambda = 0, 1, \ldots, \kappa-1)$. Then it follows from (14) ii that (15) holds for $\sigma = \kappa-1, \kappa-2, \ldots, \lambda+1$, while for $\sigma = \lambda, \lambda-1, \ldots, 0$,

$$S_n^{-\sigma}(\epsilon_\nu) = \sum_{\mu=1}^{\kappa-\sigma} \alpha_\mu A_n^{\kappa-\sigma-q} + o(n^{-\sigma-q}\log n). \tag{16a}$$

Taking $\sigma = 0$ in (16a) we find from (14) i that $\alpha_\mu = 0$ for $\mu = 1, 2, \ldots, \kappa-\lambda-1$, and hence, by (15), that $S_n^{-\sigma}(\epsilon_\nu) = o(n^{-\sigma-q})$ for $\sigma = \kappa-1, \kappa-2, \ldots, \lambda+1$, and in particular that $nS_n^{-\lambda-1}(\epsilon_\nu) = o(1)$.

Now,‡ by (7), (8) and lemma 2,

$$(n+\alpha)\, S_n^{\alpha-\lambda-1}(\epsilon_\nu) - \alpha S_n^{\alpha-\lambda}(\epsilon_\nu) = S_n^\alpha\{\nu S_\nu^{-\lambda-1}(\epsilon_\mu)\} = o(n^\alpha) \tag{16}$$

for $\alpha = 0, 1, \ldots, \lambda$, and it follows by induction from (14) i and (16) that

$$S_n^{-\sigma}(\epsilon_\nu) = o(n^{-\sigma-q}) \text{ for } \sigma = 1, 2, \ldots, \lambda.$$

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* There is no need to restrict the value of κ.

† The case $\kappa+q=1$ is the Cesàro-Tauber theorem. The case $q=0$ was proved in I (lemma 8). When $\kappa+q<1$ condition (i) is superfluous.

‡ The final step is simply the Cesàro-Tauber theorem.

LEMMA 4.* *If q is real and $\Sigma(n+1)^q \,|\, \Delta\epsilon_n \,| < \infty$, then there is a number s such that $\epsilon_n = s + o(n^{-q})$.*

Suppose that $q \geqslant 0$. Then $\Sigma \,|\, \Delta\epsilon_n \,| < \infty$, and hence there is a number s such that $\epsilon_n \to s$, and

$$|\, \epsilon_n - s \,| \leqslant \sum_{\nu=n}^{\infty} |\, \Delta\epsilon_\nu \,| \leqslant n^{-q} \sum_{\nu=n}^{\infty} \nu^q \,|\, \Delta\epsilon_\nu \,| = o(n^{-q}).$$

Suppose that $q < 0$. Then

$$|\, \epsilon_n - \epsilon_m \,| \leqslant \sum_{\nu=m}^{n} |\, \Delta\epsilon_\nu \,| \leqslant n^{-q} \sum_{\nu=m}^{n} \nu^q \,|\, \Delta\epsilon_\nu \,| < \epsilon n^{-q}$$

for some m and all $n > m$. Hence $\epsilon_n = o(n^{-q})$.

LEMMA 5.† *If $\kappa \geqslant 1$ and q is real, and if*

$$\text{(i)}\ \epsilon_n = o(n^{-q}), \quad \text{(ii)}\ \Sigma(n+1)^{\kappa+q} \,|\, \Delta^{\kappa+1}\epsilon_n \,| < \infty, \tag{17}$$

then $\Delta^\kappa \epsilon_n = o(n^{-\kappa-q})$.

The same holds with O in place of o.

It follows from (17) ii, by (11), (12) and lemma 4, that there is a number s such that

$$S_n^{-\kappa}(\epsilon_\nu) = s + o(n^{-\kappa-q}). \tag{18}$$

Hence, if $q \neq 0, -1, \ldots, -(\kappa-1)$, by (5), (10) and repeated applications of (6) i,

$$\epsilon_n = S_n^\kappa\{S_\nu^{-\kappa}(\epsilon_\mu)\} = sA_n^\kappa + O(n^{\kappa-1}) + o(n^{-q}),$$

and it follows from (17) i, by (10), that s is zero, if $q \geqslant -\kappa$, or arbitrary, if $q < -\kappa$ (and so may be taken as zero).

Also, if $q = 0, -1, \ldots$ or $-(\kappa-1)$,

$$\epsilon_n = S_n^\kappa\{S_\nu^{-\kappa}(\epsilon_\mu)\} = sA_n^\kappa + O(n^{\kappa-1}) + o(n^{-q}\log n)$$

so that, by (17) i and (10), s is zero.

Thus in all cases, by (18) and (12), $\Delta^\kappa\epsilon_n = o(n^{-\kappa-q})$.

LEMMA 6. *If $0 \leqslant \rho \leqslant \kappa$ (ρ, κ integers), p is real, $q < p+1$, and if $S_n^\kappa = O(n^{\kappa+p})$ and*

$$\text{(i)}\ \epsilon_n = o(n^{\rho-\kappa-q}), \quad \text{(ii)}\ \Delta^\kappa\epsilon_n = o(n^{-\kappa-q}), \tag{19}$$

then $S_n^\rho(s_\nu \epsilon_\nu) = o(n^{\rho+p-q})$, i.e. $s_n \epsilon_n = o(n^{p-q})$ (C, ρ).

The conclusion holds if O and o are interchanged in the hypotheses.

Suppose that $\rho = 0$. Then, by lemma 1 and (19) i,

$$s_n \epsilon_n = O(n^{\kappa+p})\, o(n^{-\kappa-q}) = o(n^{p-q}).$$

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* The case $q \geqslant 0$ was proved in I (lemma 6), where the reference to Bromwich applied only to the second part.

† This is included in a result proved for $q = 0$ by A. F. Andersen, *Studier over Cesàro's summabilitets metode* (Copenhagen, 1921), and for $q \geqslant 0$ by L. S. Bosanquet, *Journal London Math. Soc.* 17 (1942), 166–173. If $q \neq -\kappa$ it is sufficient to have O in (17) i to obtain o in the conclusion.

Suppose that $1 \leqslant \rho \leqslant \kappa$, *and assume the result with* κ *replaced by* $\kappa-1$ *and* $0 \leqslant \rho \leqslant \kappa-1$. Write

$$\sum_0^n s_\nu \epsilon_\nu = S_n^1 \epsilon_{n+1} + \sum_0^n S_\nu^1 \Delta\epsilon_\nu. \tag{20}$$

Then $\quad S_n^{\kappa-1}(S_\nu^1) = S_n^\kappa = O\{n^{(\kappa-1)+(p+1)}\}, \quad \epsilon_{n+1} = o\{n^{(\rho-1)-(\kappa-1)-q}\}$

and, by lemma 3, $\Delta^{\kappa-1}\epsilon_{n+1} = o\{n^{-(\kappa-1)-q}\}$. Thus S_n^1, ϵ_{n+1} satisfy the hypotheses of s_n, ϵ_n with κ, ρ, p replaced by $\kappa-1$, $\rho-1$, $p+1$. It follows from the case assumed that

$$S_n^1 \epsilon_{n+1} = o\{n^{(p+1)-q}\} \quad (C, \rho-1).$$

Again, by (11) and lemma 3, $\Delta\epsilon_n = o\{n^{\tau-(\kappa-1)-(q+1)}\}$, where $\tau = \min(\rho, \kappa-1)$, and $\Delta^{\kappa-1}\Delta\epsilon_n = \Delta^\kappa\epsilon_n = o\{n^{-(\kappa-1)-(q+1)}\}$. Thus S_n^1, $\Delta\epsilon_n$ satisfy the hypotheses of s_n, ϵ_n with κ, ρ, p, q replaced by $\kappa-1$, τ, $p+1$, $q+1$.

It follows from the case assumed that $S_n^1 \Delta\epsilon_n = o\{n^{(p+1)-(q+1)}\}$ (C, τ), and so $= o(n^{p-q})$ (C, ρ), by lemma 2, since $p-q > -\tau-1$.* Hence, by (6) i and (20),

$$\sum_0^n s_\nu \epsilon_\nu = o(n^{p+1-q}) \quad (C, \rho-1),$$

i.e. $\quad S_n^\rho(s_\nu \epsilon_\nu) = S_n^{\rho-1}\left(\sum_0^\nu s_\mu \epsilon_\mu\right) = o\{n^{(\rho-1)+(p+1-q)}\} = o(n^{\rho+p-q}),$

and the lemma is proved by induction.

3. *Proof of theorem A (sufficiency). Suppose that* $\rho = 0$, $\kappa = 0$. Write

$$\sum_0^n a_\nu \epsilon_\nu = S_n^0 \epsilon_{n+1} + \sum_0^n S_\nu^0 \Delta\epsilon_\nu. \tag{21}$$

Then by (13) i, since $S_n^0 = O(n^p)$,

$$S_n^0 \epsilon_{n+1} = O(n^p)\, o(n^{-p}) = o(1)$$

and, by (13) ii, $\quad \Sigma\, | S_\nu^0 \Delta\epsilon_\nu | \leqslant A\Sigma(\nu+1)^p\, |\Delta\epsilon_\nu| < \infty.$

Thus, by (21), $\Sigma a_n \epsilon_n$ converges.

Suppose that $\rho = 0$, $\kappa \geqslant 1$ *and assume the result with* κ *replaced by* $\kappa-1$ *and* $\rho = 0$. Then, by lemma 1 and (13) i,

$$S_n^0 \epsilon_{n+1} = O(n^{\kappa+p})\, o(n^{-\kappa-p}) = o(1).$$

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* Since the proof is by induction we require here the full force of the condition $p-q > -1$. When $\rho = 0$ it is not used.

When $\rho \geqslant 1$ the result is false with $q = p+1$, as is seen by taking

$$S_n^\kappa = (-1)^n (n+1)^{\kappa+p}, \quad \epsilon_n = (-1)^n n^{-\kappa-p-1}\{\log(n+1)\}^{-1},$$

so that, by (9),

$$(-1)^n s_n = (-1)^n S_n^{-\kappa}(S_\nu^\kappa) = \sum_{\mu=0}^{\kappa} \binom{\kappa}{\mu} (n+1-\mu)^{\kappa+p} \sim 2^\kappa (n+1)^{\kappa+p}$$

and $\quad s_n \epsilon_n \sim 2^\kappa (n+1)^{-1} \{\log(n+1)\}^{-1}.$

Also $\quad S_n^{\kappa-1}(S_\nu^1) = S_n^\kappa = O\{n^{(\kappa-1)+(p+1)}\}, \quad \Delta\epsilon_n = o\{n^{-(\kappa-1)-(p+1)}\}$, by (11) and (13) i, and $\Sigma(n+1)^{(\kappa-1)+(p+1)}\,|\,\Delta^\kappa(\Delta\epsilon_n)\,| < \infty$, by (11) and (13) ii. Thus S_n^0, $\Delta\epsilon_n$ satisfy the hypotheses of a_n, ϵ_n with κ, p replaced by $\kappa-1$, $p+1$, and so by our assumption $\Sigma S_n^0 \Delta\epsilon_n$ converges. Thus, by (21), $\Sigma a_n \epsilon_n$ converges, and the case $\rho = 0$ is proved by induction.

Suppose that $1 \leqslant \rho \leqslant \kappa$ *and assume the result with* κ *replaced by* $\kappa-1$ *and* $0 \leqslant \rho \leqslant \kappa-1$. Then, by (13) and lemma 5, $\Delta^\kappa \epsilon_{n+1} = o(n^{-\kappa-p})$. Thus S_n^0, ϵ_{n+1} satisfy the hypotheses of s_n, ϵ_n in lemma 6 with $q = p$, and hence

$$S_n^0 \epsilon_{n+1} = o(1) \quad (C, \rho).$$

Also, by (11) and lemmas 5 and 3,

$$\Delta\epsilon_n = o\{n^{\tau-(\kappa-1)-(p+1)}\},$$

where $\tau = \min(\rho, \kappa-1)$, and by (6) i,

$$S_n^{\kappa-1}\left(\sum_0^\nu S_\mu^0\right) = S_n^\kappa = O\{n^{(\kappa-1)+(p+1)}\}.$$

Thus S_n^0, $\Delta\epsilon_n$ satisfy the hypotheses of a_n, ϵ_n in the present theorem, with κ, p, ρ replaced by $\kappa-1$, $p+1$, τ. It follows from the case assumed that $\Sigma S_n^0 \Delta\epsilon_n$ is summable (C, τ), and hence (C, ρ). Thus, by (21), $\Sigma a_n \epsilon_n$ is summable (C, ρ), and the case $1 \leqslant \rho \leqslant \kappa$ is proved by induction.

Suppose that $\rho = -1$, $\kappa \geqslant -1$. Then, by lemma 1 and (13) i,

$$a_n \epsilon_n = S_n^{-1} \epsilon_n = O(n^{\kappa+p})\, o(n^{-1-\kappa-p}) = o(n^{-1}).$$

If $\kappa = -1$, $\rho = -1$, by (13) ii,

$$\Sigma\,|\,a_n \epsilon_n\,| = \Sigma\,|\,S_n^{-1} \epsilon_n\,| \leqslant A\Sigma(n+1)^{-1+p}\,|\,\epsilon_n\,| < \infty,$$

so that $\Sigma a_n \epsilon_n$ is convergent, and if $k \geqslant 0$, $\rho = -1$, the convergence of $\Sigma a_n \epsilon_n$ follows from the case $\rho = 0$. Thus $\Sigma a_n \epsilon_n$ is summable $(C, -1)$.

Suppose that $\rho > \kappa \geqslant -1$. Then (13) i and (13) ii, with $\rho = \kappa$, ensure that $\Sigma a_n \epsilon_n$ is summable (C, κ), and so (C, ρ). But in the case $\kappa = -1$ the second hypothesis alone is sufficient to ensure the convergence of $\Sigma a_n \epsilon_n$.

This completes the proof of the sufficiency part of the theorem.

4. In order to prove the necessity of the conditions in theorem A we require some more lemmas.

LEMMA 7*. *If* $\kappa \geqslant -1$ *and* p *is real, and if* $a_n \epsilon_n = o(1)$ *whenever* $S_n^\kappa = O(n^{\kappa+p})$, *then* $\epsilon_n = o(n^{-\kappa-p})$.

The same holds with O and o interchanged throughout.

Suppose the result false, so that $\overline{\lim}\, n^{\kappa+p}\,|\,\epsilon_n\,| > \delta > 0$, for some δ. Then there are integers n_ν such that $n_0 \geqslant 0$, $n_\nu - n_{\nu-1} \geqslant \kappa+2$ and $n_\nu^{\kappa+p}\,|\,\epsilon_{n_\nu}\,| > \delta$ for $\nu = 1, 2, \ldots$.

* Proved for $p = 0$ in I (lemmas 9 and 10).

Choose a_n so that $S^\kappa_{n_\nu} = n_\nu^{\kappa+p} (\nu \geqslant 1)$, $S^\kappa_n = 0$ $(n \neq n_\nu)$. Then $S^\kappa_n = O(n^{\kappa+p})$, and by (9), since $S^\kappa_{n_\nu-\mu} = 0$ for $1 \leqslant \mu \leqslant \kappa+1$,

$$a_{n_\nu} = S^{-\kappa-1}_{n_\nu}(S^\kappa_\mu) = S^\kappa_{n_\nu} = n_\nu^{\kappa+p} \quad (\nu \geqslant 1).$$

Hence $|a_{n_\nu}\epsilon_{n_\nu}| = n_\nu^{\kappa+p}|\epsilon_{n_\nu}| > \delta$ for $\nu = 1, 2, \ldots,$ which contradicts the hypothesis that $a_n\epsilon_n = o(1)$.

LEMMA 8. *If $\Sigma g_n(x) s_n$ converges for $0 < x < 1$ and its sum tends to a limit as $x \to 1-0$ whenever $\{s_n\}$ is convergent, then there are numbers M, X such that*

$$\Sigma |g_n(x)| \leqslant M \text{ for } X < x < 1.$$

This is due to Schur.*

LEMMA 9. *If $\kappa \geqslant -1$, p is real, and $\Sigma a_n\epsilon_n$ is summable (A) whenever $S^\kappa_n = o(n^{\kappa+p})$, then $\Sigma(n+1)^{\kappa+p}|\Delta^{\kappa+1}\epsilon_n| < \infty$.*

The series $\Sigma a_n\epsilon_n x^n$ converges for $0 < x < 1$ whenever $S^\kappa_n = o(n^{\kappa+p})$, and in particular if $S^\kappa_n = (-1)^n(n+1)^{\kappa+p-1}$, when $a_n = S_n^{-\kappa-1}(S^\kappa_\nu) \sim 2^{\kappa+1}n^{\kappa+p-1}$.† Since the same is true with x' in place of x, where $x < x' < 1$, it follows immediately that $n^q\epsilon_n x^n = o(1)$ as $n \to \infty$, for $0 < x < 1$ and every real q. Hence if $S^\kappa_n = o(n^{\kappa+p})$ we have, by $\kappa+1$ partial summations, since $S^\sigma_n = o(n^{\kappa+p})$ for $\sigma = 0, 1, \ldots, \kappa$, by lemma 1,

$$\Sigma a_n\epsilon_n x^n = \Sigma S^\kappa_n \Delta^{\kappa+1}(\epsilon_n x^n) \tag{22}$$

for $0 < x < 1$. Further, since $\Sigma a_n\epsilon_n$ is summable (A), the right-hand side of (22) tends to a limit whenever $S^\kappa_n = o(n^{\kappa+p})$. Hence, by lemma 8,

$$\Sigma(n+1)^{\kappa+p}|\Delta^{\kappa+1}(\epsilon_n x^n)| \leqslant M$$

for $X < x < 1$,‡ and so, on letting $x \to 1-0$, we obtain

$$\Sigma(n+1)^{\kappa+p}|\Delta^{\kappa+1}\epsilon_n| < \infty.$$

LEMMA 10. *If $\kappa \geqslant 0$ and p is real, and if $\Sigma a_n\epsilon_n$ is summable (A) whenever $S^\kappa_n = O(n^{\kappa+p})$, then $\epsilon_n = o(n^{-p})$.*

The same holds with O and o interchanged.

Suppose that $p \neq -1, -2, \ldots, -\kappa$. Then, by repeated applications of (6) i and (5), whenever $s_n = O(n^p)$ there are constants $\alpha_0, \alpha_1, \ldots, \alpha_{\kappa-1}$ such that

$$S^\kappa_n = \sum_{\mu=0}^{\kappa-1} \alpha_\mu A^\mu_n + O(n^{\kappa+p}),$$

* Schur, loc. cit. † Cf. footnote to lemma 6. ‡ We may take $X = 0$.

i.e. by (5) and (7), $$S_n^\kappa\left\{s_\nu - \sum_{\mu=0}^{\kappa-1}\alpha_\mu A_\nu^{\mu-\kappa}\right\} = O(n^{\kappa+p}).$$

Hence, if we alter s_n suitably for $n = 0, 1, \ldots, \kappa-1$,* i.e. alter a_n for $n = 0, 1, \ldots, \kappa$, we obtain $S_n^\kappa = O(n^{\kappa+p})$. It follows that $\Sigma a_n\epsilon_n$ is summable (A) whenever $s_n = O(n^p)$, and so, by lemma 9, with $\kappa = 0$,

$$\Sigma(n+1)^p \mid \Delta\epsilon_n \mid < \infty.$$

It follows from lemma 4 that there is a number s such that

$$\epsilon_n = s + o(n^{-p}), \tag{23}$$

and, if $p < 0$, s is arbitrary (and so may be taken as zero).

If $p \geqslant 0$ we observe that, by (23) and lemma 9, $\epsilon_n - s$ satisfies the conditions of ϵ_n in theorem A, with $\rho = \kappa$. Hence, by the sufficiency part of theorem A, $\Sigma a_n(\epsilon_n - s)$ is summable (C, κ), and so (A), whenever $S_n^\kappa = O(n^{\kappa+p})$. It then follows from the summability (A) of $\Sigma a_n\epsilon_n$ that $\Sigma s a_n$ is summable (A) whenever $S_n^\kappa = O(n^{\kappa+p})$.

Now, if $p > 0$, the example $a_n = (n+1)^{p-1}$ shows that s must be zero, and so, by (23), $\epsilon_n = o(n^{-p})$.

If $p = 0$ we may take $a_n = (n+1)^{-1}\cos\{\log(n+1)\} = \Re\{(n+1)^{i-1}\}$.† Then Σa_n is bounded $(C, 0)$, but not convergent. Hence $S_n^\kappa = O(n^\kappa)$, but Σa_n is not summable (A) since $a_n = O(n^{-1})$.‡ Hence in this case also $s = 0$, and so, by (23), $\epsilon_n = o(n^{-p})$.

Suppose that $p = -\lambda$ $(\lambda = 1, 2, \ldots, \kappa)$. Then $\Sigma a_n\epsilon_n$ is summable (A) whenever $S_n^\kappa = O(n^{\kappa-\lambda})$, and so, since $S_n^\kappa = S_n^{\kappa-\lambda}(S_\nu^\lambda)$, whenever $S_n^\lambda = O(1)$. It follows from lemma 9, with $\kappa = -p = \lambda$, that

$$\Sigma \mid \Delta^{\lambda+1}\epsilon_n \mid < \infty. \tag{24}$$

Hence, by (11), (12) and lemma 4, with $q = 0$, there is a number s such that

$$S_n^{-\lambda}(\epsilon_\nu) = s + o(1),$$

and so, by repeated applications of (6) i and (5),

$$\epsilon_n = S_n^\lambda\{S_\nu^{-\lambda}(\epsilon_\mu)\} = sA_n^\lambda + o(n^\lambda). \tag{25}$$

Since, by (12), (5) and (7), $\Delta^{\lambda+1}A_n^\lambda = (-1)^{\lambda+1}A_{n+\lambda+1}^{-1} = 0$ for $n \geqslant 0$, it follows from (24) and (25) that $\epsilon_n - sA_n^\lambda$ satisfies the conditions of ϵ_n in theorem A,

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* If σ is a positive integer, $A_n^{-\sigma} = 0$ for $n \geqslant \sigma$.

† See G. H. Hardy, *Proc. London Math. Soc.* (2), 8 (1910), 301–320.

‡ If Σa_n is summable (A) and $a_n = O(n^{-1})$, then Σa_n is convergent. J. E. Littlewood, *Proc. London Math. Soc.* (2), 9 (1910), 434–448.

with $\rho = \kappa = -p = \lambda$, and hence, by the sufficiency part of theorem A, $\Sigma a_n(\epsilon_n - sA_n^\lambda)$ is summable (C, λ), and so (A), whenever $S_n^\lambda = O(1)$. It then follows from the summability (A) of $\Sigma a_n \epsilon_n$ that $\Sigma s A_n^\lambda a_n$ is summable (A) whenever $S_n^\lambda = O(1)$.

But if

$$a_n = (n+1)^{-\lambda-1} \cos\{\log(n+1)\} = \Re\{(n+1)^{i-\lambda-1}\} \quad (n \geqslant \lambda),$$

and a_n is suitably chosen for $n = 0, 1, \ldots, \lambda-1$, we have $S_n^\lambda = O(1)$, while $\Sigma A_n^\lambda a_n$ is not summable (A), since, by (10),

$$\lambda!\, A_n^\lambda a_n = (n+1)^{-1} \cos\{\log(n+1)\} + O(n^{-2}).$$

Thus $s = 0$, and so, by (25), $\epsilon_n = o(n^\lambda) = o(n^{-p})$.

5. *Proof of theorem A (necessity).* If $-1 \leqslant \rho \leqslant \kappa$, the necessity of (13) i follows from lemma 7, with $n^{-\rho}\epsilon_n$ in place of ϵ_n, since $a_n \epsilon_n = o(n^\rho)$ whenever $\Sigma a_n \epsilon_n$ is summable (C, ρ), by lemma 1.

If $\kappa \geqslant -1$, $\rho \geqslant -1$, the necessity of (13) ii follows from lemma 9, since $\Sigma a_n \epsilon_n$ is summable (A) whenever it is summable (C, ρ).

If $0 \leqslant \kappa < \rho$, the necessity of (13) i, with $\rho = \kappa$, follows from lemma 10.

To show that the condition $\epsilon_n = o(n^{-p})$ need not hold in the case $\kappa = -1$, $\rho \geqslant 0$, we observe first that the condition $\Sigma(n+1)^{p-1} |\epsilon_n| < \infty$ has already been proved to be necessary and sufficient. Now let $\epsilon_{\nu^2} = \nu^{-2p}$ $(\nu \geqslant 1)$, $\epsilon_n = 0$ $(n \neq \nu^2)$. Then

$$\sum_1^\infty n^{p-1} |\epsilon_n| = \sum_1^\infty \nu^{-2} < \infty,$$

but $\epsilon_n \neq o(n^{-p})$.

This completes the proof of theorem A.

6. We end by giving a simple application of theorem A (in its alternative form).

THEOREM B.* *If* $-1 \leqslant \rho < \kappa$ *and* p *is real, and if* $S_n^\kappa = o(n^{\kappa+p})$, *then* $\Sigma(n+1)^{\rho-\kappa-p} a_n$ *is summable* (C, ρ).

If $\rho = \kappa$, *the result is false except when* $p = 0, -1, \ldots, -\kappa$.†

Proof. We have $\epsilon_n = (n+1)^{\rho-\kappa-p} = O(n^{\rho-\kappa-p})$, which is the alternative form of (13) i.

* The case $0 \leqslant \rho < \kappa$, $p = 0$, was given by M. Riesz, *Comptes rendus*, 148 (1909), 1658–1660, the case $\rho = \kappa$, $p > 0$, by G. H. Hardy and J. E. Littlewood, *Proc. London Math. Soc.* (2), 11 (1913), 411–478, and the cases $\rho = \kappa$, $-1 < p < 0$ and $\rho = \kappa$, $p = -1$, by J. M. Hyslop, *Proc. Edinburgh Math. Soc.* (2), 5 (1938), 182–201. The last authors proved further, in the cases $\rho = \kappa$, $p > 0$ and $\rho = \kappa$, $-1 < p < 0$, respectively, that $\Sigma(n+1)^{-p} a_n$ is either summable (C, κ) or not summable (A).

† p may be replaced by $p + iq$, where q is real, except that the result remains false when $\rho = \kappa$, $p = 0, -1, \ldots -\kappa$, $q \neq 0$.

If $-1 \leqslant \rho < \kappa$, we have

$$\Sigma(n+1)^{\kappa+p} \mid \Delta^{\kappa+1}\{(n+1)^{\rho-\kappa-p}\} \mid \leqslant A\Sigma(n+1)^{\rho-\kappa-1} < \infty,$$

while, if $\rho = \kappa$, $p = 0, -1, \ldots, -\kappa$,

$$\Sigma(n+1)^{\kappa+p} \mid \Delta^{\kappa+1}\{(n+1)^{-p}\} \mid = 0,$$

so that (13) ii is satisfied in both cases. But if $\rho = \kappa$, $p \neq 0, -1, \ldots, -\kappa$, there is a number $A > 0$ such that

$$\Sigma(n+1)^{\kappa+p} \mid \Delta^{\kappa+1}\{(n+1)^{-p}\} \mid \geqslant A\Sigma(n+1)^{-1} = \infty,$$

so that (13) ii is not satisfied. The result follows from theorem A.

University College
London

ON THE SET OF DISTANCES BETWEEN THE POINTS OF A CARATHÉODORY LINEARLY MEASURABLE PLANE POINT SET

By A. S. BESICOVITCH *and* D. S. MILLER

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1. In 1920, H. Steinhaus (**1**) proved that the set of distances between the points of a measurable set of positive measure along the line filled up an interval. Moreover, it was actually shown that this set of distances included an interval with the origin as an end-point. An application of Fubini's theorem at once establishes the same results in the plane when ordinary Lebesgue measure is used. S. Piccard (**2**, **3**) has also shown extreme interest in this subject, having in 1939 published her book *Sur les ensembles de distances des ensembles de points d'un espace Euclidien*, and in 1942 her book *Sur les ensembles parfaits*. In most of her work she deals with sets along the line.

In this paper we shall turn to Euclidean space of two dimensions, where the measure concerned is Carathéodory linear measure. The work of Steinhaus suggests the following question: Given a measurable set in the plane of positive measure, is it necessarily true that the set of distances between the points of the set will include an interval about the origin? It may be recalled that Besicovitch in 1937 (see (**6**), 324) proved that any regular set consists of a set of measure zero and a Y-set (a Y-set is a set which is contained in a denumerable set of rectifiable curves). In view of the somewhat similar behaviour of regular sets and rectifiable curves, one might expect that the set of distances between the points of a regular set would fill up an interval about the origin. We have been able to prove this to be the case.

In general, the answer to our question turns out to be in the negative. Two examples of irregular sets will be given, such that the set of distances fills up an interval about the origin for one of them, and does not for the other one. However, in these two examples both sets have inferior density zero at every point. Could it then be possible that the added restriction of

positive inferior density almost everywhere in a set would assure that the points of the set would include an interval about the origin? An example is furnished to show that this is not the case. Finally, an irregular set is introduced to show that the set of distances may fill up an interval without including any interval about the origin.

2. We now make the following definitions. Let A be a plane point set. For each sequence of bounded open sets $U_1, U_2, \ldots$ lying in the plane and satisfying the conditions: (a) $\sum_{i=1}^{\infty} U_i \supset A$ and† (b) $d(U_i) \leq \rho$, $i = 1, 2, \ldots$, where ρ is a pre-assigned positive number, form $\sum_{i=1}^{\infty} d(U_i)$. Let $L_\rho[A]$ denote the greatest lower bound of all such sums $\sum_{i=1}^{\infty} d(U_i)$. If ρ decreases, $L_\rho[A]$ cannot decrease. Define $\lim_{\rho \to 0} L_\rho[A] = L^*[A]$. $L^*[A]$ is called the outer Carathéodory linear measure of A. Such a measure function will satisfy Carathéodory's six fundamental measure postulates, from which it follows that closed sets, open sets, etc., are all measurable.‡

Define $c(p, r)$ to be the set of points lying in the closed circle with centre p and radius r. Let

$$\bar{D}(A, p) = \overline{\lim_{r \to 0}} \frac{L[Ac(p, r)]}{2r}, \quad \underline{D}(A, p) = \underline{\lim_{r \to 0}} \frac{L[Ac(p, r)]}{2r}$$

represent the superior and inferior density functions respectively of the set A. If $\bar{D}(A, p) = \underline{D}(A, p)$, then the density $D(A, p)$ of A at the point p is said to exist and has this common value. If the density at almost all points of A exists and is unity, then A is said to be regular. If the density at almost all points of A either does not exist or exists but is not unity, then A is defined to be irregular.

3. In this section we shall establish that *the set of distances between the points of a regular set contains an interval about the origin*. It has been mentioned before that a regular set consists of a set of measure zero and at most a denumerable set of subsets of rectifiable curves. Thus, we need only prove our result for the case of a linearly measurable set E, $L[E] > 0$, which is a subset of a single rectifiable curve C.

† The symbol $d(U)$ will be used to denote the diameter of the bounded plane set U. Define $d(a, b)$ to be the distance between the points a and b in the plane. For two sets A and B in the plane $A.B$, $A+B$, $A-B$ will be the ordinary point-set intersection, sum and difference.

‡ Only Carathéodory linearly measurable plane sets will be used. We shall adopt the customary notation $L[A]$ to represent the measure of a measurable set A. It may be shown that the set of points on a rectifiable curve is measurable and has measure equal to the ordinary length of the curve.

Let q and r be any two points on C. Then define $\overline{qr}$ to be the set of points on the chord joining q and r and $\widehat{qr}$ to be the set of points on the arc of C joining q and r.

Let a, b be the end-points of C and choose a positive number $\alpha < \frac{1}{2}\pi$. Denote by I the set of all points of the subarcs of C subtending chords, whose angles with $\overline{ab}$ are at least as large as α. (We shall refer to I as a set of arcs.) Obviously I is a finite or denumerable set of non-overlapping subarcs of C. Furthermore, from the very definition of I, the chord joining any point of $C-I$ to any other point of C forms an angle less than α with $\overline{ab}$.

Fix $\eta > 0$. By the definition of I we can choose a finite set $S_1, S_2, \ldots, S_k$ of non-overlapping arcs of C subtending chords whose angles with $\overline{ab}$ are as large as α, and satisfying the inequality

$$\sum_{i=1}^{k} L[S_i] > \tfrac{1}{2}L[I] - \eta. \tag{1}$$

Let $S_1', S_2', \ldots, S_{k'}'$ denote the complementary arcs of the set $S_1, S_2, \ldots, S_k$ with respect to C. The sum of the absolute values of the orthogonal projections of $S_1, S_2, \ldots, S_k, S_1', S_2', \ldots, S_{k'}'$ on $\overline{ab}$ is at least equal to $L[\overline{ab}]$, and so

$$\sum_{i=1}^{k} L[S_i]\cos\alpha + \sum_{i=1}^{k'} L[S_i'] \geqq L[\overline{ab}].$$

From this it is easy to see that

$$\sum_{i=1}^{k} L[S_i] \leqq \frac{L[\widehat{ab}] - L[\overline{ab}]}{1-\cos\alpha}.$$

Consequently, using (1), we have

$$L[I] < 2\,\frac{L[\widehat{ab}] - L[\overline{ab}]}{1-\cos\alpha} + 2\eta.$$

This being true for any $\eta > 0$, we have

$$L[I] \leqq 2\,\frac{L[\widehat{ab}] - L[\overline{ab}]}{1-\cos\alpha}. \tag{2}$$

Inscribe a polygonal line $p_0p_1\ldots p_n$ into the arc $\widehat{ab}$ so that p_0 and p_n coincide respectively with a and b. On any arc $\widehat{p_{i-1}p_i}$ consider all the arcs subtending chords whose angles with $\overline{p_{i-1}p_i}$ are as large as α. Denote by $I^{(p)}$ the set of points of such arcs corresponding to all $\widehat{p_{i-1}p_i}$. By (2)

$$L[I^{(p)}\widehat{p_{i-1}p_i}] \leqq 2\,\frac{L[\widehat{p_{i-1}p_i}] - L[\overline{p_{i-1}p_i}]}{1-\cos\alpha} \tag{3}$$

for $i = 1, 2, \ldots, n$, and thus

$$L[I^{(p)}] \leqq 2\,\frac{L[\widehat{ab}] - \sum_{i=0}^{n-1} L[\overline{p_ip_{i+1}}]}{1-\cos\alpha}.$$

$I^{(p)}$ can be considered as a finite or denumerable set of non-overlapping subarcs of $\widehat{p_{i-1}p_i}$, $i = 1, i, \ldots, n$.

We now choose α so that $\qquad \cos 2\alpha > \frac{32}{33}$, $\qquad$ (4)

and a polygonal line $p_0 p_1 \ldots p_n$ so that

$$L[I^{(p)}] < L[E]. \tag{5}$$

Denote by C_1 the curve obtained from C by replacing the arcs of $I^{(p)}$ by their chords (note that the points p_i still remain on C_1). From now on $\widehat{qr}$ will denote an arc of C_1 and not of C. Let E_1 be the set EC_1. Then by (5)

$$L[EC_1] = L[E] - L[I^{(p)} E] \geqq L[E] - L[I^{(p)}] > 0. \tag{6}$$

Let q be an interior point of the arc $\widehat{p_{i-1}p_i}$ and $0 < t \leqslant L[\overline{qp_i}]$. Denote by q' the point of $\widehat{qp_i}$ such that $L[\overline{qq'}] = t$. We shall call q' the t-transform of q. From the fact that no chord of $\widehat{p_{i-1}p_i}$ forms an angle $> \alpha$ with $\overline{p_{i-1}p_i}$, it may be shown that to every point q of $\widehat{p_{i-1}p_i}$ such that $L[\overline{qp_i}] \geqq t$, there corresponds one and only one t-transform q'.

Denote by q_i the point of $\widehat{p_{i-1}p_i}$ such that $L[\overline{q_ip_i}] = t$. If G is any set of points of $\widehat{p_{i-1}q_i}$, there exists one and only one t-transform of every point of G. The set of t-transforms of all points of G is called the t-transform of G and is denoted by G'. If G is an arc $\widehat{cd}$, then G' is the arc $\widehat{c'd'}$, c' and d' being the t-transforms of c and d respectively. As the angle between any chord $\overline{cd}$ of $\widehat{p_{i-1}p_i}$ and $\overline{p_{i-1}p_i}$ is $\leqq \alpha$, the angle between any pair of chords $\overline{c_1d_1}$, $\overline{c_2d_2}$ of $\widehat{p_{i-1}p_i}$ is $\leqq 2\alpha$. From this it follows that, if c is an interior point of $\widehat{p_{i-1}q_i}$ and d a variable point of $\widehat{p_{i-1}q_i}$, and if c', d' are the corresponding t-transforms, then

$$\cos 2\alpha \leqq \varliminf_{d \to c} \frac{L[\overline{c'd'}]}{L[\overline{cd}]} \leqslant \varlimsup_{d \to c} \frac{L[\overline{c'd'}]}{L[\overline{cd}]} \leqq \frac{1}{\cos 2\alpha}.$$

Hence, if $\widehat{cd}$ is any arc on $\widehat{p_{i-1}q_i}$ and if c', d' are the corresponding t-transforms of c and d, then

$$\cos 2\alpha \leqq \frac{L[\widehat{c',d'}]}{L[\widehat{c,d}]} \leqq \frac{1}{\cos 2\alpha}. \tag{7}$$

If $c \in E_1$ and $k \in C_1$, then, for almost all c,

$$\lim_{k \to c} \frac{L[E_1 \widehat{ck}]}{L[\widehat{ck}]} = 1. \tag{8}$$

Let $c \in E_1$ be an interior point of some $\widehat{p_{i-1}p_i}$ satisfying (8). Then there exists a point k_0 on $\widehat{cp_i}$ such that for any point $k \in \widehat{ck_0}$,

$$\frac{L[E_1\widehat{ck}]}{L[\widehat{ck}]} > \frac{3}{4}. \tag{9}$$

We shall show that any $t < \frac{1}{2}L[\overline{ck_0}] = \lambda$, and hence the open interval $(0, \lambda)$ belongs to the set of distances between the points of E.

Take the point d on $\widehat{cp_i}$ such that $L[\overline{cd}] = t$. Let d' be the t-transform of d. As any chord of $\widehat{cd}(\widehat{dd'})$ forms an angle $\leqq 2\alpha$ with the chord $\overline{cd}(\overline{dd'})$, we have

$$2t \leqq L[\widehat{cd'}] \leqq \frac{2t}{\cos 2\alpha} < \tfrac{33}{16}t, \tag{10}$$

$$L[E_1\widehat{cd'}] > \tfrac{3}{4}L[\widehat{cd'}] \geqq \tfrac{3}{2}t. \tag{11}$$

Let J, $J_1 \subset J$ be two sets of non-overlapping arcs of $\widehat{cd}$, such that

$$E_1\widehat{cd} \subset J, \quad J - J_1 \subset E_1\widehat{cd}, \quad L[J_1] < \tfrac{1}{33}t. \tag{12}$$

Also let J', J_1' be the t-transforms of J, J_1 so that $J' - J_1'$ is the t-transform of $J - J_1$. By (7)

$$L[J' - J_1'] = L[J'] - L[J_1'] > \cos 2\alpha\, L[J] - \frac{1}{\cos 2\alpha} L[J_1]. \tag{13}$$

By (9) and (12) $$L[E_1\widehat{cd}] > \tfrac{3}{4}t, \quad L[J] > \tfrac{3}{4}t. \tag{14}$$

Hence $$L[J' - J_1'] > \tfrac{8}{11}t - \tfrac{1}{32}t. \tag{15}$$

By (11) and (15)

$$L[E_1\widehat{cd'}] + L[J' - J_1'] > (\tfrac{3}{2} + \tfrac{8}{11} - \tfrac{1}{32})t > \tfrac{17}{8}t. \tag{16}$$

Each of the two sets $E_1\widehat{cd'}$ and $J' - J_1'$ are on $\widehat{cd'}$, and by (10) and (16) the sum of their measures is greater than the measure of $\widehat{cd'}$. Consequently some of the points $J' - J_1'$ belong to $E_1\widehat{cd'}$. Let f' be one of such points. It is the t-transform of a certain point f of $J - J_1$. As the whole of $J - J_1$ belongs to $E_1\widehat{cd}$, f and f' belong to $E_1\widehat{cd'}$ and they are a pair of points at a distance t apart. This completes the proof.

4. We now consider the set of points lying interior to and on the boundary of the square

$$F = \mathop{\mathscr{E}}_{(x,y)} (0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1).$$

An operation of order n on the set F is defined as follows: Divide the square F into n^4 equal closed squares. Of these the n^2 mutually exclusive closed squares having lower left-hand vertices

$$\left(\frac{kn+l}{n^2}, \frac{k+ln}{n^2}\right) \quad (k = 0, 1, \ldots, n-1,\ l = 0, 1, \ldots, n-1)$$

will be the result of the operation of order n on the set F.

On the set F perform the operation of order 3 obtaining 3^2 mutually exclusive, closed squares, each a subset of the set F and having side-length $1/3^2$. It will be said of the 3^2 closed squares that they belong to class 2. Each of the closed squares of class 2 will be denoted by β_2 and the union of the 3^2 squares β_2 by B_2. On each of the squares β_2 perform the operation of order 4 obtaining the $3^2.4^2$ mutually exclusive, closed squares of class 3 each having side-length $1/(3^2.4^2)$. Each of the closed squares of class 3 will be denoted by β_3 and the union of the $3^2.4^2$ squares β_3 by B_3. In general, on each of the $3^2.4^2.....(n+1)^2$ squares β_n of class n perform the operation of order $n+1$ obtaining the $3^2.4^2.....(n+2)^2$ mutually exclusive closed squares of class $n+1$ each having side-length $1/[3^2.4^2.....(n+2)^2]$. Each of the closed squares of class $n+1$ will be denoted by β_{n+1} and the union of the $3^2.4^2.....(n+2)^2$ squares by B_{n+1}.

Consider the set† $B = B_1.B_2.....$ The set‡ B is perfect, and since $B_1, B_2, \ldots$ form a decreasing sequence of closed sets, B is also non-empty. Define the set $F_{cn} = \underset{(x,y)}{\mathscr{E}}[x = c;\ (x,y) \in B_n]$, where $0 \leqq c \leqq 1$ $(n = 1, 2, \ldots)$. It may readily be seen that the set $F_c = F_{c1}.F_{c2}....$ is non-empty. However, the set F_c is just the intersection of the set B with the line $\underset{(x,y)}{\mathscr{E}}[x = c]$, and so the projection B_x of the set B on the x-axis completely fills the unit interval. From the§ inequality $L[B] \geqq L[B_x] = 1$, it follows that the set B is not of measure zero. Let s_n be the side-length of any square β_n of class n. Then the sum of the diameters of the $3^2.4^2.....(n+1)^2$ squares of class n is equal to $2^{\frac{1}{2}}$. Furthermore, $\lim_{n\to\infty} 2^{\frac{1}{2}} s_n = 0$. From these remarks we see that a sequence of convex sets $U_1, U_2, \ldots$ may be chosen which satisfies the following conditions:

(a) $\sum_{i=1}^{\infty} U_i \supset B$,

(b) $d(U_i) \leqq \rho$ for $i = 1, 2, \ldots$, where ρ is some preassigned positive number,

(c) $\sum_{i=1}^{\infty} d(U_i) = 2^{\frac{1}{2}}$.

Such a sequence is formed, for instance, by the squares of B_n for n satisfying the inequality $2^{\frac{1}{2}} s_n \leqq \rho$. Thus $L_\rho[B] \leqq 2^{\frac{1}{2}}$ for each positive ρ and so $L[B] \leqq 2^{\frac{1}{2}}$.

† Note that $B_1 = F$.

‡ This set was invented by Gross (see (5), 434).

§ For a plane set A, $L^*[A] \geqq L^*[A_1]$, where the set A_1 is the projection of A on a line l in the plane.

We are now in a position to show that $\underline{D}(B, p) = 0$ for all points p belonging to the set B and so consequently B will be an irregular set. Let β_n and β'_n be two squares of class n such that $d(\beta_n, \beta'_n)$ does not exceed the distance between any two other squares of class n. A simple geometric investigation will show that $d(\beta_n, \beta'_n) = ns_n$ for $n \geqq 4$. A point $p \in B$ will belong to some square β_n of class n. It is now easy to see that the circle $c(p, ns_n)$ includes only the part of B in the square β_n, and so

$$\frac{L[B.c(p, ns_n)]}{2ns_n} = \frac{L[B.\beta_n]}{2ns_n} \leqq \frac{2^{\frac{1}{2}}}{2n}.$$

Since $\lim_{n \to \infty} ns_n = 0$, it follows at once that $D(B, p) = 0$.

We now show that the set of distances between the points of the irregular set B does not include an interval about the origin. For α, an arbitrary positive number, define the interval $I_\alpha = \underset{x}{\mathscr{E}}\ [0 < x < \alpha]$. Choose $n \geqq 4$ such that $ns_n < \alpha$. Then clearly the interval $I = \underset{x}{\mathscr{E}}\ [2^{\frac{1}{2}} s_n < x < ns_n]$ is contained in I_α. Let $p, q \in B$. If p and q belong to a single square β_n of class n, clearly $d(p, q) \leqq 2^{\frac{1}{2}} s_n$. If $p \in \beta_n$ and q belongs to any other square of class n, the above remarks show that $d(p, q) \geqq ns_n$. This shows that the set of distances between the points of B will not fill up the interval I and so certainly not the interval I_α. This establishes the required result.

5. We now introduce an irregular set possessing the property that the set of distances between the points of the set actually includes an interval with the origin as an end-point. Let F be the set of points lying inside and on the boundary of an equilateral triangle. For simplicity let one side of the triangle coincide with the unit interval along the x-axis. Construct on F three closed triangles of side-length $1/3$, each one having two sides and a vertex on the boundary of the original triangle F. It will be said of the three mutually exclusive, closed triangles that they belong to class 1. Each closed triangle of class 1 will be denoted by β_1 and the union of the three triangles β_1 by B_1. On each triangle β_1 of class 1 construct three closed equilateral triangles of side-length $1/3^2$, each one having two sides and a vertex on the boundary of the original triangle β_1. It will be said of the resulting 3^2 mutually exclusive triangles that they belong to class 2. Each triangle of class 2 will be denoted by β_2 and the union of the 3^2 sets β_2 by B_2. Continuing in this manner, we arrive at the sequence of sets† $\{B_n\}$. Writing $B = B_1 . B_2 . \ldots$, we see that B is closed and non-empty.

† The set B was first invented by Sierpiński (see (7), 184).

The orthogonal projection B_x of B on the x-axis is the unit interval. It follows at once that $L[B] \geqq 1$. It may be noted that $\sum_{B_n \in B_n} d(\beta_n) = 1$, and so B_n, for sufficiently large n, represents a sequence of convex sets $U_1, U_2, \ldots$ such that

(a) $\sum_{i=1}^{\infty} U_i \supset B$,

(b) $d(U_i) < \rho$ $(i = 1, 2, \ldots)$, where ρ is a preassigned positive number,

(c) $\sum_{i=1}^{\infty} d(U_i) = 1$.

Thus $L_\rho[B] \leqq 1$ and so $L[B] = 1$.

We now show that the set B is irregular. An arbitrary point $p \in B$ will belong to some triangle β_n of class n. It is easily verified that

$$L[B.c(p, 1/3^n)] = L[B.\beta_n] = 1/3^n$$

($1/3^n$ is the side-length of the triangle β_n). From this it follows that

$$\lim_{n\to\infty} \frac{L[B.c(p, 1/3^n]}{2.(1/3^n)} = \frac{1}{2}.$$

Consequently $\underline{D}(B, p) \leqq \frac{1}{2}$.

To prove that the set of distances between the points of B includes an interval about the origin, merely observe that the Cantor perfect set in the unit interval about the x-axis is a subset of B. It has already been established (see (4)) that the set of distances between the points of the Cantor set along a line fills up an interval including the origin as an end-point.

6. Referring back to § 3, it may be noted that the Gross set is irregular, possessing inferior density zero at each one of its points. Moreover, the set of distances between the points of the set does not fill up an interval about the origin. The question now arises as to whether positive inferior density at almost all points of a set assures that the set of distances between the points of the set actually fills up an interval including the origin as an end-point. The following example due to Besicovitch (see (5), 431) is an answer to this question.

Let the set S be the interior of a circle of radius r. Define an operation of order n with respect to the circle S as follows. Construct the concentric circle of radius $(1-1/n)r$, divide its circumference into n equal parts and draw circles of radius r/n about the points of subdivision. The results of the operation of order n is n mutually exclusive, open circles, each of radius r/n and each lying interior to the open circle S.

We now introduce the following set F. Let B_1 be an open circle of radius 1. It will be said of this circle that it belongs to class 1. Perform the operation

of order 4 on the set B_1 obtaining 4 mutually exclusive, open circles, each a subset of set B_1 and having radius $\frac{1}{4}$. It will be said of the four open circles that they belong to class 4. Each of the open circles of class 4 will be denoted by β_4 and the union of the four sets β_4 by B_4. On each circle of class 4 perform the operation of order 5 obtaining 4.5 mutually exclusive open circles of class 5, each of which is a subset of some circle of class 4 and has radius $r_5 = 1/(4.5)$. Furthermore, perform this operation so that none of the circles of class 5 are tangent to the original circle B_1. In general, on each circle of class $n-1$ perform the operation of order n, obtaining $4.5.\ldots.n$ mutually exclusive, open circles of class n, each contained in a circle of class $n-1$ and having as its radius $r_n = 1/(4.5.\ldots.n)$. Each of the circles of class n will be denoted by β_n and the union of the $4.5.\ldots.n$ sets β_n by B_n. Perform this operation so that none of the circles of class n are tangent to a circle of class $n-2$.

Consider the intersection $F = B_1.B_4.B_5.\ldots$. Let I_α again represent an interval about the origin with right-hand end-point α. Choose an integer $k \geqq 5$ such that

(*a*) $r_k < \frac{1}{4}\alpha$,

(*b*) $(k-1)\sin(\pi/k) - 1 > 2$.

This is possible since $\lim_{n\to\infty} r_n = 0$, and $\lim_{n\to\infty}\{(n-1)\sin(\pi/n) - 1\} = \pi - 1$. Suppose p and q to be any two points belonging to the set F. If p and q both belong to a circle β_k of class k, clearly $d(p,q) \leqq 2r_k$. If p and q belong respectively to different circles β_k and β_k' of class k, both of which are contained in a single circle β_{k-1}, geometric considerations show that

$$2(r_{k-1} - r_k)\sin(\pi/k) - 2r_k \leqq d(p,q).$$

However, using the definition of r_k and (*b*), it follows that

$$2(r_{k-1} - r_k)\sin(\pi/k) - 2r_k = 2r_k\{(k-1)\sin(\pi/k) - 1\} > 4r_k.$$

Finally, if p belongs to a circle β_k of class k contained in a circle β_{k-1} of class $k-1$, and if q is outside β_{k-1}, it may be verified, since $k \geqq 5$, that

$$d(p,q) \geqq 2(r_{k-2} - r_{k-1})\sin\frac{\pi}{k-1} - 2r_{k-1} = 2kr_k\left\{(k-2)\sin\frac{\pi}{k-1} - 1\right\} \geqq 4r_k.$$

From (*a*) it is readily seen that the interval $I = \mathop{\mathscr{E}}_{x}[2r_k < x < 4r_k]$ is within the interval I_α. The above considerations then show that the set of distances between the points of F will not include the interval I. Since I_α is an arbitrary interval about the origin, we conclude that the set of distances between the points of the set F does not include an interval about the origin.

Besicovitch (see (5), 434) proved that the upper density of F is equal to $\frac{1}{2}$ at every point of F. Furthermore, he proved that the lower density of F is equal to $\frac{1}{\sqrt{(4\pi^2+1)}-1}$ at almost all points of F. Thus we have an example of an irregular set where the lower density is positive almost everywhere and where the set of distances between the points of the set does not fill up an interval about the origin.

7. Finally, we shall give an example of an irregular set such that the set of distances between the points will fill up an interval, and yet will not include an interval about the origin.

Consider a Cartesian coordinate system C and a polar coordinate system P set up in the plane. Let T be a transformation which takes a point (x, y) referred to C into a point $[1+x, y]$† referred to P. We shall now use the Gross set B (as defined in § 3) in the system C. Take the T-transform of B together with the origin in P and denote this set by T_B. It will be shown that the set T_B possesses the desired properties.

First of all, the set T_B is closed. For, let

$$\lim_{n\to\infty} [1+x_n, y_n] = [r, \theta], \quad 0 \leqq \theta \leqq \tfrac{1}{2}\pi, \quad (x_n, y_n) \in B_n \quad (n = 1, 2, \ldots).$$

It is then easily verified that

$$\lim_{n\to\infty} \{(1+x_n-r)^2 + 4r(1-x_n)\sin^2\tfrac{1}{2}(y_n-\theta)\} = 0,$$

from which it follows immediately that

$$\lim_{n\to\infty} (1+x_n-r) = 0 \quad \text{and} \quad \lim_{n\to\infty} (y_n-\theta) = 0.$$

Since B is closed, a point $(x, y) \in B$ exists such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Clearly $r = 1+x$ and $\theta = y$, and so $[r, \theta] = \in T_B$.

Next, we shall make an investigation of the lower density at each point of T_B. Let δ_n be the T-transform of the square β_n used in the definition of the set B. Following the procedure in § 3, we shall say that the sets δ_n belong to class n, and their union will be denoted by the symbol Δ_n. A simple verification shows that $T_B = \Delta_1 . \Delta_2 . \ldots$, and so T_B is a closed linearly measurable set. Furthermore, it follows at once that $L[T_B] \geqq 1$, and so T_B is non-empty.

For $n = 3, 4, \ldots$, define $\quad t_n = \dfrac{1}{3^2 . 4^2 . \ldots . n^2}$,

$$p_n = \sum_{i=3}^{n} t_i(ik_i + l_i), \quad q_n = \sum_{i=3}^{n} t_i(k_i + il_i), \quad (0 \leqq k_i,\ l_i \leqq i-1).$$

† The symbol (x, y) will represent Cartesian coordinates, while $[x, y]$ will denote the polar coordinates of a point.

The lowest vertex nearest the origin and the highest vertex, both of the set δ_n, will have respectively polar coordinates

$$[1+p_{n+1}, q_{n+1}] \quad \text{and} \quad [1+p_{n+1}+t_{n+1}, q_{n+1}+t_{n+1}].$$

A simple computation then shows that

$$d(\delta_n) = \{t_{n+1}^2 + 4(1+p_{n+1})(1+p_{n+1}+t_{n+1})\sin^2 \tfrac{1}{2}t_{n+1}\}^{\frac{1}{2}}.$$

From this equation it follows that for every $\delta_n \in \Delta_n$ the inequality $d(\delta_n) \leqq 5^{\frac{1}{2}} t_{n+1}$ holds. However, since there are $3^2 . 4^2 . \ldots . (n+1)^2$ sets δ_n belonging to Δ_n, and since $\lim_{n\to\infty} t_n = 0$, for an arbitrary $\rho > 0$ there exists a sequence of open sets $U_1, U_2, \ldots$ such that

$$(a) \sum_{i=1}^{\infty} U_i \supset T_B, \qquad (b)\ d(U_i) < \rho \quad (i = 1, 2, \ldots), \qquad (c) \sum_{i=1}^{\infty} d(U_i) \leqq 5^{\frac{1}{2}}.$$

This shows at once that $L_\rho[T_B] \leqq 5^{\frac{1}{2}}$, and so $L[T_B] \leqq 5^{\frac{1}{2}}$.

It is now observed that an $N > 0$ exists such that, for $n \geqq N$, the relation

$$(n-1) \min_{\delta_n \in \Delta_n} d(\delta_n) = (n-1)\{t_{n+1}^2 + 4(1+t_{n+1})\sin^2 \tfrac{1}{2}t_{n+1}\}^{\frac{1}{2}} \geqq n t_{n+1}$$

holds. For $n \geqq N$, consider two sets δ_n and δ_n', of class n, such that $d(\delta_n, \delta_n')$ does not exceed the distance between any two other sets of class n. Geometric considerations show that, for every $\delta_n \in \Delta_n$, there exists a $\delta_n'' \in \Delta_n$ such that $d(\delta_n, \delta_n'') = n t_{n+1}$, and, furthermore, that either $d(\delta_n, \delta_n')$ equals $n t_{n+1}$ or is as great as $(n-1) \min_{\delta_n \in \Delta_n} d(\delta_n)$. The previous inequality rules out the second case and so $d(\delta_n, \delta_n') = n t_{n+1}$. Let $p \in T_B$, in which case p will belong to some δ_n of class n. If $n \geqq \max(3, N)$, certainly $n t_{n+1} \geqq 5^{\frac{1}{2}} t_{n+1} \geqq \max_{\delta_n \in \Delta_n} d(\delta_n)$. Thus

$$\frac{L[T_B . c(p, n t_{n+1})]}{2 n t_{n+1}} = \frac{L[T_B . \delta_n]}{2 n t_{n+1}} \leqq \frac{5^{\frac{1}{2}}}{2n},$$

and $\lim_{n\to\infty} [5^{\frac{1}{2}}/2n] = 0$. This proves that $\underline{D}(T_B, p) = 0$ everywhere, and so the set T_B is irregular.

To show that the set of distances between the points of T_B does not include an interval about the origin, for an arbitrary $\alpha > 0$ consider the interval I_α as defined in § 3. Choose $n \geqq N$ such that $\max_{\delta_n \in \Delta_n} d(\delta_n) < n t_{n+1} < \alpha$. Let $p, q \in T_B$. If $p, q \in \delta_n$ of class n, then clearly $d(p, q) \leqq \max_{\delta_n \in \Delta_n} d(\delta_n)$. If $p \in \delta_n$ and q belongs to any other set of class n, then $d(p, q) \geqq n t_{n+1}$. The interval

$$I = \mathop{\mathscr{E}}_{x} [\max_{\delta_n \in \Delta_n} d(\delta_n) < x < n t_{n+1}]$$

is clearly contained in I_α. However, from the above consideration, the set of distances between the points of T_B will not include the interval I. This establishes the desired result.

We finally show that the set of distances between the points of T_B will fill up the interval $J = \underset{x}{\mathscr{E}}[1 < x < 2]$. Consider an arbitrary $a \in J$. Obviously $d(a, 0) = a$. Since the projection of the set B on the x-axis fills up the unit interval, there exists a point $(x, y) \in B$ such that $x = a-1$. Thus $a = x+1$ and so the point (a, y) actually belongs to the set T_B. Therefore, the distance a is actually attained by two points of T_B which proves the desired result.

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Cambridge University

Cornell University and Yale University

ON SIMPLY HARMONIC "KAPPA-SPACES" OF FOUR DIMENSIONS

By H. S. RUSE

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1. *Introduction*

A Riemannian V_n $(n > 2)$ is called *simply harmonic* (Walker, **3**, 25)* if, for every pair of points (x^i), (x_0^i), Hadamard's "elementary" solution (Hadamard, **11**) of the generalized Laplace equation $\Delta_2 V = 0$ reduces to $1/s^{n-2}$, where s is the geodesic distance between the two points. For $n = 2$ or 3 a simply harmonic space is flat, and the elementary solution reduces to $\log s$ or $1/s$ respectively, or, in a more familiar notation, to $\log r$ or $1/r$. A simply harmonic V_n, for any n, is, incidentally, an Einstein space of zero scalar curvature, the Ricci tensor being everywhere zero.

In a recent paper (Ruse, **10**) I demonstrated the existence of simply harmonic spaces for which the curvature tensor satisfies the relation

$$R_{ijkl,p} = \kappa_p R_{ijkl} \tag{1·1}$$

for some vector-field κ_p, the left-hand side being the covariant derivative of R_{ijkl}. It was shown that a V_n for which (1·1) holds is simply harmonic if, and only if, all the latent roots of the matrix $\Gamma \equiv [\Gamma^i_j]$, where

$$\Gamma^i_j = R^{i}_{\cdot pqj} r^p r^q, \tag{1·2}$$

are zero for all points (x^i) and for all vectors r^i thereat. (In the earlier paper the arbitrary vector r^i was denoted by λ^i.)

It was also shown that every V_2 satisfies a relation of type (1·1), with

$$\kappa_p = \frac{\partial}{\partial x^p} \log |K|, \tag{1·3}$$

K being the Gaussian curvature. The curvature or other geometrical properties of spaces V_n $(n > 2)$ satisfying (1·1) have not been fully investigated so far as I know, except for the case $\kappa_p \equiv 0$, when they are *symmetric* in the

* An approximately chronological list of earlier papers on harmonic spaces appears at the end of this paper.

sense of Cartan. Pending such an investigation, I call any V_n satisfying (1·1) with $\kappa_p \neq 0$, whether simply harmonic or not, by the unsatisfactory but geometrically non-committal name of *kappa-space*, and denote it by K_n. (1·1) will be called the *kappa-condition*.

This paper is confined to the case $n = 4$ because this case may be dealt with by the use of geometrical methods developed in earlier papers. The problem of determining the general nature of simply or completely harmonic spaces for $n > 4$ remains unsolved, though it is known that in certain cases they are flat or of constant curvature.

In § 2 I obtain the most general form for the curvature tensor in any K_4. In subsequent sections attention is confined to the case when the K_4 is simply harmonic. The final conclusion is that the metric of every such space is reducible to the form

$$ds^2 = \alpha dx^2 + 2\gamma dx dy + \beta dy^2 + 2dx dz + 2dy dt,$$

where α, β, γ are functions of x, y only; and, conversely, that every space having a metric of this type is a simply harmonic K_4.

2. *The Riemann tensor in a K_4*

Let g_{ij} be the fundamental tensor, g its determinant and ϵ_{ijkl}, ϵ^{ijkl} the dualizing tensors of non-zero components $\sqrt{g}$ and $1/\sqrt{g}$ respectively. If $g < 0$, the latter tensors are imaginary. The K_4 is not at present assumed to be simply harmonic, nor even an Einstein space: it is any V_4 satisfying the kappa-condition.

The dual of any bivector b_{ij} $(= -b_{ji})$ will be denoted by ${}^\circ b^{ij}$. Thus

$${}^\circ b^{ij} = \tfrac{1}{2}\epsilon^{ijkl} b_{kl}. \tag{2·1}$$

The dual of the Riemann tensor is defined by

$${}^\circ R^{ijkl} = \tfrac{1}{4}\epsilon^{ijpq}\epsilon^{klrs} R_{pqrs}. \tag{2·2}$$

If b_{ij}, c_{ij} are any two bivectors, then $\tfrac{1}{2}\epsilon^{ijkl} b_{ij} c_{kl}$, which is equal to ${}^\circ b^{kl} c_{kl}$ and to ${}^\circ c^{ij} b_{ij}$, will be denoted by $({}^\circ bc)$ or $({}^\circ cb)$. A bivector p_{ij} is *simple* if $({}^\circ pp) = 0$, a relation that may also be written

$$p_{23}p_{14} + p_{31}p_{24} + p_{12}p_{34} = 0. \tag{2·3}$$

In the projective space S_3 associated with any given point (x^i) of K_4, the components p_{ij} of a simple bivector are Plücker coordinates of a line, or, otherwise interpreted, of a special linear complex. The components of ${}^\circ p^{ij}$ are dual coordinates of the same line or complex. The equation

$$R_{ijkl}\, {}^\circ p^{ij}\, {}^\circ p^{kl} = 0$$

is that of a quadratic complex of lines, the *Riemann complex* (Ruse, **14**).

The kappa-condition $$R_{ijkl,p} = \kappa_p R_{ijkl}, \tag{2·4}$$

combined with Bianchi's identity, gives

$$R_{mnpq}\kappa_r + R_{mnqr}\kappa_p + R_{mnrp}\kappa_q = 0, \tag{2·5}$$

which may also be written

$$\tfrac{1}{2}\epsilon^{kpql}R_{mnpq}\kappa_l = 0.$$

Multiplying this by $\frac{1}{2}\epsilon^{ijmn}$ and noting that $\epsilon^{kpql} = \epsilon^{klpq}$, we get

$$^\circ R^{ijkl}\kappa_l = 0. \tag{2·6}$$

In the projective space S_3, $^\circ R^{ijkl}$ are dual coordinates of the Riemann complex and κ_l the coordinates of a plane (of equation $\kappa_l X^l = 0$ in current point-coordinates X^i). Let q^i be any plane other than κ_i. Then

$$C^{il} \equiv {^\circ R^{ijkl}} q_j q_k \tag{2·7}$$

are the coordinates of the complex conic-envelope of the plane q_i, the equation of the envelope being $C^{ij}\varpi_i\varpi_j = 0$ in current plane coordinates ϖ_i. By (2·6) and (2·7),

$$C^{il}\kappa_l = 0,$$

and so the conic C^{ij} lies in the plane κ_i. It is already known to lie in q_i, which is a different plane, and so it must consist of a pair of points (not necessarily distinct) on the line of intersection of the planes. Thus the complex conic-envelope of any plane q_i is a point-pair in the plane κ_i. The quadratic complex must therefore consist* of all the lines meeting a fixed conic in the plane κ_i. This will be called the *base-conic*.

Assume for the moment that the base-conic is non-degenerate, and let r^i be any point of S_3 not in the plane κ_i, so that $\kappa_i r^i \neq 0$. Let $S_{ij} = R_{ipqj}r^p r^q$ be its complex cone—the cone having r^i as vertex and passing through the base-conic. S_{ij} may be identified, if desired, with the Γ_{ij} of (1·2). Then it follows quickly by use of (2·5) that

$$S_{hj}\kappa_i\kappa_k + S_{ik}\kappa_h\kappa_j - S_{hk}\kappa_i\kappa_j - S_{ij}\kappa_h\kappa_k \equiv -(r^p\kappa_p)(r^q\kappa_q)R_{hijk}.$$

Therefore, if the components of r^i are normalized so as to make $(r^i\kappa_i)^2 = -1$, we have

$$R_{hijk} = S_{hj}\kappa_i\kappa_k + S_{ik}\kappa_h\kappa_j - S_{hk}\kappa_i\kappa_j - S_{ij}\kappa_h\kappa_k \tag{2·8}$$

(cf. Ruse, **13**, (2·8)).

* It can be proved independently that, when the Riemann tensor satisfies (2·6) for some κ_p, then it also satisfies the identities given in Ruse (**13**, § 4), for the coordinates of a conic defined by the quadratic complex of lines meeting it. In the latter paper pairs of skew suffixes are raised by means of the ϵ-tensor without the distinguishing mark °.

Let l_i, m_i, n_i be three planes through the vertex r^i of the cone S_{ij}, forming a self-polar trihedral with respect to the cone. Then

$$S_{ij} = l_i l_j + m_i m_j + n_i n_j, \tag{2·9}$$

if l_i, m_i, n_i be suitably normalized: for the equation of the cone is $S_{ij}X^iX^j = 0$, and so, if we change the tetrahedron of reference to that formed by the planes l_i, m_i, n_i and any fourth plane, say κ_i itself, we get new coordinates ξ^i, where

$$\xi^1 = l_i X^i, \quad \xi^2 = m_i X^i, \quad \xi^3 = n_i X^i, \quad \xi^4 = \kappa_i X^i,$$

in terms of which the equation of the cone reduces to $(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 = 0$. Equation (2·9) follows at once.

Substituting from (2·9) in (2·8), we get

$$R_{hijk} = l_{hi} l_{jk} + m_{hi} m_{jk} + n_{hi} n_{jk}, \tag{2·10}$$

where $\quad l_{hi} = l_h \kappa_i - l_i \kappa_h, \quad m_{hi} = m_h \kappa_i - m_i \kappa_h, \quad n_{hi} = n_h \kappa_i - n_i \kappa_h.$

Thus l_{ij}, m_{ij}, n_{ij} are Plücker coordinates of three lines in the plane κ_i of the base-conic, forming a triangle self-polar with respect to the conic. Since they are lines, we have

$$({}^\circ ll) = 0 = ({}^\circ mm) = ({}^\circ nn) \tag{2·11}$$

in the notation of § 2, and since they intersect,

$$({}^\circ mn) = 0 = ({}^\circ nl) = ({}^\circ lm). \tag{2·12}$$

Further, as they all lie in the plane κ_i, we have

$${}^\circ l^{ij}\kappa_j = 0 = {}^\circ m^{ij}\kappa_j = {}^\circ n^{ij}\kappa_j. \tag{2·13}$$

It has therefore been shown that *the most general form for the Riemann tensor in a* K_4 *is that given by* (2·10), *where the simple bivectors* l_{ij}, m_{ij}, n_{ij} *satisfy* (2·11), (2·12), (2·13). The bivectors are not unique because they may represent the sides of any triangle self-polar with respect to the base-conic.

Other equivalent forms for R_{hijk} may be found by taking, instead of a self-polar triangle, some other triangle specially related to the conic, for example one consisting of a pair of tangents and the chord of contact (cf. Ruse, **13**, (3·34)).

Hitherto the base-conic has been assumed non-degenerate. If it consists of a pair of distinct lines, the Riemann complex then consisting of the pair of special linear complexes having these lines as directrices, the curvature tensor is reducible to the form

$$R_{hijk} = l_{hi} l_{jk} + m_{hi} m_{jk}. \tag{2·14}$$

And lastly, if the conic consists of a repeated line, the Riemann complex then being a linear complex taken twice, the tensor assumes the form

$$R_{hijk} = m_{hi} m_{jk}. \tag{2·15}$$

3. *The Riemann tensor in a simply harmonic K_4*

If the K_4 is simply harmonic, it is among other things an Einstein space for which

$$R_{ij} = 0, \tag{3·1}$$

and the Riemann complex in S_3 is self-polar with respect to the fundamental quadric g_{ij} (Ruse, **14**, 71).

Now a system of lines meeting a conic cannot be self-polar with respect to a quadric unless the conic either: (*a*) consists of a pair of lines which are either generators of the quadric (of opposite systems) or else conjugate tangents; or (*b*) consists of a repeated line which is a generator of the quadric.

It has been shown elsewhere (Ruse, **6**, 161) that case (*a*) is impossible in a non-flat completely harmonic space, and hence, in particular, in a simply harmonic space. We therefore conclude that the Riemann tensor must have the form (2·15), where m_{ij} represents a generator of the fundamental quadric and is therefore such that

$$({}^\circ mm) = 0 \tag{3·2}$$

and

$${}^\circ m_{ij} = \pm m_{ij}. \tag{3·3}$$

Equation (3·2) expresses the fact that m_{ij} represents a line of S_3 (simple bivector of K_4), and (3·3) that it is a generator of one or other system (self-dual or anti-self-dual bivector of K_4). Thus

$$R_{ijkl} = m_{ij}m_{kl}, \tag{3·4}$$

where there is now no distinction, except perhaps for sign, between m_{ij} and its covariant dual, and none at all between R_{ijkl} and *its* dual; thus

$${}^\circ R_{ijkl} = R_{ijkl}. \tag{3·5}$$

Because of these facts, equation (2·6) now gives

$$m^{kl}\kappa_l = 0, \tag{3·6}$$

which states that, in S_3, the plane κ_l contains the generator m_{ij}. Therefore it touches the quadric, and so

$$g^{ij}\kappa_i\kappa_j = 0. \tag{3·7}$$

Thus κ_i is a null vector of K_4.

Substituting from (3·4) in the kappa-equation (2·4), we obtain

$$m_{ij,p}m_{kl} + m_{ij}m_{kl,p} = \kappa_p m_{ij}m_{kl}. \tag{3·8}$$

If this is multiplied by μ^{kl}, where μ^{kl} is an arbitrary contravariant bivector, we get

$$m_{ij,p} = \nu_p m_{ij}, \tag{3·9}$$

where

$$\nu_p = \kappa_p - (m_{kl,p}\mu^{kl})/(m_{rs}\mu^{rs}).$$

Substitution of (3·9) in (3·8) at once gives $\nu_p = \frac{1}{2}\kappa_p$, and so

$$m_{ij,p} = \tfrac{1}{2}\kappa_p m_{ij}. \tag{3·10}$$

Now by the permutation-formula for covariant derivatives,

$$\begin{aligned} m_{ij,p,q} - m_{ij,q,p} &= R^h_{.ipq} m_{hj} + R^h_{.jpq} m_{ih} \\ &= m_{pq}(m^h_{.i} m_{hj} + m^h_{.j} m_{ih}) \\ &\equiv 0, \end{aligned}$$

because of the skew symmetry of m_{ij}. Thus

$$m_{ij,p,q} = m_{ij,q,p}. \tag{3·11}$$

But by (3·10),
$$\begin{aligned} m_{ij,p,q} &= \tfrac{1}{2}(\kappa_{p,q} m_{ij} + \kappa_p m_{ij,q}) \\ &= \tfrac{1}{2}(\kappa_{p,q} + \tfrac{1}{2}\kappa_p \kappa_q)\, m_{ij}, \end{aligned}$$

whence, by (3·11),
$$\kappa_{p,q} = \kappa_{q,p},$$

that is,
$$\frac{\partial \kappa_p}{\partial x^q} - \frac{\partial \kappa_q}{\partial x^p} = 0.$$

κ_p is therefore the gradient of a scalar κ, and we have*

$$\kappa_p = \partial\kappa/\partial x^p. \tag{3·12}$$

Equation (3·7) now becomes $\Delta_1 \kappa = 0.$ (3·13)

Now differentiate (2·5) covariantly with respect to x^s. In the equation so obtained, the terms involving the covariant derivative of the Riemann tensor vanish because of the kappa-condition and because of (2·5) itself, and we are left with

$$R_{mnpq}\kappa_{,r,s} + R_{mnqr}\kappa_{,p,s} + R_{mnrp}\kappa_{,q,s} = 0. \tag{3·14}$$

Multiply this by g^{nq}, summing for n, q. The first two terms vanish because the Ricci tensor is zero, and so we obtain

$$R_{mnrp}\kappa^n_{\ ,s} = 0. \tag{3·15}$$

Now multiply (3·14) by g^{rs}. Then, because of the symmetry properties of the Riemann tensor in its suffixes and also because $\kappa_{,p,s}$ is symmetric in p, s, we get, using (3·15),
$$R_{mnpq} \cdot g^{rs}\kappa_{,r,s} = 0,$$

whence
$$\Delta_2 \kappa = 0. \tag{3·16}$$

Both differential parameters formed from κ are therefore zero.

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* Consistency of notation now really requires κ_p to be denoted by $\kappa_{,p}$, but as a rule the comma will be omitted.

4. *Fields of parallel vectors in a simply harmonic K_4*

A set of functions λ^i defines a parallel vector field if

$$\lambda^i{}_{,j} = 0. \tag{4·1}$$

The conditions of integrability of these equations are

$$R_{ijkl}\lambda^l = 0, \quad R_{ijkl,p}\lambda^l = 0, \quad R_{ijkl,p,q}\lambda^l = 0, \quad \ldots \tag{4·2}$$

(cf. Eisenhart, **12**, 67). If the first of these is satisfied, then, in a K_4, all the subsequent conditions are satisfied because of the kappa-condition, covariant differentiation of which gives $R_{ijkl,p,q} = (\kappa_{,p,q} + \kappa_p \kappa_q) R_{ijkl}$, and so on. Now by (3·4), equation (4·2) is equivalent to

$$m_{kl}\lambda^l = 0. \tag{4·3}$$

In the projective space S_3, this means that λ^i is a point on the generator m_{ij} of the fundamental quadric. Therefore (4·3), and hence (4·2), admits two distinct solutions, say $\lambda^i_\sigma \equiv (\lambda^i_1, \lambda^i_2)$ $(\sigma = 1, 2)$, since a straight line is defined by any two points upon it. Any other vector satisfying (4·3) is a linear combination $\phi^\sigma \lambda^i_\sigma$ of the λ^i_σ, ϕ^1 and ϕ^2 being scalars. Hence, as in Eisenhart, *loc. cit.* 68, 69, with his p equal to 2, it follows that K_4 admits fields of parallel vectors depending upon two arbitrary constants. In other words, we can find two independent vectors λ^i, μ^i, both satisfying (4·1), and hence also (4·2) and (4·3), such that any other vector satisfying (4·1) is of the form $a\lambda^i + b\mu^i$, where a, b are constants.

Because λ^i, μ^i both satisfy (4·3), they represent points in S_3 that lie on the same generator of the fundamental quadric. Therefore they lie on the quadric and are conjugate with respect to it. Hence

$$g_{ij}\lambda^i\lambda^j = 0 = g_{ij}\mu^i\mu^j = g_{ij}\lambda^i\mu^j. \tag{4·4}$$

Thus λ^i and μ^i are both null vectors in K_4 and are perpendicular to one another.

By (2·6), (3·5) and (3·4), the vector κ^i also satisfies (4·3). Therefore it, too, represents in S_3 a point on the generator m_{ij}. Hence it is expressible in the form

$$\kappa^i = \sigma\lambda^i + \tau\mu^i. \tag{4·5}$$

Here, however, it is to be presumed that σ and τ are not constants, but are scalar functions of the x^i, since there is no reason for supposing that the covariant derivative of κ^i is zero. It turns out, indeed, that it is in general not so.

5. *Special coordinate-systems*

Again following the argument given by Eisenhart (this time from the middle of his p. 69 to the middle of p. 70), we can deduce that it is possible to choose a coordinate-system (x^i) in the simply harmonic K_4 so that all

components of λ^i are zero except one, say λ^3, and also so that all components of μ^i are zero except one, say μ^4. Writing p, q for λ^3 and μ^4, we have, in that coordinate-system,

$$\lambda^i = p\,\delta_3^i, \tag{5·1}$$

$$\mu^i = q\,\delta_4^i. \tag{5·2}$$

From (4·4) it at once follows that

$$g_{33} = 0 = g_{44} = g_{34}, \tag{5·3}$$

and hence that

$$g^{11} = 0 = g^{22} = g^{12}. \tag{5·4}$$

By (4·5), (5·1) and (5·2),

$$\kappa^1 = 0 = \kappa^2,$$

and hence, by (5·3),

$$\kappa_3 = 0 = \kappa_4. \tag{5·5}$$

The last equations show that, in the special coordinate-system, *the scalar* κ *is a function of* x^1, x^2 *only.*

Moreover, since λ^i and μ^i both satisfy (4·2), we have, by (5·1) and (5·2),

$$R_{ijk3} = 0 = R_{ijk4}.$$

Because of the symmetry properties of R_{ijkl} in its suffixes, it follows that any component of the Riemann tensor having one or more of its suffixes equal to 3 or 4 is zero. Hence *the only* non-zero component is* R_{1212}.

The covariant derivatives of λ^i and μ^i being zero, we have, from (5·1) and (5·2),

$$\delta_3^i \frac{\partial p}{\partial x^j} + \begin{Bmatrix} i \\ 3j \end{Bmatrix} p = 0, \tag{5·6}$$

$$\delta_4^i \frac{\partial q}{\partial x^j} + \begin{Bmatrix} i \\ 4j \end{Bmatrix} q = 0. \tag{5·7}$$

Taking $i = 1, 2, 4$ in (5·6) and $i = 1, 2, 3$ in (5·7), we get

$$\begin{Bmatrix} 1 \\ 3j \end{Bmatrix} = 0 = \begin{Bmatrix} 2 \\ 3j \end{Bmatrix} = \begin{Bmatrix} 4 \\ 3j \end{Bmatrix}, \tag{5·8}$$

$$\begin{Bmatrix} 1 \\ 4j \end{Bmatrix} = 0 = \begin{Bmatrix} 2 \\ 4j \end{Bmatrix} = \begin{Bmatrix} 3 \\ 4j \end{Bmatrix}, \tag{5·9}$$

in each case for all values of j. If we now put $i = 3$ in (5·6) and $i = 4$ in (5·7), we obtain

$$\begin{Bmatrix} 3 \\ 3j \end{Bmatrix} = -\frac{\partial}{\partial x^j} \log p, \tag{5·10}$$

$$\begin{Bmatrix} 4 \\ 4j \end{Bmatrix} = -\frac{\partial}{\partial x^j} \log q \tag{5·11}$$

for all j.

* To save constant reference to the possibility of permuting suffixes, the components for which $(ijkl) = (2121)$, (2112) or (1221) will be regarded as effectively the same as that for which $(ijkl) = (1212)$. They differ from it at most in sign.

It will be convenient hereafter to denote (x^1, x^2, x^3, x^4) by (x, y, z, t).

From the last of equations (5·9), with $j = 3$, and from (5·10) with $j = 4$, we have

$$0 = -\frac{\partial}{\partial t}\log p,$$

and similarly, from (5·8) and (5·11),

$$0 = -\frac{\partial}{\partial z}\log q.$$

Hence $$p = p(x,y,z), \quad q = q(x,y,t).$$

Now transform to a new coordinate-system $(x'^i) \equiv (x', y', z', t')$ defined by

$$x' = x, \quad y' = y, \quad z' = \int \frac{dz}{p(x,y,z)} + \phi(x,y), \quad t' = \int \frac{dt}{q(x,y,t)} + \psi(x,y),$$

where $\phi(x,y)$ and $\psi(x,y)$ are arbitrarily chosen functions of x, y, and the integrals are taken with x, y constant. Then from the ordinary law of transformation of contravariant vectors, it follows at once that, in the new coordinate-system, the vectors λ^i, μ^i have components

$$\lambda'^i = (0,0,1,0), \quad \mu'^i = (0,0,0,1).$$

Dropping the dashes, we may conclude that there exists a coordinate-system in the simply harmonic K_4 in which the functions p, q introduced in (5·1) and (5·2) are equal to unity. By (5·10) and (5·11), the Christoffel symbols $\begin{Bmatrix} 3 \\ 3j \end{Bmatrix}$, $\begin{Bmatrix} 4 \\ 4j \end{Bmatrix}$ are zero in this coordinate-system, as well as those given in (5·8), (5·9). So we now have

$$\begin{Bmatrix} i \\ 3j \end{Bmatrix} = 0 = \begin{Bmatrix} i \\ 4j \end{Bmatrix} \tag{5·12}$$

for all i, j. Thus all Christoffel symbols having a lower index 3 or 4 are zero. Equations (5·3), (5·4) and (5·5) of course still hold. That R_{1212} is the only non-zero component of the Riemann tensor may now be fairly easily verified by direct calculation.

The covariant derivative of g_{ij} is zero. Hence

$$\frac{\partial g_{ij}}{\partial x^k} = \begin{Bmatrix} l \\ ik \end{Bmatrix} g_{lj} + \begin{Bmatrix} l \\ jk \end{Bmatrix} g_{il}. \tag{5·13}$$

If $k = 3$ or 4, the Christoffel symbols on the right are zero, by (5·12). Hence *all the non-zero components of g_{ij}, and therefore of g^{ij}, are functions of x, y only.* Consequently R_{1212} *is a function of x, y only.* It has already been shown that the same is true of the scalar κ.

Now

$$R_{ijkl,p} \equiv \frac{\partial R_{ijkl}}{\partial x^p} - \begin{Bmatrix} m \\ ip \end{Bmatrix} R_{mjkl} - \begin{Bmatrix} m \\ jp \end{Bmatrix} R_{imkl} - \begin{Bmatrix} m \\ kp \end{Bmatrix} R_{ijml} - \begin{Bmatrix} m \\ lp \end{Bmatrix} R_{ijkm}. \tag{5·14}$$

If one or more of the suffixes i, j, k, l is 3 or 4, this is identically zero because R_{ijkl} and the Christoffel symbols are then zero. Hence the kappa-condition

$$R_{ijkl,p} = \kappa_p R_{ijkl} \tag{5·15}$$

is satisfied trivially, both sides being zero. Again, by (5·14),

$$\begin{aligned} R_{1212,p} &\equiv \frac{\partial R_{1212}}{\partial x^p} - 2\begin{Bmatrix} m \\ 1p \end{Bmatrix} R_{m212} - 2\begin{Bmatrix} m \\ 2p \end{Bmatrix} R_{1m12} \\ &\equiv \frac{\partial R_{1212}}{\partial x^p} - 2\left[\begin{Bmatrix} 1 \\ 1p \end{Bmatrix} + \begin{Bmatrix} 2 \\ 2p \end{Bmatrix}\right] R_{1212}. \end{aligned} \tag{5·16}$$

If $p = 3$ or 4, this is zero because R_{1212} depends on x, y only, and because of (5·12). Hence, inasmuch as $\kappa_3 = 0 = \kappa_4$, (5·15) is again satisfied trivially.

There remain only the cases in which $p = 1$ or 2 in (5·16). Now

$$\begin{Bmatrix} 1 \\ 1p \end{Bmatrix} + \begin{Bmatrix} 2 \\ 2p \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1p \end{Bmatrix} + \begin{Bmatrix} 2 \\ 2p \end{Bmatrix} + \begin{Bmatrix} 3 \\ 3p \end{Bmatrix} + \begin{Bmatrix} 4 \\ 4p \end{Bmatrix},$$

by (5·12), that is,

$$\begin{aligned} \begin{Bmatrix} 1 \\ 1p \end{Bmatrix} + \begin{Bmatrix} 2 \\ 2p \end{Bmatrix} &= \begin{Bmatrix} i \\ ip \end{Bmatrix} \\ &= \frac{\partial}{\partial x^p} \log \sqrt{|g|}. \end{aligned}$$

Hence by (5·16),

$$\begin{aligned} R_{1212,p} &= \frac{\partial}{\partial x^p} R_{1212} - R_{1212} \frac{\partial}{\partial x^p} \log |g| \\ &= \kappa_p R_{1212}, \end{aligned}$$

where

$$\kappa_p = \frac{\partial}{\partial x^p} \log \left| \frac{R_{1212}}{g} \right|. \tag{5·17}$$

This verifies that the Riemann tensor does in fact satisfy the kappa-condition.

It is of interest to note that (5·17) bears a formal resemblance to (1·3), which holds in any V_2. For the Gaussian curvature of a V_2 is precisely

$$K = R_{1212}/g,$$

R_{1212} then being the only component of the Riemann tensor.

6. *Further particularization of the coordinate-system*

It has been found that coordinate-systems exist in which g_{33}, g_{44}, g_{34}, g^{11}, g^{22}, g^{12} are all zero and in which the remaining g's are functions of x, y only. The question arises whether these conditions are alone sufficient to ensure that the space should be a simply harmonic K_4. The answer is no,

because it is also necessary that the Christoffel symbols $\left\{ {i \atop 3j} \right\}$, $\left\{ {i \atop 4j} \right\}$ should all be zero for all i, j. Most of these do in fact vanish by reason of the above conditions, but not all. Those not identically zero are

$$\left.\begin{aligned} \left\{ {3 \atop 31} \right\} = \left\{ {3 \atop 13} \right\} &= \tfrac{1}{2} g^{23} \left(\frac{\partial g_{23}}{\partial x} - \frac{\partial g_{13}}{\partial y} \right), \\ \left\{ {3 \atop 32} \right\} = \left\{ {3 \atop 23} \right\} &= -\tfrac{1}{2} g^{13} \left(\frac{\partial g_{23}}{\partial x} - \frac{\partial g_{13}}{\partial y} \right), \end{aligned}\right\} \tag{6·1}$$

$$\left.\begin{aligned} \left\{ {4 \atop 41} \right\} = \left\{ {4 \atop 14} \right\} &= \tfrac{1}{2} g^{24} \left(\frac{\partial g_{24}}{\partial x} - \frac{\partial g_{14}}{\partial y} \right), \\ \left\{ {4 \atop 42} \right\} = \left\{ {4 \atop 24} \right\} &= -\tfrac{1}{2} g^{14} \left(\frac{\partial g_{24}}{\partial x} - \frac{\partial g_{14}}{\partial y} \right). \end{aligned}\right\} \tag{6·2}$$

For the space to be a simply harmonic K_4 these, too, must vanish. So by (6·1) we must have

$$\frac{\partial g_{23}}{\partial x} = \frac{\partial g_{13}}{\partial y}. \tag{6·3}$$

The only alternative would be $g^{23} = 0 = g^{31}$, but if this were so the determinant $|g^{ij}|$ would vanish because we also have $g^{11} = 0 = g^{22} = g^{12}$. Similarly, from (6·2),

$$\frac{\partial g_{24}}{\partial x} = \frac{\partial g_{14}}{\partial y}. \tag{6·4}$$

But because g_{33}, g_{44}, g_{34} are all zero, the fundamental quadratic form is

$$ds^2 = g_{11} dx^2 + 2g_{12} dx\,dy + g_{22} dy^2 + 2(g_{13} dx + g_{23} dy)\,dz + 2(g_{14} dx + g_{24} dy)\,dt. \tag{6·5}$$

By (6·3) and (6·4), $g_{13}dx + g_{23}dy$ and $g_{14}dx + g_{24}dy$ are both exact differentials, say $d\xi$ and $d\eta$. ξ is determined as a function of x, y except for an additive constant which may be chosen at convenience, and the same is true of η. Taking ξ, η as new coordinates in place of x, y, remembering that the g's are functions of x, y only, we get ds^2 in the form

$$ds^2 = \gamma_{11} d\xi^2 + 2\gamma_{12} d\xi\,d\eta + \gamma_{22} d\eta^2 + 2d\xi\,dz + 2d\eta\,dt,$$

where the γ's are functions of ξ, η only. Writing x, y in place of ξ, η and α, β, γ for $\gamma_{11}, \gamma_{22}, \gamma_{12}$, we have:

The metric of any simply harmonic K_4 is reducible to the form

$$ds^2 = \alpha\,dx^2 + 2\gamma\,dx\,dy + \beta\,dy^2 + 2dx\,dz + 2dy\,dt, \tag{6·6}$$

where α, β, γ are functions of x, y only. The only non-zero component of R_{ijkl} is R_{1212}, and the scalar κ is also a function of x, y only.

For the space (6·6),

$$[g_{ij}] = \begin{bmatrix} \alpha & \gamma & 1 & 0 \\ \gamma & \beta & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad [g^{ij}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -\alpha & -\gamma \\ 0 & 1 & -\gamma & -\beta \end{bmatrix}, \tag{6·7}$$

the determinant g is equal to unity, and the only non-zero Christoffel symbols are

$$\left.\begin{aligned} \begin{Bmatrix} 3 \\ 11 \end{Bmatrix} &= \frac{1}{2}\frac{\partial\alpha}{\partial x}, \quad \begin{Bmatrix} 3 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 21 \end{Bmatrix} = \frac{1}{2}\frac{\partial\alpha}{\partial y}, \quad \begin{Bmatrix} 3 \\ 22 \end{Bmatrix} = \frac{\partial\gamma}{\partial y} - \frac{1}{2}\frac{\partial\beta}{\partial x}, \\ \begin{Bmatrix} 4 \\ 11 \end{Bmatrix} &= \frac{\partial\gamma}{\partial x} - \frac{1}{2}\frac{\partial\alpha}{\partial y}, \quad \begin{Bmatrix} 4 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 21 \end{Bmatrix} = \frac{1}{2}\frac{\partial\beta}{\partial x}, \quad \begin{Bmatrix} 4 \\ 22 \end{Bmatrix} = \frac{1}{2}\frac{\partial\beta}{\partial y}. \end{aligned}\right\} \tag{6·8}$$

Calculation shows that

$$R_{1212} = -\frac{1}{2}\left(\frac{\partial^2\alpha}{\partial y^2} + \frac{\partial^2\beta}{\partial x^2} - 2\frac{\partial^2\gamma}{\partial x\,\partial y}\right). \tag{6·9}$$

7. *The converse*

It remains only to show that every V_4 whose metric is reducible to the form (6·6) is a simply harmonic K_4.

That it is a K_4 follows at once from the work of § 5, in which κ_p was determined for a more general coordinate-system than that of (6·6).

That it is simply harmonic follows almost at once. Writing

$$\Gamma^i_j = R^i_{\cdot pqj} r^p r^q$$

as in (1·2), we find immediately that, whatever r^i, all the Γ^i_j are zero except Γ^3_1, Γ^3_2, Γ^4_1 and Γ^4_2. It is therefore obvious that the latent roots of the matrix $[\Gamma^i_j]$ are all zero. Consequently the space is simply harmonic.

The conclusions of this paper are all illustrated by the example of a simply harmonic K_4 given in Ruse (**10**). The K_4 in question was given by

$$ds^2 = F(\eta)\,x^4 d\eta^2 + 2dx\,dz + 2\eta\,dx\,dt + 2x\,d\eta\,dt,$$

where η is written for the y of the previous paper, $F(\eta)$ being any function of η. If we put

$$\eta = y/x,$$

this becomes $$ds^2 = F\left(\frac{y}{x}\right)(y\,dx - x\,dy)^2 + 2dx\,dz + 2dy\,dt,$$

which is of the form (6·6).

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The University
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ON THE SUMMABILITY (C) OF ALLIED SERIES AND THE EXISTENCE OF $(CP)\int_0^\pi \frac{f(x+t)-f(x-t)}{t}\,dt$

By W. L. C. SARGENT

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1. *Introduction*

The following theorems concerning the Cesàro summability of the allied series and Fourier series of a function $f(t)$ integrable in the Lebesgue sense are known.* In stating the theorems we suppose that $f(t)$ has period 2π and is integrable L in $(0, 2\pi)$. We write

$$\psi(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\}, \quad \phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)\},$$

and denote the allied series and Fourier series of $f(t)$ for $t = x$ by $\sum_{n=1}^{\infty} B_n(x)$ and $\sum_{n=0}^{\infty} A_n(x)$ respectively.

THEOREM A. *If* $\beta > \alpha \geqslant 0$ *and*†

$$\text{(i)} \lim_{t\to+0} \psi(t) = 0(C,\alpha), \quad \text{(ii)} \lim_{t\to+0} \frac{1}{\pi}\int_t^\pi \psi(u)\cot\tfrac{1}{2}u\,du = s(C),$$

then $\sum_{n=1}^{\infty} B_n(x)$ *is summable* (C,β) *to* s.

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* See L. S. Bosanquet and J. M. Hyslop (8, 491–492), and the references there given.

† By $\lim_{t\to+0} g(t) = l(C,\alpha)$, we mean that

$$\lim_{t\to+0} \frac{\alpha}{t^\alpha}\int_0^t (t-u)^{\alpha-1} g(u)\,du = l \quad \text{or} \quad \lim_{t\to+0} g(t) = l,$$

according as $\alpha > 0$ or $\alpha = 0$; by $\lim_{t\to+0} g(t) = l(C)$, we mean that $\lim_{t\to+0} g(t) = l(C,\alpha)$ for some $\alpha \geqslant 0$. In theorems A, B and C we replace the more usual conditions involving $\int_t^\infty \frac{\psi(u)}{u}\,du$ by equivalent conditions involving $\int_t^\pi \psi(u)\cot\tfrac{1}{2}u\,du$; in this form the theorems can be more easily extended to functions integrable in the Cesàro-Perron sense.

THEOREM B. *If** $\alpha > \beta + 1 \geqslant 0$ *and* $\sum_{n=1}^{\infty} B_n(x)$ *is summable* (C, β) *to* s, *then*

$$\text{(i)} \lim_{t\to+0} \psi(t) = 0(C,\alpha), \quad \text{(ii)} \lim_{t\to+0} \frac{1}{\pi}\int_t^\pi \psi(u)\cot\tfrac{1}{2}u\,du = s(C).$$

THEOREM C. *For* $\sum_{n=1}^{\infty} B_n(x)$ *to be summable*† (C) *to* s, *it is necessary and sufficient that*

$$\lim_{t\to+0} \frac{1}{\pi}\int_t^\pi \psi(u)\cot\tfrac{1}{2}u\,du = s(C),$$

or, what is equivalent, that $\psi(t)\cot\frac{1}{2}t$ *be integrable*‡ (CL) *in* $(0, \pi)$ *and*

$$\frac{1}{\pi}\int_{\to 0}^{\pi} \psi(t)\cot\tfrac{1}{2}t\,dt = s.$$

THEOREM D. *If* $\beta > \alpha \geqslant 0$ *and* $\lim_{t\to+0} \phi(t) = s(C,\alpha)$, *then* $\sum_{n=0}^{\infty} A_n(x)$ *is summable* (C, β) *to* s.

THEOREM E. *If* $\alpha > \beta + 1 \geqslant 0$ *and* $\sum_{n=0}^{\infty} A_n(x)$ *is summable* (C, β) *to* s, *then* $\lim_{t\to+0} \phi(t) = s\,(C, \alpha)$.

THEOREM F. *For* $\sum_{n=0}^{\infty} A_n(x)$ *to be summable* (C) *to* s, *it is necessary and sufficient that* $\lim_{t\to+0} \phi(t) = s\ (C)$.

These theorems have been extended by various writers to cases when $f(t)$ is not necessarily integrable L in $(0, 2\pi)$. In particular, it follows from a result due to Bosanquet§ that theorem C holds in the form stated for functions $f(t)$ integrable (CL) in $(0, 2\pi)$. Further, theorems D and E have been extended by Burkill|| and Bosanquet¶ to functions integrable in the Cesàro-Perron sense of integral order. Their results, for a function $f(t)$ integrable $C_\lambda P$ (λ a positive integer), can be stated in the following form:

THEOREM D_1. *If* $\beta > \alpha \geqslant \lambda + 1$ *and* $\lim_{t\to+0} \phi(t) = s(C,\alpha)$, *then* $\sum_{n=0}^{\infty} A_n(x)$ *is summable* (C, β) *to* s.

THEOREM E_1. *If* $\alpha > \beta + 1 \geqslant \lambda + 1$ *and* $\sum_{n=0}^{\infty} A_n(x)$ *is summable* (C, β) *to* s, *then* $\lim_{t\to+0} \phi(t) = s\ (C, \alpha)$.

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* Theorem B was obtained by Paley (**13**, theorem IV) in an equivalent form in the case $\beta \geqslant 0$; the case $-1 \leqslant \beta < 0$ can be obtained by adapting a method used by Bosanquet (**2**, theorem 4) and is included in theorem II proved below.

† I.e. summable by Cesàro means of some order.

‡ For the definition of the Cesàro-Lebesgue integral, see L. S. Bosanquet (**6**).

§ L. S. Bosanquet (**7**, 71).

|| See J. C. Burkill (**10**) for theorem D_1.

¶ See L. S. Bosanquet (**4**, 282) for theorem E_1; for similar extensions of theorems D and E to functions integrable in the general Denjoy sense, see L. S. Bosanquet (**2**, theorems 2 and 4).

It follows from these results, and from consistency theorems, that theorem F holds in the form stated for functions integrable (CP) in $(0, 2\pi)$.

In this paper, we suppose that $f(t)$ is integrable* $C_\lambda P$, where λ may be any positive number (not necessarily integral) or zero, and obtain extensions of theorems A, B and C. We show, in fact, that theorem B holds in the form stated† for functions integrable in the $C_\lambda P$ sense, whilst theorem A holds with the additional hypothesis‡ $\beta \geqslant \lambda+1$. There is then a corresponding extension of theorem C.

Theorems D and E can be extended in a similar way;§ we prove the results for the allied series, as they involve extra considerations concerning the integrability of $\psi(t)\cot\frac{1}{2}t$.

Finally, we deduce a result concerning the existence of

$$\int_0^\pi \frac{f(x+t)-f(x-t)}{t}\,dt. \tag{1·1}$$

It is well known that if $f(t)$ is absolutely integrable, then (1·1) exists as a Cauchy integral for almost all values of x. This result was obtained by Plessner,|| who later¶ extended his proof to show that if $f(t)$ is integrable in the C_0P or special Denjoy sense, then (1·1) exists as a special Denjoy integral for almost all values of x. In view of a result due to Marcinkiewicz and Zygmund,** it appears that this result also holds for the $C_\lambda P$ integral; if $f(t)$ is integrable in the $C_\lambda P$ sense, then (1·1) exists as a $C_\lambda P$ integral for almost all values of x.

2. *Notation and preliminary lemmas involving* (CP) *integrals*

The notation corresponds to that given by Bosanquet†† for the (CL) integral. We suppose throughout that $\lambda \geqslant 0$ and $\mu = \max(\lambda-1, 0)$.

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* For the definition of the $C_\lambda P$ integral, see J. C. Burkill (9).

† I am indebted to Dr Bosanquet for the suggestion that it should be possible to dispense with a restriction involving order of integrability in the extension of theorem B, and with the corresponding restriction $\beta \geqslant \lambda$ in theorem E_1. Cf. footnote † on p. 333 and footnote † on p. 334.

‡ It is shown below that the condition $\beta \geqslant \lambda+1$ is required.

§ See theorems IV and V below; these theorems also extend theorems D_1 and E_1.

|| A. Plessner (14). See also A. S. Besicovitch (1).

¶ A. Plessner (15). It has been shown by Marcinkiewicz (12, 65–68) that the result does not hold for the general Denjoy integral.

** J. Marcinkiewicz and A. Zygmund (11, 6); if a trigonometric series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is summable (C, k) in a measurable set E, where $k > -1$, then $\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx)$ is summable (C, k) almost everywhere in E.

†† L. S. Bosanquet (6).

If $f(t)$ is integrable $C_\lambda P$ in (a,b), we write $f(t)\in C_\lambda P$ in (a,b), and denote the value of the integral by $\int_a^b f(t)\,dt$, or, if we wish to emphasize the existence of an integral of order λ, by $C_\lambda P\int_a^b f(t)\,dt$. If $f(t)\in C_\lambda P$ in (a,b) for some unspecified λ, we write $f(t)\in(CP)$ in (a,b).

If $\alpha>0$ and $f(t)$ and $|d-t|^{\alpha-1}f(t)\in(CP)$ in* (c,d), then $C_\alpha(f,c,d)$, the αth mean of $f(x)$ in (c,d), is defined† by the equation

$$C_\alpha(f,c,d)=\int_c^d|d-t|^{\alpha-1}f(t)\,dt\Big/\int_c^d|d-t|^{\alpha-1}\,dt.$$

The mean of order 0, $C_0(f,c,d)$, is defined to be equal to $f(d)$.

If $\alpha\geqslant 0$, $p>-1$, and

$$C_\alpha(f,x,x+h)\sim lh^p\Gamma(p+1)\,\Gamma(\alpha+1)/\Gamma(\alpha+p+1)$$

as $h\to 0$, we write‡ $$f(x+h)\sim lh^p\quad(C,\alpha)$$

as $h\to 0$. The usual related notation is defined in the obvious way.

If $\lim_{h\to 0}f(x+h)=f(x)$ (C,α), $f(t)$ is said to be C_α continuous at the point x.

In most of this paper we are concerned with functions integrable (CP) in an interval $(0,a)$, where $a>0$. If $\alpha>0$ and $f(t)$ and $(x-t)^{\alpha-1}f(t)\in(CP)$ in $(0,x)$, where $x>0$, we write

$$F_\alpha(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)\,dt.$$

We also write $F_0(x)=f(x)$, and use a similar notation with other functions.

Since $F_\alpha(x)=x^\alpha C_\alpha(f,0,x)/\Gamma(\alpha+1)$, it follows that $f(x)\sim lx^p$ (C,α) as $x\to+0$ is equivalent to

$$F_\alpha(x)\sim lx^{\alpha+p}\,\Gamma(p+1)/\Gamma(\alpha+p+1)$$

as $x\to+0$.

We now state some well-known properties§ of the $C_\lambda P$ integral.

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* Or in (d,c) if $d<c$.

† By lemma II below, the value of $C_\alpha(f,c,d)$ does not depend on the order of integrability of $f(t)$. In the definition of C_α means given by Burkill (9, 220–221), and hence in the definitions of C_α limits and C_α continuity given previously, it was assumed that the integrals involved were of order $\max(\alpha-1,0)$; in view of lemma V, stated below, this restriction is no longer necessary.

‡ Thus $f(x+h)\sim lh^p\,(C,\alpha)$ as $h\to 0$ implies that $f(t)$ and $|x+h-t|^{\alpha-1}f(t)\in(CP)$ in $(x,x+h)$ for all sufficiently small values of $|h|$.

§ For lemma I, see L. S. Bosanquet (5); there is, of course, an analogue in which $f(t)\in C_\lambda P$ in $(a,b-\epsilon)$, and $\int_a^t f(u)\,du$ tends to a limit (C,λ) as $t\to b-0$. For lemmas II and III, see J. C. Burkill (9). Lemma IV follows by induction from lemma III and the definition of the functions $F_r(t)$.

LEMMA I. *In order that* $f(t) \in C_\lambda P$ *in* (a, b), *it is necessary and sufficient that* (i) $f(t) \in C_\lambda P$ *in* $(a+\epsilon, b)$ *whenever* $0 < \epsilon < b-a$, (ii) $\int_t^b f(u)\,du \in C_\mu P$ *in* (a, b) *and tends to a limit* (C, λ) *as* $t \to a+0$. *The value of the limit is then equal to* $C_\lambda P \int_a^b f(t)\,dt$.

LEMMA II. *If* $\lambda < \lambda'$ *and* $f(t) \in C_\lambda P$ *in* (a, b), *then* $f(t) \in C_{\lambda'} P$ *in* (a, b).

LEMMA III. *If* $f(t) \in C_\lambda P$ *in* $(0, a)$, *and if* $g(t)$ *and its successive derivatives are all bounded in* $(0, a)$, *then* $f(t)\,g(t) \in C_\lambda P$ *in* $(0, a)$, *and*

$$C_\lambda P \int_0^a f(t)\,g(t)\,dt = F_1(a)\,g(a) - C_\mu P \int_0^a F_1(t)\,g'(t)\,dt.$$

LEMMA IV. *If* $f(t) \in C_\lambda P$ *in* $(0, a)$ *and* r *is zero or a positive integer, then* $F_r(t)$ *is defined for* $0 \leqslant t \leqslant a$, *and*

$$F_{r+1}(t) = C_\nu P \int_0^t F_r(u)\,du \quad (0 \leqslant t \leqslant a),$$

where $\nu = \max(\lambda - r, 0)$.

We next state some results* which will be given in a subsequent paper.†

LEMMA V. *If* $f(t) \in (CP)$ *in* $(0, a)$, *then* $F_\alpha(t)$ *exists for almost all* t *in* $(0, a)$, *if* $\alpha > 0$, *and for all* t *in* $(0, a)$, *if* $\alpha \geqslant 1$. *Further, if* $F_\alpha(a)$ *exists and* $\alpha > \beta > 0$, *then*

$$F_\alpha(a) = \frac{1}{\Gamma(\alpha - \beta)} (CP) \int_0^a (a-t)^{\alpha-\beta-1} F_\beta(t)\,dt. \tag{2·1}$$

LEMMA VI. *If* $0 \leqslant \alpha < \alpha'$, $p > -1$ *and* $f(t) \sim lt^p\,(C, \alpha)$ *as* $t \to +0$, *then* $f(t) \sim lt^p\,(C, \alpha')$ *as* $t \to +0$.

LEMMA VII. *If* $\alpha \geqslant 0$, $p > -1$ *and* $f(t) \sim lt^p\,(C, \alpha)$ *as* $t \to +0$, *then*

$$F_1(t) \sim \frac{lt^{p+1}}{p+1}\,(C, \beta)$$

as $t \to +0$, *where* $\beta = \max(\alpha - 1, 0)$. *Conversely, if* $f(t) \in (CP)$ *in* $(0, a)$, $a > 0$, *and if* $F_1(t) \sim \dfrac{lt^{p+1}}{p+1}(C, \alpha - 1)$ *as* $t \to +0$, *where* $\alpha \geqslant 1$, $p > -1$, *then* $f(t) \sim lt^p\,(C, \alpha)$ *as* $t \to +0$.

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* For similar results involving (CL) integrals, see L. S. Bosanquet (**6**, theorems 8 and 9 and lemma 1). For the proof of (2·1) in the case when $f(t)$ and $(x-t)^{\beta-1} f(t) \in C_\gamma P$ in $(0, x)$ whenever $0 < x \leqslant a$, where $\gamma = \max(\beta - 1, 0)$, see J. C. Burkill (**9**, 221).

† The main results obtained in this paper are contained in lemma V, and arose out of a suggestion due to Dr Bosanquet (cf. footnote † on p. 332 and footnote † on p. 333). Lemmas VI and VII can be deduced without difficulty, and the proof of lemma VIII is based on lemma V and is similar to one already given for the $C_0 P$ integral (cf. W. L. C. Sargent (**17**) and the references there given).

LEMMA VIII. *If* $0<\alpha\leqslant 1$, $0<a<b$, *and* $f(t)\in(CP)$ *in* $(0,a)$, *then*

$$\left|\frac{1}{\Gamma(\alpha)}\int_0^a (b-t)^{\alpha-1}f(t)\,dt\right| \leqslant \operatorname*{ess-\overline{bound}}_{0<x<a} |F_\alpha(x)|.$$

The next results are similar to results obtained by Bosanquet* for the (CL) integral.

LEMMA IX. *If* $\alpha\geqslant 0$, $\lambda'=\max(\lambda,\alpha-1)$, $p>-1$, $q<p+1$ *and* (i) $f(t)\in C_\lambda P$ *in* $(0,a)$ *and is* $o(t^p)$ (C,α) *as* $t\to+0$, (ii) $k(t)$ *is defined in* $(0,a)$ *and* $k^r(t)$ *is* $O(t^{-q-r})$ *in* $(0,a)$, *for* $r=0,1,2,\ldots$, *then* $f(t)\,k(t)\in C_{\lambda'}P$ *in* $(0,a)$ *and is* $o(t^{p-q})$ (C,α) *as* $t\to+0$.

The proof is by induction with regard to α. It is based on the lemmas already stated, and follows the same lines as that of a similar result due to Bosanquet.†

LEMMA X. *If n is a positive integer,* $p>-1$, *and* (i) $f(t)\in(CP)$ *in* $(0,a)$ *and is* $o(t^p)$ (C) *as* $t\to+0$, (ii) $k(t)$ *is defined in* $(0,a)$ *and* $k^r(t)$ *is* $O(t^{-p-1-r})$ *in* $(0,a)$, *for* $r=0,1,2,\ldots$, *then in order that* $f(t)\,k(t)\in(CP)$ *in* $(0,a)$, *it is necessary and sufficient that* $F_n(t)\,k^n(t)\in(CP)$ *in* $(0,a)$. *Further, if* $f(t)\,k(t)\in(CP)$ *in* $(0,a)$, *then*

$$\int_0^a f(t)\,k(t)\,dt=\sum_{r=1}^n(-1)^{r-1}F_r(a)\,k^{r-1}(a)+(-1)^n\int_0^a F_n(t)\,k^n(t)\,dt.$$

It follows from lemma III that, whenever $0<t<a$,

$$\int_t^a f(u)\,k(u)\,du=\left[\sum_{r=1}^n(-1)^{r-1}F_r(u)\,k^{r-1}(u)\right]_t^a+(-1)^n\int_t^a F_n(u)\,k^n(u)\,du.$$

Since $f(t)$ is $o(t^p)$ (C) as $t\to+0$, it follows from lemma VII that $F_r(t)$ is $o(t^{p+r})$ (C) as $t\to+0$, r being a positive integer. Since $F_r(t)\in(CP)$ in $(0,a)$, it follows from lemma IX that $F_r(t)\,k^{r-1}(t)\in(CP)$ in $(0,a)$ and is $o(1)$ (C) as $t\to+0$. Hence‡ if one of the functions $\int_t^a f(u)\,k(u)\,du$, $\int_t^a F_n(u)\,k^n(u)\,du\in(CP)$ in $(0,a)$ and tends to a limit (C) as $t\to+0$, then so does the other. The results stated follow, by lemma I.

LEMMA XI. *If*§ $f(t)\in(CP)$ *in* $(0,a)$, *then in order that*

$$\lim_{t\to+0}\int_t^a\frac{f(u)}{u}\,du=s\quad(C,\delta),\tag{2·2}$$

* Cf. L. S. Bosanquet (**6**, theorems 2, 18 and 19).

† L. S. Bosanquet (**6**, 46–48).

‡ We use the consistency results given by lemmas II and VI. These results are implied by the notation, and we shall not refer to the lemmas explicitly every time.

§ Cf. R. E. A. C. Paley (**13**, 175–176) and L. S. Bosanquet (**6**, 56) for similar results.

where $\delta \geqslant 0$, it is necessary and sufficient that

$$\lim_{t\to+0} f(t) = 0 \quad (C, \delta+1), \tag{2·3}$$

and

$$\lim_{t\to+0} \int_t^a \frac{f(u)}{u} du = s \quad (C). \tag{2·4}$$

It follows from lemma III that $t^{-1}f(t) \in (CP)$ in (η, a) whenever $0 < \eta < a$. Since (2·4) implies that $\int^a \frac{f(u)}{u} du \in (CP)$ in some interval $(0, \eta)$, $\eta > 0$, whilst, by lemma I, $\int_t^a \frac{f(u)}{u} du \in (CP)$ in (η, a), it follows from lemma I that (2·4) is equivalent to

$$(CP)\int_0^a \frac{f(u)}{u} du = s. \tag{2·5}$$

Since (2·4) is included in (2·2), we assume throughout that (2·5) is satisfied; (2·2) is then equivalent to

$$\lim_{t\to+0} \int_0^t \frac{f(u)}{u} du = 0 \quad (C, \delta). \tag{2·6}$$

We therefore have to show that (2·3) follows from (2·6) and vice versa.

Write $g(t) = \int_0^t \frac{f(u)}{u} du$ for $0 \leqslant t \leqslant a$. Then, by lemma III,

$$F_1(t) = tg(t) - \int_0^t g(u)\, du. \tag{2·7}$$

If (2·6) is satisfied, it follows from (2·7) and lemmas VI, VII and IX that $F_1(t)$ is $o(t)\ (C, \delta)$ as $t \to +0$, and hence, by lemma VII, (2·3) is satisfied.

It remains to show that (2·6) follows from (2·3). Since $g(t)$ is $o(1)\ (C)$ as $t \to +0$, by* lemma I, we may suppose that $g(t)$ is $o(1)\ (C, \delta+n)$ as $t \to +0$, where n is a positive integer. Since

$$g(t) = \frac{1}{t}\int_0^t \{f(u) + g(u)\} du \quad (0 < t \leqslant a)$$

by (2·7), it follows from (2·3) and lemmas VI, VII and IX that $g(t)$ is $o(1)$ $(C, \delta+n-1)$ as $t \to +0$. By $n-1$ further applications of the argument used above, it follows that (2·6) is satisfied.

LEMMA XII. *If $f(t) \in C_\lambda P$ in $(0, a)$, then $\int_t^a \frac{f(u)}{u} du \in C_\mu P$ in $(0, a)$.*

Let n be a positive integer such that $n \geqslant \max(\lambda+1, 2)$. Write

$$\theta(t) = C_\lambda P \int_t^a \frac{f(u)}{u} du$$

* Or by (2·4) and (2·5).

for $0 < t \leqslant a$, and take v such that $0 < v < a$. Then, by lemma III,

$$\begin{aligned} C_\mu P\int_v^a \theta(t)\,dt &= [t\theta(t)]_v^a + C_\lambda P\int_v^a f(t)\,dt \\ &= -v\int_v^a \frac{f(t)}{t}\,dt + F_1(a) - F_1(v) \\ &= \left(1-\frac{v}{a}\right)F_1(a) - v\left[\sum_{r=2}^{n}(r-1)!\,\frac{F_r(t)}{t^r}\right]_v^a - vn!\int_v^a \frac{F_n(t)}{t^{n+1}}\,dt. \end{aligned}$$

If r is a positive integer greater than 2, it follows from lemmas II and IV that $F_r(t) \in C_\nu P$ in $(0,a)$, where $\nu = \max(\mu-1, 0)$. Also $F_n(t)$ is continuous for $0 \leqslant t \leqslant a$. Further, since $F_1(t)$ is $o(1)$ (C,λ) as $t \to +0$, by lemma I, it follows from lemmas VI and VII that $F_r(t)$ is $o(t^{r-1})$ (C,μ), whilst $F_n(t)$ is $o(t^{n-1})$ $(C, 0)$, as $t \to +0$. It therefore follows from lemma IX that $\sum_{r=2}^{n}(r-1)!\,v^{1-r}F_r(v)$, considered as a function of v, is integrable $C_\nu P$ in $(0,a)$ and is $o(1)$ (C,μ) as $v \to +0$. Further, it follows from the properties of $F_n(t)$ stated above that $v\int_v^a t^{-(n+1)}F_n(t)\,dt$ is bounded in $(0,a)$ and is $o(1)$ as $v \to +0$. Hence $C_\mu P\int_v^a \theta(t)\,dt \in C_\nu P$ in $(0,a)$ and tends to $F_1(a)$ (C,μ) as $v \to +0$. It therefore follows from lemma I that $\theta(t) \in C_\mu P$ in $(0,a)$.

LEMMA XIII. *If* $\alpha \geqslant 0$, $f(t) \in C_\lambda P$ *in* $(0,a)$, *and*

$$\lim_{t\to+0} f(t) = 0 \quad (C,\alpha), \tag{2·8}$$

$$\lim_{t\to+0}\int_t^a \frac{f(u)}{u}\,du = s \quad (C), \tag{2·9}$$

then $t^{-1}f(t) \in C_{\lambda'}P$ *in* $(0,a)$, *where* $\lambda' = \max(\lambda, \alpha-1)$, *and*

$$\int_0^a \frac{f(u)}{u}\,du = s.$$

Since $f(t)$ is $o(1)$ $(C,\lambda'+1)$ as $t \to +0$, by (2·8) and lemma VI, it follows from (2·9) and lemma XI that

$$\lim_{t\to+0}\int_t^a \frac{f(u)}{u}\,du = s \quad (C,\lambda').$$

Since $\int_t^a \frac{f(u)}{u}\,du \in C_{\mu'}P$ in $(0,a)$, where $\mu' = \max(\lambda'-1, 0)$, by lemmas II and XII, the result stated follows from lemma I.

We obtain one more lemma* which involves integrals of a sequence of functions.

* This lemma is fundamental in the proof that summability (C,β) of the Fourier series and allied series of a function integrable $C_\lambda P$ is a local property if $\beta \geqslant \lambda+1$.

LEMMA XIV. *If* $0<q\leqslant 1$ *and* $F(t) = C_0 P\int_a^t f(u)\,du$ *for* $a\leqslant t\leqslant b$, *where* $f(t)$ *is* C_q*-continuous for** $a\leqslant t\leqslant b$, *and if, for* $a\leqslant t\leqslant b$,

$$|\,k_n(t)\,|\leqslant An^{1-q} \quad (n=1,2,3,\ldots), \tag{2·10}$$

$$\left|\int_a^t k_n(u)\,du\right|\leqslant An^{-q} \quad (n=1,2,3,\ldots), \tag{2·11}$$

where A *is constant, then* $\lim_{n\to\infty}\int_a^b F(t)\,k_n(t)\,dt = 0$.

If $q=1$, the result stated follows from Lebesgue's convergence theorem.†

We therefore suppose that $0<q<1$. For convenience we shall also suppose that $f(t)$ vanishes for $t<a$ and for $t>b$. We proceed to show that every point c of the closed interval (a,b) is contained in an open interval $(c-\delta, c+\delta)$, such that

$$\lim_{n\to\infty}\int_c^d F(t)\,k_n(t)\,dt = 0, \tag{2·12}$$

whenever $c-\delta<d<c+\delta$. The result stated follows without difficulty, by means of Borel's covering theorem.

Since $f(x)$ is C_q-continuous at the point c, we can find δ such that $C_q(f,c,x)$ is bounded for $0<|\,x-c\,|\leqslant\delta$. We shall suppose that $c<d<c+\delta$; the case when $c-\delta<d<c$ can be considered in a similar way. We shall also suppose that $F(c)=0$; this is clearly legitimate, in view of (2·11).

Write
$$g(x) = \frac{1}{\Gamma(q)}\int_c^x (x-t)^{q-1} f(t)\,dt \quad (c<x<d),$$

$$h_n(x) = \frac{1}{\Gamma(1-q)}\int_x^d (t-x)^{-q}\,k_n(t)\,dt \quad (c<x<d).$$

Then $g(x)$ is bounded in (c,d), and it can be shown by a standard method‡ that $h_n(x)$ is bounded in (c,d). Further, if $c\leqslant\gamma<d$, then, by lemma V,

$$\int_\gamma^d h_n(t)\,dt = \frac{1}{\Gamma(2-q)}\int_\gamma^d (t-\gamma)^{1-q}\,k_n(t)\,dt$$
$$= \frac{(d-\gamma)^{1-q}}{\Gamma(2-q)}\int_\xi^d k_n(t)\,dt,$$

* It is understood that $f(t)$ need only be C_q-continuous on the right at a and on the left at b.

† This may be stated as follows: if $F(t)\in L$ in (a,b) and the functions $k_n(t)$ are uniformly bounded in (a,b), and if $\lim_{n\to\infty}\int_a^\lambda k_n(t)\,dt = 0$ whenever $a\leqslant\lambda\leqslant b$, then $\lim_{n\to\infty}\int_a^b F(t)k_n(t)\,dt = 0$.

‡ If $x<d-1/n$, we consider the integrals in $(x, x+1/n)$, $(x+1/n, d)$ separately. See L. S. Bosanquet (**3**, 196–197).

where $\gamma < \xi < d$, so that, by (2·11),

$$\lim_{n\to\infty}\int_{\gamma}^{d} h_n(t)\,dt = 0.$$

It follows from lemma V that

$$F(x) = \frac{1}{\Gamma(1-q)}\int_{c}^{x}(x-t)^{-q}g(t)\,dt$$

for $c < x < d$, and hence, by Fubini's theorem,

$$\int_{c}^{d} F(t)\,k_n(t)\,dt = \frac{1}{\Gamma(1-q)}\int_{c}^{d} g(t)\,dt\int_{t}^{d}(u-t)^{-q}k_n(u)\,du$$
$$= \int_{c}^{d} g(t)\,h_n(t)\,dt.$$

It therefore follows from Lebesgue's convergence theorem that (2·12) is satisfied. The lemma is thus established.

3. *Results involving allied series*

We suppose throughout this section that $f(t)$ is periodic, with period 2π, and that $f(t)\in C_\lambda P$ in $(0, 2\pi)$. We write

$$\psi(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\},\quad a_n+ib_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\,e^{nit}\,dt,$$
$$B_n(x) = b_n\cos nx - a_n\sin nx,\quad s_n = \sum_{\nu=0}^{n} B_\nu(x),$$

and denote the nth Cesàro mean* of order β ($\beta > -1$) of s_n by s_n^β. Then $\sum_{n=1}^{\infty} B_n(x)$ is the allied series of $f(t)$ for $t = x$, and

$$\psi(t) \sim \sum_{n=1}^{\infty} B_n(x)\sin nt, \tag{3·1}$$

$$s_n = \frac{1}{\pi}C_\lambda P\int_0^{\pi}\psi(t)\,\{\cot\tfrac{1}{2}t - \cos(n+\tfrac{1}{2})\,t\,\mathrm{cosec}\,\tfrac{1}{2}t\}\,dt, \tag{3·2}$$

$$s_n^\beta = \frac{1}{\pi}C_\lambda P\int_0^{\pi}\psi(t)\,\{\cot\tfrac{1}{2}t - k_n(t)\}\,dt, \tag{3·3}$$

where $k_n(t)$ is the nth Cesàro mean of order β of $\cos(n+\tfrac{1}{2})\,t\,\mathrm{cosec}\,\tfrac{1}{2}t$.

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* I.e. $s_n^\beta = \dfrac{1}{A_n^\beta}\sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1}s_\nu$, where A_n^r is the coefficient of x^n in the expansion of $(1-x)^{-r-1}$ in ascending powers of x; it is clear, of course, that $B_0(x) = 0$ and $s_0 = 0$.

If $\beta \geqslant 0$, it can be shown* that

$$\left| k_n^r(t) - \frac{d^r}{dt^r}\left(\frac{2}{t}\right) \right| \leqslant A n^{r+1} \quad (0 < t \leqslant \pi), \tag{3·4}$$

$$| k_n^r(t) | \leqslant B n^{r+1} (nt)^{-\rho} \quad \left(\frac{1}{n} \leqslant t \leqslant \pi\right), \tag{3·5}$$

where r is zero or a positive integer, $\rho = \min(r+2, \beta+1)$, and A and B are constants, which may depend on r but are independent of n and t.

If $p > 0$, we write $C_p(t)$, $\bar{C}_p(t)$ for Young's functions, defined by the equation

$$C_p(t) + i\bar{C}_p(t) = \frac{1}{\Gamma(p)} \int_0^t (t-u)^{p-1} e^{iu}\, du.$$

It is well known that $C_p(t)$ *is bounded for* $t \geqslant 0$, *if* $0 < p \leqslant 2$, whilst $t^{2-p} C_p(t)$ *is bounded for* $t > 0$, *if* $p > 2$. Further, if m is an odd positive integer, then†

$$C_p(t) = \sum_{r=0}^{\frac{1}{2}(m-1)} \frac{(-1)^r t^{p+2r}}{\Gamma(p+2r+1)} + (-1)^{\frac{1}{2}(m+1)} \bar{C}_{m+p}(t). \tag{3·6}$$

We next give three more lemmas.

LEMMA XV. *If* $\alpha > \beta + 1 \geqslant 0$ *and* Σu_n *is summable* (C, β), *then*

$$\lim_{\rho \to 1-0} \sum_{n=1}^{\infty} \frac{u_n}{n^\alpha} \bar{C}_\alpha(nt) \rho^n = o(t^\alpha)$$

as $t \to +0$, *and*

$$\frac{1}{t^{\alpha+1}} \lim_{\rho \to 1-0} \sum_{n=1}^{\infty} \frac{u_n}{n^\alpha} \bar{C}_\alpha(nt) \rho^n \in C_0 P$$

in $(0, \pi)$.

Lemma XV has been obtained by Bosanquet‡ in the case when $\beta \geqslant 0$, and can be obtained without difficulty when§ $-1 \leqslant \beta < 0$.

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* (3·4) is easily verified since $\cot \frac{1}{2}t - \cos(n+\frac{1}{2})t \operatorname{cosec} \frac{1}{2}t = 2 \sum_{\nu=0}^{n} \sin \nu t$, whilst (3·5) follows from working due to Zygmund (**20**, 258–259).

† Easily verified by integration by parts.

‡ L. S. Bosanquet (**7**, lemmas 7 and 8). It is shown in lemma 8 that

$$\int_t^\infty \frac{1}{t^{\alpha+1}} \left\{ \lim_{\rho \to 1-0} \sum_{n=1}^{\infty} \frac{u_n}{n^\alpha} \bar{C}_\alpha(nt) \rho^n \right\} dt$$

tends to a finite limit as $t \to +0$, and hence $\frac{1}{t^{\alpha+1}} \lim_{\rho \to 1-0} \sum_{n=1}^{\infty} \frac{u_n}{n^\alpha} \bar{C}_\alpha(nt) \rho^n \in C_0 P$ in any interval $(0, a)$, where $a > 0$.

§ Since $\bar{C}_\alpha(nt) = C_{\alpha+1}(nt)$, it can easily be verified that the series $\sum_{n=1}^{\infty} \frac{u_n}{n^\alpha} \bar{C}_\alpha(nt)$ is, in this case, uniformly convergent for $t \geqslant 0$. Since, for fixed n, $\bar{C}_\alpha(nt)$ is $O(t^{\alpha+1})$ as $t \to +0$, the results stated follow without difficulty.

LEMMA XVI. *In order that*

$$\lim_{t\to+0} \frac{1}{\pi}\int_t^\pi \psi(u)\cot\tfrac{1}{2}u\,du = s \quad (C,\delta),$$

where $\delta \geqslant 0$, it is necessary and sufficient that

$$\lim_{t\to+0} \psi(t) = 0 \quad (C,\delta+1),$$

and

$$\lim_{t\to+0} \frac{1}{\pi}\int_t^\pi \psi(u)\cot\tfrac{1}{2}u\,du = s \quad (C).$$

Write $g(t) = t\psi(t)\cot\frac{1}{2}t$ for $0<t\leqslant\pi$. Then, by lemma III, $g(t)\in C_\lambda P$ in $(0,\pi)$. Also, by lemma IX, $\lim_{t\to+0}\psi(t) = 0\ (C,\delta+1)$ is equivalent to $\lim_{t\to+0} g(t) = 0$ $(C,\delta+1)$. The result stated therefore follows from lemma XI.

LEMMA XVII. *If $\alpha\geqslant 0$ and*

$$\lim_{t\to+0} \psi(t) = 0 \quad (C,\alpha),$$

$$\lim_{t\to+0} \frac{1}{\pi}\int_t^\pi \psi(u)\cot\tfrac{1}{2}u\,du = s \quad (C),$$

then $\psi(t)\cot\frac{1}{2}t\in C_{\lambda'}P$ in $(0,a)$, where $\lambda' = \max(\lambda,\alpha-1)$, and

$$\frac{1}{\pi}\int_0^\pi \psi(t)\cot\tfrac{1}{2}t\,dt = s.$$

Define $g(t)$ as in lemma XVI. Then $\lim_{t\to+0} g(t) = 0\ (C,\alpha)$, and the result stated follows from lemma XIII.

We now obtain the main theorems.

THEOREM I. *If $\alpha\geqslant 0$ and**

$$\lim_{t\to+0} \psi(t) = 0 \quad (C,\alpha), \tag{3·7}$$

$$\lim_{t\to+0} \frac{1}{\pi}\int_t^\pi \psi(u)\cot\tfrac{1}{2}u\,du = s \quad (C), \tag{3·8}$$

then $\sum_{n=1}^{\infty} B_n(x)$ is summable (C,β) to s, where $\beta>\alpha$ if $\alpha\geqslant\lambda+1$ and $\beta\geqslant\lambda+1$ if $0\leqslant\alpha<\lambda+1$.

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* If $\alpha\geqslant 1$, it follows from lemma XVI that (3·7) and (3·8) are equivalent to

$$\lim_{t\to+0} \frac{1}{\pi}\int_t^\pi \psi(u)\cot\tfrac{1}{2}u\,du = s\,(C,\alpha-1);$$

if $0<\alpha<1$, then $\psi(t) = o(1)\,(C,1)$ as $t\to+0$, by (3·7) and lemma VI, and hence the limit in (3·8) exists in the ordinary sense.

We first point out that the condition* $\beta \geqslant \lambda+1$ is required. If $0<\beta<\lambda+1$, there is a function† $f(t)$ integrable in the $C_\lambda P$ sense in $(0, 2\pi)$, and continuous except at the origin, for which‡ $a_n \neq o(n^\beta)$ and $b_n \neq o(n^\beta)$ as $n \to \infty$. Thus (3·7) is satisfied with $\alpha = 0$, whenever $0<x<2\pi$, and it can be deduced from the result mentioned above due to Plessner§ that (3·8) is satisfied for almost all values of x in $(0, 2\pi)$. On the other hand, since $a_n \neq o(n^\beta)$ and $b_n \neq o(n^\beta)$ as $n\to\infty$, $\Sigma B_n(x)$ can only be summable (C, β) is a set of measure zero.

In view of consistency theorems for Cesàro sums and Cesàro limits, it is sufficient to consider the case when $\alpha = m+p$, $\beta = m+q$, where m is zero‖ or a positive integer, $0<p<q\leqslant 1$ and $m+q\geqslant \lambda+1$.

Since $m+q \geqslant \lambda+1$, it follows from lemmas II and IV that

$$\Psi_{m+1}(t) = C_0 P\int_0^t \Psi_m(u)\,du$$

and, in the case when $m>0$, $\Psi_m(t) = C_q P\int_0^t \Psi_{m-1}(u)\,du$ for $0\leqslant t\leqslant \pi$. Thus $\Psi_{m+1}(t)$ *is continuous for* $0\leqslant t\leqslant\pi$ and, if $m>0$, $\Psi_m(t)$ is C_q-*continuous for* $0\leqslant t\leqslant \pi$. Further, since $m+1>\alpha$, it follows from (3·7) and lemma VII that, as $t\to+0$,

$$\Psi_{m+1}(t) = o(t^{m+1}) \quad (C, 0). \tag{3·9}$$

It follows from (3·7), (3·8) and lemma XVII that $\psi(t)\cot\frac{1}{2}t \in (CP)$ in $(0,\pi)$ and $\frac{1}{\pi}\int_0^\pi \psi(t)\cot\frac{1}{2}t\,dt = s$. It therefore follows from (3·3) that $\psi(t)\,k_n(t) \in (CP)$ in $(0,\pi)$, and

$$\pi(s_n^\beta - s) = -\int_0^\pi \psi(t)\,k_n(t)\,dt.$$

Hence, by (3·4), (3·7) and lemma X, $\Psi_{m+1}(t)\,k_n^{m+1}(t)\in(CP)$ in $(0,\pi)$, and

$$\begin{aligned}\pi(s_n^\beta - s) &= \sum_{r=1}^{m+1}(-1)^r\,\Psi_r(\pi)\,k_n^{r-1}(\pi) + (-1)^m\int_0^\pi \Psi_{m+1}(t)\,k_n^{m+1}(t)\,dt\\ &= o(1) + (-1)^m\int_0^\pi \Psi_{m+1}(t)\,k_n^{m+1}(t)\,dt,\end{aligned}$$

as $n\to\infty$, by (3·5).

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* Cf. S. Pollard (**16**, 216) for a similar argument. It can be seen from the arguments given below that, if $\beta\geqslant\lambda+1$, summability (C,β) of $\Sigma B_n(x)$ is a local property.

† W. L. C. Sargent (**18**); for a similar construction in the case $\lambda = 0$, see E. C. Titchmarsh (**19**).

‡ It can be deduced from lemma XIV that a_n and b_n are $o(n^{\lambda+1})$ as $n\to\infty$.

§ If r is any positive integer, then $f(t)$ is absolutely integrable in $(1/r, 2\pi)$, and hence the limit in (3·8) exists in the ordinary sense for almost all x in $(1/r, 2\pi)$.

‖ m can be zero only if $\lambda = 0$ and $q = 1$.

Since, by (3·4), (3·9) and the continuity of $\Psi_{m+1}(t)$, $t\Psi_{m+1}(t)\,k_n^{m+1}(t)\in C_0P$ in $(0,\pi)$ and is $o(1)$ $(C,0)$ as $t\to+0$, whilst, by lemma I, $\int_t^\pi \Psi_{m+1}(u)\,k_n^{m+1}(u)\,du$ tends to a limit (C) as $t\to+0$, it follows from lemma XIII that

$$\Psi_{m+1}(t)\,k_n^{m+1}(t)\in C_0P$$

in $(0,\pi)$. It therefore follows from (3·4) and the continuity of $\Psi_{m+1}(t)$ that $t^{-(m+2)}\Psi_{m+1}(t)\in C_0P$ in $(0,\pi)$, and hence, by (3·4) and (3·9),

$$\lim_{n\to\infty}\int_0^{1/n}\Psi_{m+1}(t)\,k_n^{m+1}(t)\,dt=0.$$

If $m>0$, it follows from (3·5), lemma XIV and the C_q-continuity of $\Psi_m(t)$ that, whenever $0<\eta<\pi$,

$$\lim_{n\to\infty}\int_\eta^{\pi}\Psi_{m+1}(t)\,k_n^{m+1}(t)\,dt=0,$$

whilst, if $m=0$, this result follows from Lebesgue's convergence theorem.

In order to prove that $\sum_{n=1}^{\infty}B_n(x)$ is summable (C,β) to s, it therefore remains to show that

$$\lim_{\eta\to 0}\overline{\lim_{n\to\infty}}\left|\int_{1/n}^{\eta}\Psi_{m+1}(t)\,k_n^{m+1}(t)\,dt\right|=0. \tag{3·10}$$

The proof of (3·10) follows standard lines. In view of (3·7), we can take η so that $\Psi_\alpha(t)$ is bounded in $(0,\eta)$. Since

$$\Psi_{m+1}(t)=\frac{1}{\Gamma(1-p)}\int_0^t (t-u)^{-p}\,\Psi_\alpha(u)\,du$$

for $0<t\leqslant\pi$, by lemma V, it follows from Fubini's theorem that

$$\int_{1/n}^{\eta}\Psi_{m+1}(t)\,k_n^{m+1}(t)\,dt=\frac{1}{\Gamma(1-p)}\int_0^{1/n}\Psi_\alpha(u)\,du\int_{1/n}^{\eta}(t-u)^{-p}\,k_n^{m+1}(t)\,dt$$
$$+\frac{1}{\Gamma(1-p)}\int_{1/n}^{\eta}\Psi_\alpha(u)\,du\int_u^{\eta}(t-u)^{-p}\,k_n^{m+1}(t)\,dt.$$

It follows from (3·5) and the second mean value theorem that, whenever $0\leqslant u<1/n$,

$$\left|\int_{1/n}^{\eta}(t-u)^{-p}\,k_n^{m+1}(t)\,dt\right|\leqslant 2Bn^{m+1}(1/n-u)^{-p},$$

and it can be shown by a standard method* that

$$\left|\int_u^{\eta}(t-u)^{-p}\,k_n^{m+1}(t)\,dt\right|\leqslant Cn^{1+\alpha}(nu)^{-1-\beta}$$

for $1/n\leqslant u\leqslant\eta$, where C is a constant independent of n and η.

Since $\Psi_\alpha(t)$ is $o(t^\alpha)$ as $t\to+0$, by (3·7), it follows without difficulty that (3·10) is satisfied.

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* Cf. L. S. Bosanquet (3, 196–197).

THEOREM II. *If* $\alpha > \beta + 1 \geqslant 0$ *and* $\sum_{n=1}^{\infty} B_n(x)$ *is summable* (C, β) *to* s, *then*

$$\lim_{t \to +0} \psi(t) = 0 \quad (C, \alpha), \tag{3·7}$$

$$\lim_{t \to +0} \frac{1}{\pi} \int_t^{\pi} \psi(u) \cot \tfrac{1}{2} u \, du = s \quad (C). \tag{3·8}$$

By increasing β if necessary, it is sufficient to consider the case when $\beta + 1 < \alpha \leqslant [\beta] + 2$. Then $\alpha = m + p$, where $m = [\beta] + 1$ and $0 < p \leqslant 1$. We shall suppose that m is odd; if m is even (or zero) the procedure is similar.

It can be deduced* from (3·1) that there is a function $G(t)$ which differs from $\Psi_m(t)$ by a polynomial of degree $m-1$ and is such that†

$$G(t) \sim (-1)^{\frac{1}{2}(m+1)} \sum_{n=1}^{\infty} \frac{B_n}{n^m} \cos nt, \quad G_1(t) \sim (-1)^{\frac{1}{2}(m+1)} \sum_{n=1}^{\infty} \frac{B_n}{n^{m+1}} \sin nt,$$

where $B_n = B_n(x)$.

Since the (CP) integral $\Psi_{m+1}(t)$ is C-continuous, so is $G_1(t)$. It therefore follows from theorem‡ D_1, due to Burkill, that the Fourier series of $G_1(t)$ is summable (C) to $G_1(t)$ for all t. Since B_n is $o(n^{\beta})$ as $n \to \infty$, and $m > \beta$, this series is uniformly convergent for any range of values of t, and

$$G_1(t) = (-1)^{\frac{1}{2}(m+1)} \sum_{n=1}^{\infty} \frac{B_n}{n^{m+1}} \sin nt.$$

Take u, t such that $0 < u < t \leqslant \pi$. Then, by lemma III and the uniform convergence of the series on the right,

$$\begin{aligned}
&\int_0^u (t-v)^{p-1} G(v)\, dv \\
&\quad = (t-u)^{p-1} G_1(u) - (1-p) \int_0^u (t-v)^{p-2} G_1(v)\, dv \\
&\quad = (t-u)^{p-1} G_1(u) - (-1)^{\frac{1}{2}(m+1)} (1-p) \sum_{n=1}^{\infty} \frac{B_n}{n^{m+1}} \int_0^u (t-v)^{p-2} \sin nv\, dv \\
&\quad = (t-u)^{p-1} G_1(u) - (-1)^{\frac{1}{2}(m+1)} \sum_{n=1}^{\infty} \frac{B_n}{n^{m+1}} \\
&\qquad\qquad \times \left\{ (t-u)^{p-1} \sin nu - n \int_0^u (t-v)^{p-1} \cos nv\, dv \right\} \\
&\quad = (-1)^{\frac{1}{2}(m+1)} \sum_{n=1}^{\infty} \frac{B_n}{n^m} \int_0^u (t-v)^{p-1} \cos nv\, dv.
\end{aligned}$$

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* If $f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$, where $f(t)$ has period 2π and $f(t) \in (CP)$ in $(0, 2\pi)$, integration by parts gives

$$\frac{a_n}{n} = \frac{1}{\pi} \int_0^{2\pi} \{F_1(t) - \tfrac{1}{2} a_0 t\} \sin nt\, dt, \quad \frac{b_n}{n} = -\frac{1}{\pi} \int_0^{2\pi} \{F_1(t) - \tfrac{1}{2} a_0 t\} \cos nt\, dt.$$

† $G_1(t)$ is defined as $\int_0^t G(u)\, du$, in the usual way, and hence $G_1(t)$ differs from $\Psi_{m+1}(t)$ by a polynomial of degree m.

‡ Stated in the Introduction.

If $0<u\leqslant t$, then, by lemma VIII,

$$\left|\int_0^u (t-v)^{p-1}\cos nv\,dv\right| \leqslant \overline{\operatorname*{bound}_{0<y\leqslant t}}\left|\int_0^y (y-v)^{p-1}\cos nv\,dv\right|$$
$$= \overline{\operatorname*{bound}_{0<y\leqslant t}}\,|\,\Gamma(p)\,n^{-p}\,C_p(ny)\,|$$
$$\leqslant Kn^{-p},$$

where K is constant. Since B_n is $o(n^\beta)$ as $n\to\infty$, and $m+p-\beta>1$, it follows that $\int_0^u (t-v)^{p-1}G(v)\,dv$ is bounded in $(0,t)$ and tends to

$$(-1)^{\frac{1}{2}(m+1)}\Gamma(p)\sum_{n=1}^{\infty} n^{-(m+p)}B_n C_p(nt)$$

as $u\to t-0$. Hence, by lemma I and (3·6), $(t-v)^{p-1}G(v)\in(CP)$ in $(0,t)$, and

$$\frac{1}{\Gamma(p)}\int_0^t (t-v)^{p-1}G(v)\,dv$$
$$=(-1)^{\frac{1}{2}(m+1)}\sum_{n=1}^{\infty}\frac{B_n}{n^{m+p}}\left\{\sum_{r=0}^{\frac{1}{2}(m-1)}\frac{(-1)^r(nt)^{p+2r}}{\Gamma(p+2r+1)}+(-1)^{\frac{1}{2}(m+1)}\bar{C}_{m+p}(nt)\right\}.$$

Since $\Sigma n^{2r-m}B_n$ is summable* (C) and therefore summable (A) for $r=0,1,2,\ldots,\frac{1}{2}(m-1)$, it follows that

$$\frac{1}{\Gamma(p)}\int_0^t (t-v)^{p-1}G(v)\,dv=\sum_{r=0}^{\frac{1}{2}(m-1)}c_r t^{p+2r}+\lim_{\rho\to1-0}\sum_{n=1}^{\infty}\frac{B_n}{n^{m+p}}\bar{C}_{m+p}(nt)\,\rho^n,$$

where $$c_r=\lim_{\rho\to1-0}\left\{\sum_{n=1}^{\infty}(-1)^{\frac{1}{2}(2r+m+1)}n^{2r-m}B_n\rho^n\right\}\Big/\Gamma(p+2r+1),$$

and hence†

$$\frac{1}{\Gamma(p)}\int_0^t (t-v)^{p-1}\Psi_m(v)\,dv=t^pP(t)+\lim_{\rho\to1-0}\sum_{n=1}^{\infty}\frac{B_n}{n^{m+p}}\bar{C}_{m+p}(nt)\,\rho^n, \quad (3\cdot11)$$

where $P(t)$ is a polynomial in t of degree $m-1$.

It follows from (3·11) and lemma XV that, as $t\to+0$,

$$\frac{1}{\Gamma(p)}\int_0^t (t-v)^{p-1}\Psi_m(v)\,dv=t^pP(t)+o(t^{m+p}).$$

Since, however, $\Psi_1(t)$ is $o(1)$ (C) as $t\to+0$, by lemma I, it follows from lemma VII that‡ $\Psi_m(t)$ is $o(t^{m-1})$ (C) as $t\to+0$. In view of lemma VI, the polynomial $P(t)$ must therefore be identically zero, and hence $\Psi_m(t)$ is $o(t^m)$ (C,p) as $t\to+0$. It follows from lemma VII that (3·7) is satisfied.

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* Cf. L. S. Bosanquet (**4**, 277).

† (3·11) can be established whether m is even or odd, $P(t)$ being zero if m is zero.

‡ If $m=0$, this argument is not required.

Since $P(t)$ is identically zero, it follows from (3·11) and lemma XV that $t^{-(m+p+1)}\int_0^t (t-u)^{p-1}\Psi_m(u)\,du \in C_0 P$ in $(0,\pi)$. This result holds whenever $m+1 \geqslant m+p > \beta+1$, and hence, taking $p = 1$, $t^{-(m+2)}\Psi_{m+1}(t) \in C_0 P$ in $(0,\pi)$. Since (3·7) has already been established, it follows from lemma X that $t^{-1}\psi(t) \in (CP)$ in $(0,\pi)$; (3·8) then follows from lemmas III and I and theorem I.

THEOREM III. *In order that* $\sum_{n=1}^{\infty} B_n(x)$ *be summable* (C) *to* s, *it is necessary and sufficient that* $\psi(t)\cot\frac{1}{2}t \in (CP)$ *in* $(0,\pi)$, *and*

$$\frac{1}{\pi}\int_0^{\pi} \psi(t)\cot\tfrac{1}{2}t\,dt = s.$$

By theorems I and II and lemma XVI, in order that $\sum_{n=1}^{\infty} B_n(x)$ be summable (C) to s, it is necessary and sufficient that

$$\lim_{t\to+0}\frac{1}{\pi}\int_t^{\pi}\psi(u)\cot\tfrac{1}{2}u\,du = s \quad (C),$$

and this is equivalent* to the condition given.

4. *Results involving Fourier series*

The following theorems can be obtained by methods similar to those used for the corresponding theorems on allied series. In stating the theorems we suppose, as before, that $f(t)$ has period 2π and is integrable $C_\lambda P$ in $(0, 2\pi)$. We write $\phi(t) = \frac{1}{2}\{f(x+t)+f(x-t)\}$, and denote the Fourier series of $f(t)$ for $t = x$ by $\sum_{n=0}^{\infty} A_n(x)$.

THEOREM IV. *If* $\alpha \geqslant 0$ *and* $\lim_{t\to+0}\phi(t) = s\,(C,\alpha)$, *then* $\sum_{n=0}^{\infty} A_n(x)$ *is summable* (C,β) *to* s, *where* $\beta > \alpha$ *if* $\alpha \geqslant \lambda+1$ *and*† $\beta \geqslant \lambda+1$ *if* $0 \leqslant \alpha < \lambda+1$.

THEOREM V. *If* $\alpha > \beta+1 \geqslant 0$ *and* $\sum_{n=1}^{\infty} A_n(x)$ *is summable* (C,β) *to* s, *then* $\lim_{t\to+0}\phi(t) = s\,(C,\alpha)$.

THEOREM VI. *In order that* $\sum_{n=0}^{\infty} A_n(x)$ *be summable* (C) *to* s, *it is necessary and sufficient that* $\lim_{t\to+0}\phi(t) = s\,(C)$.

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* By lemma I; cf. the argument at the beginning of lemma XI.

† By arguments similar to those used in theorem I, the condition $\beta \geqslant \lambda+1$ is required.

It should be observed that theorem IV extends theorem D_1 to the case $\alpha < \lambda + 1$, as well as to non-integral values of λ; theorem V extends theorem E_1 to the case $-1 \leqslant \beta < \lambda$. Theorem VI follows from theorems D_1 and E_1 or from theorems IV and V.

5. *Existence of* $(CP)\int_0^\pi \frac{f(x+t)-f(x-t)}{t}\,dt$

THEOREM VII. *If $f(t) \in C_\lambda P$ in $(-\pi, 2\pi)$, then*

$$\int_0^\pi \frac{f(x+t)-f(x-t)}{t}\,dt$$

exists as a $C_\lambda P$ integral for almost all x in $(0, \pi)$.

Since $\{f(x+t)-f(x-t)\}/t \in C_\lambda P$ in $(\frac{1}{2}\pi, \pi)$, by lemma III, we may alter the values of $f(t)$ outside the interval $(-\frac{1}{2}\pi, \frac{3}{2}\pi)$, so that it has period 2π.

Since $F_1(x) = C_\lambda P \int_0^x f(t)\,dt$, $f(x)$ is the C_λ derivative of $F_1(x)$ almost everywhere,* and hence

$$\lim_{t \to +0} f(x+t) = f(x) \quad (C, \lambda+1) \tag{5·1}$$

for almost all values of x. If we define $\phi(t)$ and $\psi(t)$ as before, it follows that, for almost all values of x,

$$\lim_{t \to +0} \phi(t) = f(x) \quad (C, \lambda+1), \tag{5·2}$$

and

$$\lim_{t \to +0} \psi(t) = 0 \quad (C, \lambda+1). \tag{5·3}$$

It follows from (5·2) and theorem† IV that, for almost all values of x, the Fourier series of $f(t)$ is summable (C) to $f(x)$ for $t = x$. By a result due to Marcinkiewicz and Zygmund,‡ the allied series of $f(t)$ is summable (C) almost everywhere. Hence, by theorem II, (5·3) and lemma XVII,

$$\psi(t) \cot \tfrac{1}{2}t \in C_\lambda P$$

in $(0, \pi)$ for almost all values of x. The result stated follows from lemma III.

Finally, I should like to thank Dr Bosanquet for many valuable criticisms and suggestions.

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* I.e. $f(x) = \lim_{h \to 0} \{[C_\lambda(F_1, x, x+h) - F_1(x)]/[h/(\lambda+1)]\}$. This is equivalent to $\int_x^{x+h} f(t)\,dt \sim hf(x)\,(C, \lambda)$ as $h \to 0$, and hence, by lemma VII, it is equivalent to (5·1).

† Or from theorem D_1.

‡ J. Marcinkiewicz and A. Zygmund (**11**, 6 and 25–30).

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Royal Holloway College
Englefield Green
Surrey

INVARIANT THEORY UNDER ORTHOGONAL GROUPS

By D. E. LITTLEWOOD

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1. *Introduction*

The bulk of the traditional work on invariant theory is concerned with the full linear group. Concomitants under the orthogonal groups have been studied† by adjoining a fundamental quadric, and seeking the simultaneous concomitants of the ground forms and the quadric. This procedure adds considerably to the complexity of the problems. An alternative method of considering the invariant theory of orthogonal groups, among other restricted groups, was considered in a previous paper,‡ to which frequent reference will be made. It will be referred to by the shortened title of "Restricted Groups". The method of characteristic analysis employed involved the expression of the characters of the orthogonal group in terms of S-functions, and the calculation was in terms of S-functions. This conversion to S-functions involved considerable complexity as compared with the case of the full linear group.

The invariant theory of spinors§ was discussed in another paper and the concomitants found of a basic spinor in 3–8 variables.

All methods seemed over-elaborate in comparison with the simplicity of the results obtained, and search was made for a simpler procedure. This search has brought some success in the case of orthogonal groups in a small number of variables, more especially in 3 and 4 variables.

The fundamental principle employed in this paper is the comparison of an orthogonal space with an auxiliary space whose coordinates are the components of a basic spinor, or, in the case of the 4-dimensional rotation group, of two conjugate basic spinors. There is a complete correspondence, usually two-one, between these spaces, and to each algebraic form, tensor

† Turnbull (**14**).
‡ Littlewood (**8, 9**).
§ Littlewood (**10**).

or spinor, in the orthogonal space, there corresponds a form or tensor in the auxiliary space. The invariant theories of the two spaces are in complete correspondence, and a problem in the orthogonal space may be translated into a problem in the auxiliary space.

Hence the invariant theory of ternary orthogonal forms is identical with the invariant theory of binary forms under the full linear group. The group of the spin transformations is, strictly, the unitary group, but invariant theory is identical with that under the full linear group save that the determinant of the transformation is unity.

The rotation group in 4 variables is similarly found to correspond to the double binary group. Concomitants under the full orthogonal group, that is, the group including transformations of negative determinant, are easily deduced from a knowledge of concomitants under the rotation group.

In a like manner the orthogonal group in 5 variables corresponds to the symplectic group in 4 variables; the rotation group in 6 variables to the full linear group (unitary group) in 4 variables. The last correspondence has been noticed by Turnbull and employed to find the concomitants of a quadratic complex in 4 variables, though all the possibilities of the correspondence have not yet been exploited.

In 8 variables the rotation group possesses an automorphism in which a basic spinor, or rather one of the conjugate components of a basic spinor, corresponds to a simple vector. In this correspondence the orthogonal group in 7 variables corresponds to a sub-group of the 8-variable rotation group in which this component of the basic spinor is kept invariant.

2. *Note on two results which led to the central theory*

A very short account is given first of two significant results which led to the more general theory on which the paper is based. Since the two results are consequences of the more general theory discussed later, it is not thought worth while to give more than the most brief description here.

A method of studying the invariant-theory of a group which is a direct product was mentioned in "Restricted Groups". It is known that every 4-dimensional rotation is expressible as the resultant of a right-handed and a left-handed rotation which mutually commute.† The rotation group is not quite the direct product of the two groups of right-handed and left-handed rotations respectively, since the rotation J with matrix diag $(-1\ -1\ -1\ -1)$ belongs to both groups. But the rotation J commutes with every rotation, and generates a self-conjugate sub-group of order 2, the quotient group corresponding to which is a direct product. In fact, it

† Littlewood (3).

is the direct product of two groups, each simply isomorphic with the ternary rotation group.†

Hence the group of rotations in 4 variables is in 2 : 1 correspondence with the double ternary rotation group. The two-valued nature of this correspondence is in no way embarrassing, and fits in perfectly with the two-valued nature of the spin representations. Invariant theory under the quaternary rotation group proves to be identical with the invariant theory of the double ternary rotation group. Results for this group are readily deducible by the methods given in "Restricted Groups" from the invariant-theory of the simple ternary rotation group.

To make the method really effective, a simple technique for the ternary rotation group was thought to be necessary.

It was remarked that the latent roots of a ternary orthogonal matrix were of the form

$$1, \quad e^{i\theta}, \quad e^{-i\theta},$$

and could thus be arranged to be in geometric progression. Further, the character of any irreducible representation was expressible as

$$\begin{aligned}[n] &= 1+2\cos\theta+2\cos 2\theta+\ldots+2\cos n\theta \\ &= e^{-ni\theta}+e^{(1-n)i\theta}+\ldots+e^{-i\theta}+1+e^{i\theta}+\ldots+e^{ni\theta},\end{aligned}$$

and the latent roots again are in geometric progression.

In a paper by Littlewood and Richardson‡ a formula was given for the S-functions of a set of quantities in geometric progression. From this may be obtained very easily the following generating function for concomitants under the ternary orthogonal group:

If $$[n]\otimes\{p\} = \Sigma K_r[r],$$

then K_p *is the coefficient of* ρ^{-r} *in the expansion of*

$$\rho^{-np}\frac{(1-\rho^{2n+1})(1-\rho^{2n+2})\ldots(1-\rho^{2n+p})}{(1-\rho^2)(1-\rho^3)\ldots(1-\rho^p)}. \qquad (2\cdot 1)$$

This generating function, which holds also for spinors if n is half an odd integer, is very similar to a generating function obtained for binary concomitants.§ Closer examination showed that the differences were only differences of convention, and that the generating functions were essentially identical. Thus the following theorem came to light:

The concomitants of a form of type $[n]$ *under the ternary orthogonal group are in complete correspondence with the concomitants of a form of type* $\{2n\}$ *under the binary full linear group.*

† Littlewood (**3**).
‡ Littlewood and Richardson (**11**).
§ Littlewood (**4**).

In the correspondence a spinor of type $[n+\frac{1}{2}]$ corresponds to a form of type $\{2n+1\}$. The correspondence also extends to the simultaneous concomitants of several ground-forms.

In particular, a basic spinor of type $[\frac{1}{2}]$, which has two components, corresponds to the binary vector of type $\{1\}$. The identification of the binary variables with the two components of a basic spinor led immediately to a direct proof of the correspondence.

The case of the 4-variable rotation group then appeared as an obvious extension of the same principle, and further extensions to 5, 6 and 8 variables followed.

The principal theory will now be developed with more careful detail.

3. *The ternary orthogonal group*

There is a basic spin representation of degree 2 of the ternary orthogonal group. The second induced matrix of this spin transformation matrix is equivalent to the original orthogonal matrix. Denote a character of the orthogonal group by $[n]$, and a binary S-function by $\{m\}$. Then the corresponding equation is

$$[\tfrac{1}{2}]\otimes\{2\} = [1].$$

Hence if (ξ, η) are the components of a basic spinor, the coefficients of the quadratic form

$$a\xi^2+b\eta^2+h\xi\eta \tag{3·1}$$

transform in an equivalent manner to the coefficients in the ternary orthogonal form. The variables in the ternary form may be taken so that the transforming matrix is the same as for the quadratic form (3·1). Then if the ternary form is

$$ax+by+hz, \tag{3·2}$$

the coefficients in the two forms will remain the same under all transformations.

The invariant quadratic, or metric, associated with the orthogonal form must then clearly be

$$xy-z^2, \tag{3·3}$$

since

$$\xi^2\eta^2-(\xi\eta)^2$$

is identically zero and thus invariant.

A complex transformation would, of course, transform (3·3) into the more familiar metric

$$x^2+y^2+z^2 \tag{3·4}$$

if this special form were required.

An algebraic form of degree $2n$ in ξ, η may thus be associated with an algebraic form of degree n in x, y, z, the coefficients being the same in the two forms. Further, since xy may be replaced by z^2, the linear invariants

of the n-ic in x, y, z, i.e. those terms which involve the factor $(xy - z^2)$, do not appear. Thus the principal part of the n-ic only is involved, and it is an algebraic form of type $[n]$. To every form of type $\{2n\}$ in ξ, η there is a form of type $[n]$ in x, y, z which has the same coefficients.

The invariant theory of the coefficients is thus identical for the binary $2n$-ic and for the ternary orthogonal n-ic.

The spinors fit completely into the theory, for a spinor of type $[n + \frac{1}{2}]$ could be regarded as the coefficients of a polynomial of degree n in the variables, and of degree 1 in ξ, η. Such a form is in obvious correspondence with a form of degree $(2n + 1)$ in ξ, η.

It may be remarked here that the method is already well known of using the various terms in an algebraic form as new variables, so as to associate the form with another algebraic form of a different degree in a different number of variables. Thus with two variables x, y, if the powers and products of degree 2, namely, x^2, xy, y^2, are treated as new variables, the binary quartic can be associated with a ternary quadratic. Such a treatment is the basis of Sylvester's "unravelment",† and recently Edge‡ has further discussed the procedure.

Sylvester points out that a concomitant of the form in the new variables necessarily gives a concomitant in the original variables. The identity of the systems of irreducible concomitants in the two sets of variables does not, however, follow from this. In general, the group of transformations induced in the new variables is some sub-group of the full linear group, and Sylvester's result concerning the concomitants depends on the fact that the concomitants under one group are necessarily concomitants under a sub-group. The identity of the systems of irreducible concomitants, however, depends on the complete isomorphism between the groups. It is such isomorphisms which in the various instances are established here. The isomorphism is not, in any of the instances, complete, without the introduction of spinors. These were, of course, unknown to Sylvester.

Examples. The concomitants up to degree 5 in the coefficients were given in "Restricted Groups" for a ternary orthogonal cubic, the types being as follows:

Degree 1; [3].

Degree 2; [4], [2], [0].

Degree 3; [6], [4], [3], [1].

Degree 4; [5], [3], [2], [0].

Degree 5; [4], [2], [1].

† Sylvester (**12**), vol. 1, 284.

‡ Edge (**1**).

The generating function (2·1) for concomitants of degree 6 gives

$$\rho^{-18}\frac{(1-\rho^{7})(1-\rho^{8})(1-\rho^{9})(1-\rho^{10})(1-\rho^{11})(1-\rho^{12})}{(1-\rho^{2})(1-\rho^{3})(1-\rho^{4})(1-\rho^{5})(1-\rho^{6})}$$
$$= \rho^{-18}+\rho^{-16}+\rho^{-15}+2\rho^{-14}+2\rho^{-13}+4\rho^{-12}+2\rho^{-11}+5\rho^{-10}+4\rho^{-9}$$
$$+6\rho^{-8}+4\rho^{-7}+7\rho^{-6}+3\rho^{-5}+6\rho^{-4}+3\rho^{-3}+4\rho^{-2}+3-3\rho$$
$$-4\rho^{3}-3\rho^{4}-6\rho^{5}-3\rho^{6}-7\rho^{7}-4\rho^{8}-6\rho^{9}-4\rho^{10}-5\rho^{11}-2\rho^{12}-4\rho^{13}$$
$$-2\rho^{14}-2\rho^{15}-\rho^{16}-\rho^{17}-\rho^{19}.$$

Thus
$$[3]\otimes\{6\} = [18]+[16]+[15]+2[14]+2[13]+4[12]+2[11]$$
$$+5[10]+4[9]+6[8]+4[7]+7[6]+3[5]+6[4]$$
$$+3[3]+4[2]+3[0].$$

Deleting the reducible concomitants, we obtain for the types of irreducibles:

Degree 6; [3], [3], [0].

Order for sextic	0	2	4	6	8	10	12
Order for cubic	0	1	2	3	4	5	6
Degree							
1	—	—	—	f	—	—	—
2	$(ff)^6$	—	$(ff)^4=i$	—	$(ff)^2=H$	—	—
3	—	$(fi)^4=l$	—	$(fi)^2$	(fi)	—	$(fH)=t$
4	$(ii)^4$	—	$(fl)^2$	(fl)	—	(Hi)	—
5	—	$(il)^2$	(il)	—	(Hl)	—	—
6	$(ll)^2$	—	—	$((fi)^2 l)$	—	—	—
	—	—	—	$((fi)l)^2$	—	—	—
7	—	$(fl^2)^4$	$(fl^2)^3$	—	—	—	—
8	—	$(il^2)^3$	—	—	—	—	—
9	—	—	$((fi)e^2)^4$	—	—	—	—
10	$(fl^3)^6$	$(fl^3)^5$	—	—	—	—	—
12	—	$((fi)l^3)^6$	—	—	—	—	—
15	$((fi)l^4)^8$	—	—	—	—	—	—

There is no need, however, to proceed with this method. Grace and Young† give the complete system of 26 irreducible concomitants for a binary sextic. The table is reproduced above. By reason of the correspondence between the ternary orthogonal group and the binary full linear group, the same table gives the complete set of 26 concomitants of a ternary orthogonal cubic. The orders of the forms, instead of reading respectively 0, 2, 4, 6, 8, 10, 12, should then read 0, 1, 2, 3, 4, 5, 6. The symbol $(f,h)^i$, instead

† Grace and Young (2).

of denoting the ith transvectant of the product fh, would then denote the tensor obtained by contracting the tensor product fh with the metric tensor i times. Thus, if the coefficients of the ternary cubic are f^{ijk}, and g_{ij} denotes the metric tensor, the $(f\,f)^6$ denotes the contraction

$$g_{ip}\, g_{jq}\, g_{kr} f^{ijk} f^{pqr}.$$

In a similar manner it follows that, since a binary cubic has 4 irreducible concomitants of the following types:

Degree 1; $\{3\}$,
Degree 2; $\{2\}$,
Degree 3; $\{3\}$,
Degree 4; $\{0\}$,

then a ternary spinor of type $[\frac{3}{2}]$ has 4 irreducibles of types:

Degree 1; $[\frac{3}{2}]$,
Degree 2; $[1]$,
Degree 3; $[\frac{3}{2}]$,
Degree 4; $[0]$.

The irreducible systems for the ternary orthogonal group may be taken as known from the known results for binary forms, for the following types:†

$$[\tfrac{1}{2}],\ [1],\ [\tfrac{3}{2}],\ [2],\ [\tfrac{5}{2}],\ [3],\ [\tfrac{7}{2}],\ [4].$$

Simultaneous systems are known for $[\frac{1}{2}]$ and $[\frac{3}{2}]$, $[1]$ and $[\frac{3}{2}]$, $[1]$ and $[2]$.

4. *The quaternary orthogonal group*

Two groups are called orthogonal in 4 variables, that which comprises the transformations of positive determinant only, and that which includes also transformations of negative determinant. The former will be called the *rotation group* irrespective of whether the metric is the sum of the squares, as is usually understood in a rotation, or any other non-singular quadratic form; the latter the full orthogonal group.

The basic spinor for the full orthogonal group is of degree 4; for the rotation group the spinor separates into two conjugate parts each of degree 2. This makes possible a considerable simplification for the rotation group, and this group will be mainly considered here. Invariants under the full orthogonal group can be readily deduced from a knowledge of the invariants under the rotation group.

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† Turnbull (14), 246.

The character $[\lambda_1, \lambda_2]$ of the full orthogonal group separates into two conjugate characters for the rotation group if $\lambda_2 \neq 0$. It is convenient to denote these by $[\lambda_1, \lambda_2]$ and $[\lambda_1, -\lambda_2]$. The distinction between the two is conventional. Using a dash to denote a character of the full orthogonal group, this separation can be expressed by the equation

$$[\lambda_1, \lambda_2]' = [\lambda_1, \lambda_2] + [\lambda_1, -\lambda_2].$$

In particular $$[1\ 1]' = [1\ 1] + [1\ -1].$$

Thus a 6-vector under the full orthogonal group separates into two parts with 3 components each, under the rotation group. The 6 independent rotations in 4 dimensions thus separate into two sets of 3 which are called respectively *right-handed* and *left-handed* rotations. A right-handed rotation consists of a rotation in one plane combined with an equal rotation about this plane as axis; for a left-handed rotation the direction of the rotation about the plane as axis is reversed.

If the latent roots of the orthogonal matrix are

$$e^{i\alpha}, \quad e^{-i\alpha}, \quad e^{i\beta}, \quad e^{-i\beta},$$

then the basic spin character is given by

$$\begin{aligned}[\tfrac{1}{2}\ \tfrac{1}{2}]' &= 4\cos\tfrac{1}{2}\alpha\ \cos\tfrac{1}{2}\beta \\ &= e^{\frac{1}{2}i(\alpha+\beta)} + e^{-\frac{1}{2}i(\alpha+\beta)} + e^{\frac{1}{2}i(\alpha-\beta)} + e^{\frac{1}{2}i(\beta-\alpha)}.\end{aligned}$$

For the rotation group this separates as follows,

$$\begin{aligned}[\tfrac{1}{2}\ \tfrac{1}{2}] &= e^{\frac{1}{2}i(\alpha+\beta)} + e^{-\frac{1}{2}i(\alpha+\beta)}, \\ [\tfrac{1}{2}\ -\tfrac{1}{2}] &= e^{\frac{1}{2}i(\alpha-\beta)} + e^{\frac{1}{2}i(\beta-\alpha)}.\end{aligned}$$

The signs attached to α and β are conventional, as is the distinction between $[\frac{1}{2}\ \frac{1}{2}]$ and $[\frac{1}{2}\ -\frac{1}{2}]$. A convention is adopted concerning the relative signs of α and β which ensures that $\alpha = \beta$ for a right-handed rotation and $\alpha = -\beta$ for a left-handed rotation.

It follows that for a right-handed rotation $[\frac{1}{2}\ -\frac{1}{2}] = 2$, and the corresponding spinor is transformed by the identical matrix. Similarly for a left-handed rotation $[\frac{1}{2}\ \frac{1}{2}] = 2$.

Thus, if any rotation is expressed as the resultant of a right-handed and a left-handed rotation, the spin matrix of character $[\frac{1}{2}\ \frac{1}{2}]$ depends only on the right-handed part, and the spin matrix of character $[\frac{1}{2}\ -\frac{1}{2}]$ on the left-handed part.

The two conjugate parts of the basic spinor are thus algebraically independent. The simultaneous concomitants of two spinors of types $[\frac{1}{2}\ \frac{1}{2}]$ and $[\frac{1}{2}\ -\frac{1}{2}]$ respectively are next considered.

Since $$[\tfrac{1}{2}\ \tfrac{1}{2}] = e^{\frac{1}{2}i(\alpha+\beta)} + e^{-\frac{1}{2}i(\alpha+\beta)},$$

it follows that
$$\begin{aligned}[\tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{2\} &= 1+e^{i(\alpha+\beta)}+e^{i(\alpha+\beta)}\\ &= 1+2\cos(\alpha+\beta).\end{aligned}$$

Thus
$$\begin{aligned}[\tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{2\}+[\tfrac{1}{2}\ -\tfrac{1}{2}]\otimes\{2\} &= 1+2\cos(\alpha+\beta)+1+2\cos(\alpha-\beta)\\ &= 2+4\cos\alpha\cos\beta\\ &= [1\ 1]'\\ &= [1\ 1]+[1\ -1].\end{aligned}$$

The identification
$$\begin{aligned}[\tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{2\} &= [1\ 1],\\ [\tfrac{1}{2}\ -\tfrac{1}{2}]\otimes\{2\} &= [1\ -1]\end{aligned}$$
is inevitable.

Similarly,
$$\begin{aligned}&[\tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{p\}+[\tfrac{1}{2}\ -\tfrac{1}{2}]\otimes\{p\}\\ &\quad= 2\cos\tfrac{1}{2}p(\alpha+\beta)+2\cos\tfrac{1}{2}(p-2)(\alpha+\beta)+2\cos\tfrac{1}{2}(p-4)(\alpha+\beta)+\dots\\ &\qquad+2\cos\tfrac{1}{2}p(\alpha-\beta)+2\cos\tfrac{1}{2}(p-2)(\alpha-\beta)+2\cos\tfrac{1}{2}(p-4)(\alpha-\beta)+\dots\\ &\quad= 4\cos\tfrac{1}{2}p\alpha\cos\tfrac{1}{2}p\beta+4\cos\tfrac{1}{2}(p-2)\alpha\cos\tfrac{1}{2}(p-2)\beta+\dots\\ &\quad= [\tfrac{1}{2}p,\tfrac{1}{2}p]'\\ &\quad= [\tfrac{1}{2}p,\tfrac{1}{2}p]+[\tfrac{1}{2}p,-\tfrac{1}{2}p].\end{aligned}$$

Hence
$$\begin{aligned}[\tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{p\} &= [\tfrac{1}{2}p,\tfrac{1}{2}p],\\ [\tfrac{1}{2}\ -\tfrac{1}{2}]\otimes\{p\} &= [\tfrac{1}{2}p,-\tfrac{1}{2}p].\end{aligned}$$

Lastly
$$\begin{aligned}&[\tfrac{1}{2}p,\tfrac{1}{2}p][\tfrac{1}{2}q,-\tfrac{1}{2}q]+[\tfrac{1}{2}p,-\tfrac{1}{2}p][\tfrac{1}{2}q,\tfrac{1}{2}q]\\ &\quad=\frac{\sin(p+\frac{1}{2})(\alpha+\beta)}{\sin\frac{1}{2}(\alpha+\beta)}\frac{\sin(q+\frac{1}{2})(\alpha-\beta)}{\sin\frac{1}{2}(\alpha-\beta)}+\frac{\sin(p+\frac{1}{2})(\alpha-\beta)}{\sin\frac{1}{2}(\alpha-\beta)}\frac{\sin(q+\frac{1}{2})(\alpha+\beta)}{\sin\frac{1}{2}(\alpha+\beta)}\end{aligned}$$

This is easily shown to be expressible as
$$\begin{vmatrix}\cos\frac{1}{2}(p+q+2)\alpha, & \cos\frac{1}{2}(p-q)\alpha\\ \cos\frac{1}{2}(p+q+2)\beta, & \cos\frac{1}{2}(p-q)\beta\end{vmatrix}\Bigg/\begin{vmatrix}\cos\alpha, & 1\\ \cos\beta, & 1\end{vmatrix},$$

which is a recognized form for
$$[\tfrac{1}{2}(p+q),\tfrac{1}{2}(p-q)]' = [\tfrac{1}{2}(p+q),\tfrac{1}{2}(p-q)]+[\tfrac{1}{2}(p+q),-\tfrac{1}{2}(p-q)].$$

Thence it may be inferred that
$$\begin{aligned}[\tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{p\}[\tfrac{1}{2}\ -\tfrac{1}{2}]\otimes\{q\} &= [\tfrac{1}{2}p,\tfrac{1}{2}p][\tfrac{1}{2}q,-\tfrac{1}{2}q]\\ &= [\tfrac{1}{2}(p+q),\tfrac{1}{2}(p-q)].\end{aligned}$$

Thus to each quaternary orthogonal type there exists exactly one concomitant of the two algebraically independent spinors. The coefficients of a quaternary orthogonal form of type $[r, s]$ could be expressed as the coefficients of a polynomial, of degree $(r+s)$ in the two components of the spinor of type $[\frac{1}{2}\ \frac{1}{2}]$, and of degree $(r-s)$ in the two components of the spinor of type $[\frac{1}{2}\ -\frac{1}{2}]$.

The correspondence is thus exhibited between the quaternary orthogonal form of type $[r, s]$ and the double binary form of type $\{r+s\}\{r-s\}'$. Concomitants for the quaternary rotation group are thus in complete correspondence with concomitants for the double binary group, and are easily deducible from the theory of concomitants for binary forms.

The extension to forms under the full orthogonal group presents no great labour. For a simple quantic of type $[n]'$, the character is the same as for the rotation group, i.e.

$$[n]' = [n].$$

Hence

$$[n]' \otimes \{p\} = [n] \otimes \{p\}.$$

The concomitants are the same as for the rotation group save that a pair of conjugate concomitants, of types $[r, s]$ and $[r, -s]$ respectively, combine to form a single concomitant of type $[r, s]'$, if $s \neq 0$.

For $s = 0$ the theory does not determine whether a concomitant which under the rotation group is of type $[r]$ will be, under the full orthogonal group, of type $[r]'$ or of the *associated* type $[r]'^*$. A method of distinguishing the two cases, however, was described in "Restricted Groups" by the evaluation of $[\lambda] \otimes \{p\}$ for the special transformation of negative determinant for which the matrix is diag $(1\ \ 1\ \ 1\ \ -1)$.

For an orthogonal form of type $[r, s]'$, since

$$[r, s]' = [r, s] + [r, -s],$$

the concomitants are deduced from the simultaneous concomitants of a pair of conjugate forms of types $[r, s]$ and $[r, -s]$ respectively under the rotation group.

Reducibility. To obtain a finite basis for the concomitants of a system of ground forms the concept of *reducibility* is necessary. A concomitant is regarded as reducible if it is the *principal part* of the product of two concomitants, or if it is a linear combination of such principal parts. The definition of principal part is therefore fundamental. For consistency this definition for the rotation group must be different from that for the full orthogonal group.

The principal part of the product of forms of types $[p, q]'$ and $[r, s]'$ for the full orthogonal group is a form of type $[p+r, q+s]'$. For the rotation group, if q and s are positive, consistently with this, the principal part of

the product of forms of types $[p,q]$ and $[r,s]$ is of type $[p+r,q+s]$. By proceeding to concomitants of the conjugate types it is clear that the principal part of the product of forms of types $[p,-q]$ and $[r,-s]$ must be of type $[p+r,-q-s]$.

The product of forms of types $[p,q]$ and $[r,-s]$, however, has no analogue for the full orthogonal group. The obvious definition of the principal part, however, is that form in the product of type $[p+r,q-s]$, and this definition seems quite satisfactory. This form always differs from zero, and in some cases it is the only non-zero form in the product. Thus

$$[1\ 1][1\ -1]=[2].$$

An example will illustrate the significance. Suppose that in 4 variables the Riemann-Chrystoffel tensor R_{ijkp} is expressible as a direct product of two 6-vectors,

$$R_{ijkp}=A_{ij}B_{kp},$$

or, to ensure the correct symmetry, say

$$R_{ijkp}=A_{ij}B_{kp}+A_{kp}B_{ij}.$$

Then, if A_{ij} corresponds to a right-handed rotation and B_{ij} to a left-handed rotation, both the conformal curvature tensor and the total curvature R will be identically zero. The Riemann-Chrystoffel tensor is completely defined by its Ricci tensor R_{ik} which is of type [2].

This definition of reducibility is particularly useful as it corresponds completely with the definition of reducibility for the corresponding double binary forms.

Examples. The concomitants of a quaternary orthogonal quadratic are first considered.

Since $$[2]=[1\ 1][1\ -1]=[\tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{2\}[\tfrac{1}{2}-\tfrac{1}{2}]\otimes\{2\},$$

the quadratic corresponds to the double binary form of type $\{2\}\{2\}'$.

Then

$$\begin{aligned}(\{2\}\{2\}')\otimes\{2\}&=(\{2\}\otimes\{2\})(\{2\}'\otimes\{2\})+(\{2\}\otimes\{1^2\})(\{2\}'\otimes\{1^2\})\\&=(\{4\}+\{2^2\})(\{4\}'+\{2^2\}')+\{3\ 1\}\{3\ 1\}'\\&=\{4\}\{4\}'+\{4\}\{0\}'+\{0\}\{4\}'+\{0\}\{0\}'+\{2\}\{2\}',\end{aligned}$$

since, the determinant being unity, the S-functions $\{1\ 1\}$ and $\{2\ 2\}$ are taken to be unity or $\{0\}$.

$$\begin{aligned}(\{2\}\{2\}')\otimes\{3\}=&\ \{2\}\otimes\{3\}\{2\}'\otimes\{3\}+\{2\}\otimes\{2\ 1\}\{2\}'\otimes\{2\ 1\}\\&+\{2\}\otimes\{1^3\}\{2\}'\otimes\{1^3\}\\=&\ \{6\}\{6\}'+\{6\}\{2\}'+\{2\}\{6\}'+\{2\}\{2\}'+\{4\}\{4\}'+\{4\}\{2\}'\\&+\{2\}\{4\}'+\{2\}\{2\}'+\{0\}\{0\}'.\end{aligned}$$

Thus the irreducible concomitants are of types:

Degree 1; $\{2\}\{2\}'$.

Degree 2; $\{4\}\{0\}', \{0\}\{4\}', \{0\}\{0\}', \{2\}\{2\}'$.

Degree 3; $\{4\}\{2\}', \{2\}\{4\}', \{2\}\{2\}', \{0\}\{0\}'$.

Similarly we obtain:

Degree 4; $\{4\}\{0\}', \{0\}\{4\}', \{4\}\{2\}', \{2\}\{4\}', \{0\}\{0\}'$.

Hence the types of irreducible concomitants up to degree 4 for a quadratic under the quaternary rotation group are:

Degree 1; $[2]$.

Degree 2; $[2\ 2], [2\ -2], [0], [2]$.

Degree 3; $[3\ 1], [3\ -1], [2], [0]$.

Degree 4; $[2\ 2], [2\ -2], [3\ 1], [3\ -1], [0]$.

For the full orthogonal group conjugate characters are coupled, e.g.

$$[2\ 2]+[2\ -2] = [2\ 2]'.$$

To the list must then be added those concomitants which, though reducible under the rotation group, become irreducible under the full orthogonal group. The types are:

Degree 1; $[2]'$.

Degree 2; $[2\ 2]', [2]', [0]'$.

Degree 3; $[3\ 1]', [2]', [0]'$.

Degree 4; $[2\ 2]', [3\ 1]', [0]', [4]'$.

The irreducible concomitants go up to degree 6, but these have been obtained by other methods† and will not be examined further here. The results are in agreement.

The quadratic complex is next considered. Under the rotation group this separates into two conjugate parts according to the equation

$$[2\ 2]' = [2\ 2]+[2\ -2].$$

Consider first the right-handed form of type $[2\ 2]$.

Since $$[2\ 2] = [\tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{4\},$$

† Littlewood (9).

the concomitants correspond to those of a single binary quartic. The irreducible concomitants are known to be:

Degree 1; $\{4\}$.
Degree 2; $\{4\}, \{0\}$.
Degree 3; $\{6\}, \{0\}$.

Thus the irreducible concomitants of a quaternary orthogonal form of type $[2\ 2]$ are of types:

Degree 1; $[2\ 2]$.
Degree 2; $[2\ 2], [0]$.
Degree 3; $[3\ 3], [0]$.

Corresponding results hold for the left-handed quadratic complex of type $[2\ -2]$.

The concomitants of the complete quadratic complex are the simultaneous concomitants of these two forms. Clearly, since

$$[p\ p]\,[q\ -q] = [p+q,\ p-q],$$

all concomitants which contain both are reducible. Hence the complete system consists of 10 forms.

Under the full orthogonal group these combine to form 7 irreducible forms, but to these must be added 10 other forms which, reducible under the rotation group, are yet irreducible under the full linear group. These are:

Degree 2;	$[4]'$	from	$[2\ 2]\,[2\ -2]$.
Degree 3;	$[4]'$	from	$[2\ 2]\,[2\ -2]$,
	$[4]'$	from	$[2\ 2]\,[2\ -2]$.
Degree 4;	$[4]'$	from	$[2\ 2]\,[2\ -2]$,
	$[5\ 1]'$	from	$[3\ 3]\,[2\ -2]$.
Degree 5;	$[5\ 1]'$	from	$[3\ 3]\,[2\ -2]$,
	$[7\ 1]'$	from	$[3\ -3]\,[2\ 2]\,[2\ 2]$.
Degree 6;	$[7\ 1]'$	from	$[3\ -3]\,[2\ 2]\,[2\ 2]$,
	$[6]'$	from	$[3\ 3]\,[3\ -3]$.
Degree 7;	$[7\ 1]'$	from	$[3\ -3]\,[2\ 2]\,[2\ 2]$.

For partitions in two parts, one only of the conjugate products is given. The products corresponding to $[3\ 3]\,[3\ 3]\,[2\ -2]\,[2\ -2]\,[2\ -2]$ would have been eligible for this list except for the syzygy which connects the square of the form of type $[3\ 3]$, or correspondingly the square of the sextic

concomitant of the binary quartic, with other reducible forms. This syzygy establishes the reducibility of the corresponding invariants. Thus the complete list of irreducible concomitants of a quadratic complex under the full quaternary orthogonal group consists of the 17 forms:

Degree 1; [2 2].
Degree 2; [2 2], [0], [0]*, [4].
Degree 3; [3 3], [0], [0]*, [4], [4]*.
Degree 4; [5 1], [4].
Degree 5; [7 1], [5 1].
Degree 6; [7 1], [6].
Degree 7; [7 1].

Since the binary sextic has 26 irreducible concomitants, the right-handed and left-handed *cubic complexes* under the quaternary rotation group have each 26 irreducible concomitants. The complete cubic complex has thus 52. Under the full orthogonal group, because of the difference of reducibility there are several hundreds more.

The determination and enumeration of these is quite straightforward and only laborious because they are so numerous. It is, however, necessary to know the syzygies of the binary sextic in order to determine which are reducible.

The spinors are next considered. The case of the basic spinor is known† for the full orthogonal group. Under the rotation group, since the basic spinor corresponds to the double binary form $\{1\}+\{1\}'$, it is obvious that all concomitants except the spinor itself are reducible. The only case of a concomitant reducible over the rotation group but irreducible over the full orthogonal group is clearly that corresponding to

$$[\tfrac{1}{2}\ \tfrac{1}{2}]\,[\tfrac{1}{2}\ -\tfrac{1}{2}] = [1],$$

so that the 4-vector of degree 2 in the spinor is the one irreducible concomitant over the full orthogonal group.

The spinor of type $[\tfrac{3}{2}\ \tfrac{3}{2}]$ corresponds to the binary cubic. The irreducible concomitants of a binary cubic are as follows:

Degree 1; {3}.
Degree 2; {2}.
Degree 3; {3}.
Degree 4; {0}.

† Littlewood (10).

There is a sextic syzygy of degree 6. The irreducible concomitants of a right-handed spinor, and those of a left-handed spinor, are therefore as follows:

Degree 1;	$[\frac{3}{2}\ \frac{3}{2}]$.	Degree 1;	$[\frac{3}{2}\ -\frac{3}{2}]$.
Degree 2;	$[1\ 1]$.	Degree 2;	$[1\ -1]$.
Degree 3;	$[\frac{3}{2}\ \frac{3}{2}]$.	Degree 3;	$[\frac{3}{2}\ \frac{3}{2}]$.
Degree 4;	$[0]$.	Degree 4;	$[0]$.

These are the 8 irreducible concomitants of the complete spinor under the rotation group. For the full orthogonal group must be added:

Degree 2;	$[3]'$	from	$[\frac{3}{2}\ \frac{3}{2}]\,[\frac{3}{2}\ -\frac{3}{2}]$.
Degree 3;	$[\frac{5}{2}\ \frac{1}{2}]'$	from	$[\frac{3}{2}\ \frac{3}{2}]\,[1\ -1]$.
Degree 4;	$[3]'$	from	$[\frac{3}{2}\ \frac{3}{2}]\,[\frac{3}{2}\ -\frac{3}{2}]$,
	$[3]'$	from	$[\frac{3}{2}\ -\frac{3}{2}]\,[\frac{3}{2}\ \frac{3}{2}]$,
	$[2]'$	from	$[1\ 1]\,[1\ -1]$.
Degree 5;	$[\frac{5}{2}\ \frac{1}{2}]$	from	$[\frac{3}{2}\ \frac{3}{2}]\,[1\ -1]$,
	$[\frac{7}{2}\ \frac{1}{2}]$	from	$[1\ 1]\,[1\ 1]\,[\frac{3}{2}\ -\frac{3}{2}]$.
Degree 6;	$[3]$	from	$[\frac{3}{2}\ \frac{3}{2}]\,[\frac{3}{2}\ -\frac{3}{2}]$.

The complete list of irreducible concomitants of a spinor of type $[\frac{3}{2}\ \frac{3}{2}]'$ under the full orthogonal group thus contains 14 forms of the following types:

Degree 1;	$[\frac{3}{2}\ \frac{3}{2}]'$.
Degree 2;	$[1\ 1]'$, $[3]'$.
Degree 3;	$[\frac{3}{2}\ \frac{3}{2}]'$, $[\frac{5}{2}\ \frac{1}{2}]'$.
Degree 4;	$[3]'$, $[3]'^*$, $[2]'$, $[0]'$, $[0]'^*$.
Degree 5;	$[\frac{5}{2}\ \frac{1}{2}]'$, $[\frac{7}{2}\ \frac{1}{2}]'$.
Degree 6;	$[3]'$.
Degree 7;	$[\frac{7}{2}\ \frac{1}{2}]'$.

Lastly the spinor of type $[\frac{3}{2}\ \frac{1}{2}]$ is considered, for which the calculation is much more complex. The corresponding double binary character is $\{2\}\,\{1\}'$. Hence

$$\begin{aligned}
(\{2\}\,\{1\}')\otimes\{2\} &= \{2\}\otimes\{2\}\,\{2\}'+\{2\}\otimes\{1^2\}\,\{1^2\}'\\
&= \{4\}\,\{2\}'+\{0\}\,\{2\}'+\{2\}\,\{0\}',\\
(\{2\}\,\{1\}')\otimes\{3\} &= \{2\}\otimes\{3\}\,\{3\}'+\{2\}\otimes\{2\ 1\}\,\{2\ 1\}'\\
&= \{6\}\,\{3\}'+\{2\}\,\{3\}'+\{4\}\,\{1\}'+\{2\}\,\{1\}',\\
(\{2\}\,\{1\}')\otimes\{4\} &= \{2\}\otimes\{4\}\{4\}'+\{2\}\otimes\{3\ 1\}\,\{3\ 1\}'+\{2\}\otimes\{2^2\}\,\{2^2\}'\\
&= \{8\}\,\{4\}'+\{4\}\,\{4\}'+\{0\}\,\{4\}'+\{6\}\,\{2\}'+\{4\}\,\{2\}'\\
&\qquad +\{2\}\,\{2\}'+\{4\}\,\{0\}'+\{0\}\,\{0\}',\\
(\{2\}\,\{1\}')\otimes\{5\} &= \{10\}\,\{5\}'+\{6\}\,\{5\}'+\{2\}\,\{5\}'+\{8\}\,\{3\}'+\{6\}\,\{3\}'+\{4\}\,\{3\}'\\
&\qquad +\{2\}\,\{3\}'+\{6\}\,\{1\}'+\{4\}\,\{1\}'+\{2\}\,\{1\}'.
\end{aligned}$$

Deleting reducible concomitants, we see that the types of irreducibles are:

Degree 2; $\{2\}\{0\}'+\{0\}\{2\}'$.

Degree 3; $\{2\}\{1\}'$.

Degree 4; $\{0\}\{0\}'$.

The irreducible concomitants of the right-hand spinor of type $[\frac{3}{2}\ \frac{1}{2}]$ are therefore:

Degree 1; $[\frac{3}{2}\ \frac{1}{2}]$.

Degree 2; $[1\ 1]\,[1\ -1]$.

Degree 3; $[\frac{3}{2}\ \frac{1}{2}]$.

Degree 4; $[0]$.

This set is complete for all degrees.

There are 5 corresponding concomitants of the left-handed spinor. The simultaneous concomitants of the complete spinor however are not all reducible, even for the rotation group, as they were for the other spinors considered. The following analyses give the concomitants for the respective degrees in the two parts of the spinor; $1+1$, $2+1$, $3+1$, $2+2$:

$$\{2\}\{1\}'\{1\}\{2\}' = \{3\}\{3\}'+\{3\}\{1\}'+\{1\}\{3\}'+\{1\}\{1\}',$$

$$\begin{aligned}(\{2\}\{1\}')\otimes\{2\}\{1\}\{2\}' &= (\{4\}\{2\}'+\{0\}\{2\}'+\{2\}\{0\}')\{1\}\{2\}'\\ &= \{5\}\{4\}'+\{3\}\{4\}'+\{5\}\{2\}'+\{3\}\{2\}'\\ &\quad+\{5\}\{0\}'+\{3\}\{0\}'+\{1\}\{4\}'+\{1\}\{2\}'\\ &\quad+\{1\}\{0\}'+\{3\}\{2\}'+\{1\}\{2\}',\end{aligned}$$

$$\begin{aligned}(\{2\}\{1\}')\otimes\{3\}\{1\}\{2\}' &= \{7\}\{5\}'+\{7\}\{3\}'+\{7\}\{1\}'+\{5\}\{5\}'+\{5\}\{3\}'\\ &\quad+\{5\}\{1\}'+\{3\}\{5\}'+\{3\}\{3\}'+\{3\}\{1\}'\\ &\quad+\{1\}\{5\}'+\{1\}\{3\}'+\{1\}\{1\}'+\{5\}\{3\}'\\ &\quad+\{3\}\{3\}'+\{5\}\{1\}'+\{3\}\{1\}'+\{3\}\{3\}'\\ &\quad+\{3\}\{1\}'+\{1\}\{3\}'+\{1\}\{1\}',\end{aligned}$$

$$\begin{aligned}(\{2\}\{1\}')\otimes\{2\}\,(\{1\}\{2\}')\otimes\{2\} &= \{6\}\{6\}'+\{6\}\{4\}'+\{6\}\{2\}'+\{4\}\{6\}'+\{4\}\{4\}'\\ &\quad+\{4\}\{2\}'+\{2\}\{6\}'+\{2\}\{4\}'+\{2\}\{2\}'\\ &\quad+\{4\}\{4\}'+\{4\}\{2\}'+\{4\}\{0\}'+\{6\}\{2\}'\\ &\quad+\{4\}\{2\}'+\{2\}\{2\}'+\{2\}\{6\}'+\{2\}\{4\}'\\ &\quad+\{2\}\{2\}'+\{4\}\{4\}'+\{2\}\{4\}'+\{0\}\{4\}'\\ &\quad+\{0\}\{4\}'+\{0\}\{2\}'+\{0\}\{0\}'+\{4\}\{0\}'\\ &\quad+\{2\}\{0\}'+\{0\}\{0\}'+2\{2\}\{2\}'.\end{aligned}$$

Deleting the reducible concomitants, we obtain for the irreducibles:

Degree $1+1$; $\{3\}\{1\}'$, $\{1\}\{3\}'$, $\{1\}\{1\}'$.

Degree $2+1$; $\{5\}\{0\}'$, $\{3\}\{0\}'$, $\{1\}\{0\}'$, $\{1\}\{2\}'$, $\{1\}\{2\}'$.

Degree $3+1$; $\{3\}\{1\}'$, $\{1\}\{3\}'$, $\{1\}\{1\}'$, $\{1\}\{1\}'$.

Degree $2+2$; $\{4\}\{0\}'$, $\{0\}\{4\}'$, $\{2\}\{0\}'$, $\{0\}\{2\}'$, $\{0\}\{0\}'$, $\{0\}\{0\}'$.

The irreducible concomitants of degrees $4+1$ and $3+2$ are obtained similarly. The list of irreducibles up to degree 5 in the coefficients for the complete spinor of type $[\frac{3}{2}\ \frac{1}{2}]+[\frac{3}{2}\ -\frac{1}{2}]$ over the rotation group are as follows:

Degree $1+0$; $[\frac{3}{2}\ \frac{1}{2}]$.

$0+1$; $[\frac{3}{2}\ -\frac{1}{2}]$.

$2+0$; $[1\ 1]$, $[1\ -1]$.

$0+2$; $[1\ 1]$, $[1\ -1]$.

$1+1$; $[2\ 1]$, $[2\ -1]$, $[1]$.

$3+0$; $[\frac{3}{2}\ \frac{1}{2}]$.

$2+1$; $[\frac{5}{2}\ \frac{5}{2}]$, $[\frac{3}{2}\ \frac{3}{2}]$, $[\frac{1}{2}\ \frac{1}{2}]$, $[\frac{3}{2}\ -\frac{1}{2}]$, $[\frac{3}{2}\ -\frac{1}{2}]$.

$1+2$; $[\frac{5}{2}\ -\frac{5}{2}]$, $[\frac{3}{2}\ -\frac{3}{2}]$, $[\frac{1}{2}\ -\frac{1}{2}]$, $[\frac{3}{2}\ \frac{1}{2}]$, $[\frac{3}{2}\ \frac{1}{2}]$.

$0+3$; $[\frac{3}{2}\ -\frac{1}{2}]$.

$4+0$; $[0]$.

$3+1$; $[2\ 1]$, $[2\ -1]$, $[1]$, $[1]$.

$2+2$; $[2\ 2]$, $[2\ -2]$, $[1\ 1]$, $[1\ -1]$, $[0]$, $[0]$.

$1+3$; $[2\ 1]$, $[2\ -1]$, $[1]$, $[1]$.

$0+4$; $[0]$.

$4+1$; $[\frac{5}{2}\ \frac{5}{2}]$, $[\frac{3}{2}\ -\frac{1}{2}]$, $[\frac{1}{2}\ \frac{1}{2}]$.

$3+2$; $[\frac{5}{2}\ -\frac{5}{2}]$, $[\frac{3}{2}\ -\frac{3}{2}]$, $[\frac{3}{2}\ -\frac{3}{2}]$, $[\frac{1}{2}\ -\frac{1}{2}]$, $[\frac{1}{2}\ -\frac{1}{2}]$, $[\frac{3}{2}\ \frac{1}{2}]$.

$2+3$; $[\frac{5}{2}\ \frac{5}{2}]$, $[\frac{3}{2}\ \frac{3}{2}]$, $[\frac{3}{2}\ \frac{3}{2}]$, $[\frac{1}{2}\ \frac{1}{2}]$, $[\frac{1}{2}\ \frac{1}{2}]$, $[\frac{3}{2}\ -\frac{1}{2}]$.

$1+4$; $[\frac{5}{2}\ -\frac{5}{2}]$, $[\frac{3}{2}\ \frac{1}{2}]$, $[\frac{1}{2}\ -\frac{1}{2}]$.

There are further irreducibles of degree 6 and more.

5. *Five variables*

The orthogonal group in 5 variables has a basic spin representation of degree 4. The invariant matrices of this spin matrix are now examined.

The latent roots of the 5-dimensional orthogonal matrix may be expressed as

$$e^{i\alpha}, \quad e^{-i\alpha}, \quad e^{i\beta}, \quad e^{-i\beta}, \quad 1.$$

The latent roots of the spin representation are†

$$e^{\frac{1}{2}i(\alpha+\beta)}, \quad e^{\frac{1}{2}(\alpha-\beta)}, \quad e^{\frac{1}{2}i(\beta-\alpha)}, \quad e^{-\frac{1}{2}i(\alpha+\beta)}.$$

Hence

$$[\tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{2\} = e^{i(\alpha+\beta)}+e^{i(\alpha-\beta)}+e^{i(\beta-\alpha)}+e^{-i(\alpha+\beta)}+e^{i\alpha}+e^{-i\alpha}+e^{i\beta}+e^{-i\beta}+2$$

$$= [1\ 1].$$

$$[\tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{1^2\} = e^{i\alpha}+e^{-i\alpha}+e^{i\beta}+e^{-i\beta}+2$$

$$= [1]+[0].$$

It follows that the spin representation has an invariant linear complex. Thus the group of the spin transformations is either the 4-dimensional symplectic group or else some sub-group of this. Since every character of a group is also a character, simple or compound, of its sub-group, it follows that the operation $\otimes$ may be followed by a character of the 4-dimensional symplectic group. Such a character is denoted by $\langle\lambda\rangle \equiv \langle\lambda_1, \lambda_2\rangle$.

Thus $$[\tfrac{1}{2}\ \tfrac{1}{2}]\otimes\langle 1\ 1\rangle = [1].$$

By considering the principal parts involved, it is clear that $[\frac{1}{2}\ \frac{1}{2}]\otimes\langle p, q\rangle$ includes the character $[\frac{1}{2}(p+q), \frac{1}{2}(p-q)]$ of the orthogonal group. The number of terms involved may be determined‡ by the evaluation of $\langle p, q\rangle$ and $[\frac{1}{2}(p+q), \frac{1}{2}(p-q)]$ for the identical transformation. Using the formula

$$\langle\lambda_1, \ldots, \lambda_\nu\rangle = \frac{\Pi b_r \prod\limits_{r<s}(b_r^2-b_s^2)}{3!\,5!\ldots(n-1)!},$$

where $$b_s = \lambda_s+\nu-s+1,$$

it follows that $$\langle p, q\rangle = \frac{(p+2)(q+1)[(p+2)^2-(q+1)^2]}{3!}.$$

Again, for the orthogonal group, the formula

$$[\lambda_1, \ldots, \lambda_\nu] = \frac{\Pi[(a_p-a_q)(a_p+a_q+n-2)]\,\Pi(2a_p+n-2)}{(n-2)!\,(n-4)!\ldots 3!},$$

where $a_s = \lambda_s - s + 1$, is used, which holds also for spinors. This gives

$$[\tfrac{1}{2}(p+q), \tfrac{1}{2}(p-q)] = \frac{(q+1)(p+2)(p+q+3)(p-q+1)}{3!}$$

$$= \langle p, q\rangle.$$

† Littlewood (5), Chap. XI.
‡ Littlewood (6, 7).

Since $[\frac{1}{2}\ \frac{1}{2}] \otimes \langle p, q\rangle$ includes $[\frac{1}{2}(p+q),\ \frac{1}{2}(p-q)]$ and has the same number of terms, it follows that the two representations are identical, and

$$[\tfrac{1}{2}\ \tfrac{1}{2}] \otimes \langle p, q\rangle = [\tfrac{1}{2}(p+q),\ \tfrac{1}{2}(p-q)].$$

Again, since

$$[\tfrac{1}{2}\ \tfrac{1}{2}] \otimes \langle 1\ 1\rangle = [1],$$

it follows that

$$\begin{aligned}[\tfrac{1}{2}\ \tfrac{1}{2}] \otimes \langle p, q\rangle &= [\tfrac{1}{2}(p+q),\ \tfrac{1}{2}(p-q)] \\ &= [1] \otimes [\tfrac{1}{2}(p+q),\ \tfrac{1}{2}(p-q)] \\ &= [\tfrac{1}{2}\ \tfrac{1}{2}] \otimes \langle 1\ 1\rangle \otimes [\tfrac{1}{2}(p+q),\ \tfrac{1}{2}(p-q)].\end{aligned}$$

Thus

$$\langle p, q\rangle = \langle 1\ 1\rangle \otimes [\tfrac{1}{2}(p+q),\ \tfrac{1}{2}(p-q)].$$

The 5-dimensional orthogonal group is thus isomorphic with the 4-dimensional symplectic group. The isomorphism is not simple but $2:1$, since the two-valued spinors correspond to simple representations of the symplectic group.

This isomorphism does not give a markedly easier method of finding concomitants under the 5-dimensional orthogonal group as do previous results in fewer variables. It is true that the 4-dimensional symplectic group is easier to deal with than the 5-dimensional orthogonal, but to counterbalance this the degrees of the corresponding forms are higher for the symplectic group.

The advantages are about equally divided. For some forms the symplectic representation is easier, for others the orthogonal. But the labour of finding the concomitants is at least halved. If the concomitants of a form or system of ground forms is known for the one group, then the concomitants of the corresponding form or set of forms in the other group is immediately deducible.

Thus, in "Restricted Groups" the complete set of concomitants of a quaternary symplectic quadratic was obtained. The types are as follows:

Degree 1; $\langle 2\rangle$.

Degree 2; $\langle 2^2\rangle, \langle 1^2\rangle, \langle 0\rangle$.

Degree 3; $\langle 2\rangle$.

Degree 4; $\langle 3\ 1\rangle, \langle 0\rangle$.

From these may be deduced the complete system of concomitants of a linear complex under the orthogonal group in 5 variables. The types are as follows:

Degree 1; $[1^2]$.

Degree 2; $[2], [1], [0]$.

Degree 3; $[1^2]$.

Degree 4; $[2\ 1], [0]$.

On the other hand, in the same paper the concomitants of a 5-dimensional orthogonal quadratic were determined up to degree 5 in the coefficients, the types being:

Degree 1; $[2]$.
Degree 2; $[2^2], [2], [0]$.
Degree 3; $[3\ 1], [2^2], [2], [0]$.
Degree 4; $[3\ 2], [3\ 1], [2^2], [2^2], [2], [0]$.
Degree 5; $[4\ 3], [3\ 2], [3\ 1], [3\ 1], [2^2], [0]$.

The concomitants of the quaternary symplectic quadratic complex were obtained up to degree 3 only in the coefficients:

Degree 1; $\langle 2^2 \rangle$.
Degree 2; $\langle 4 \rangle, \langle 2^2 \rangle, \langle 0 \rangle$.
Degree 3; $\langle 4\ 2 \rangle, \langle 4 \rangle, \langle 2^2 \rangle, \langle 0 \rangle$.

The extensions to degrees 4 and 5 in the variables follow immediately from the results for the quadratic:

Degree 4; $\langle 5\ 1 \rangle, \langle 4\ 2 \rangle, \langle 4 \rangle, \langle 4 \rangle, \langle 2^2 \rangle, \langle 0 \rangle$.
Degree 5; $\langle 7\ 1 \rangle, \langle 5\ 1 \rangle, \langle 4\ 2 \rangle, \langle 4\ 2 \rangle, \langle 4 \rangle, \langle 0 \rangle$.

The advantages of the symplectic representation, however, are much more in evidence when dealing with 5-dimensional spinors, for these present considerable difficulty by other methods.

The concomitants of a basic spinor are known.† The spinor of type $[\frac{3}{2}\ \frac{1}{2}]$ corresponds to the symplectic form of type $\langle 2\ 1 \rangle$. This is not an easy form to deal with, but the types of concomitants can be found by means of an artifice, using the results for the corresponding orthogonal form in 4 variables.

Denote for the present the characters of the 4-dimensional orthogonal and rotation groups by $[\lambda]'$ and $[\lambda]$ respectively.

Then, since $$\langle 2\ 1 \rangle = \{2\ 1\} - \{1\} = [2\ 1]',$$
it follows that $$\langle 2\ 1 \rangle \otimes \{n\} = [2\ 1]' \otimes \{n\}.$$
The orthogonal character $[2\ 1]'$ corresponds to the double binary character
$$\{3\}\{1\}' + \{1\}\{3\}'.$$
Hence
$$\begin{aligned}(\{3\}\{1\}' + \{1\}\{3\}') \otimes \{2\} &= (\{3\}\{1\}') \otimes \{2\} + (\{1\}\{3\}') \otimes \{2\} + \{3\}\{1\}'\{1\}\{3\}' \\ &= \{6\}\{2\}' + \{2\}\{2\}' + \{4\}\{0\}' + \{0\}\{0\}' + \{2\}\{6\}' \\ &\quad + \{2\}\{2\}' + \{0\}\{4\}' + \{0\}\{0\}' \\ &\quad + \{4\}\{4\}' + \{4\}\{2\}' + \{2\}\{4\}' + \{2\}\{2\}'.\end{aligned}$$

† Littlewood (10).

Thus
$$[2\ 1]'\otimes\{2\} = [4\ 2]+[2]+[2\ 2]+[0]+[4\ -2]+[2] \\ +[2\ -2]+[0]+[4]+[3\ 1]+[3\ -1]+[2] \\ = [4\ 2]'+[2\ 2]'+[3\ 1]'+[4]'+3[2]'+2[0]'.$$
Since
$$\{4\ 2\} = [4\ 2]'+[4]'+[3\ 1]'+[2^2]'+2[2]'+[0]',$$
$$\{2\} = [2]'+[0]',$$
it follows that
$$\langle 2\ 1\rangle\otimes\{2\} = [2\ 1]'\otimes\{2\} \\ = \{4\ 2\}+\{2\} \\ = \langle 4\ 2\rangle+\langle 3\ 1\rangle+\langle 2\rangle+\langle 2\rangle.$$
Similarly
$$(\{3\}\{1\}'+\{1\}\{3\}')\otimes\{3\} \\ = (\{3\}\{1\}')\otimes\{3\}+(\{3\}\{1\}')\otimes\{2\}\{1\}\{3\}'+\{3\}\{1\}'(\{1\}\{3\}')\otimes\{2\} \\ +(\{1\}\{3\}')\otimes\{3\} \\ = \{9\}\{3\}'+\{5\}\{3\}'+\{3\}\{3\}'+\{7\}\{1\}'+\{5\}\{1\}'+\{3\}\{1\}'+\{1\}\{1\}' \\ +\{7\}\{5\}'+\{7\}\{3\}'+\{7\}\{1\}'+\{5\}\{5\}'+\{5\}\{3\}'+\{5\}\{1\}' \\ +\{3\}\{5\}'+\{3\}\{3\}'+\{3\}\{1\}'+\{1\}\{5\}'+\{1\}\{3\}'+\{1\}\{1\}' \\ +\{5\}\{3\}'+\{3\}\{3\}'+\{1\}\{3\}'+\{5\}\{7\}'+\{3\}\{7\}'+\{1\}\{7\}' \\ +\{5\}\{5\}'+\{3\}\{5\}'+\{1\}\{5\}'+\{5\}\{3\}'+\{3\}\{3\}'+\{1\}\{3\}' \\ +\{5\}\{1\}'+\{3\}\{1\}'+\{1\}\{1\}'+\{3\}\{5\}'+\{3\}\{3\}'+\{3\}\{1\}' \\ +\{3\}\{9\}'+\{3\}\{5\}'+\{3\}\{3\}'+\{1\}\{7\}'+\{1\}\{5\}'+\{1\}\{3\}' \\ +\{1\}\{1\}'.$$
Hence
$$[2\ 1]'\otimes\{3\} = [6\ 3]'+[4\ 1]'+2[3]'+[4\ 3]'+[3\ 2]'+[2\ 1]'+2[1]'+[6\ 1]' \\ +[5\ 2]'+[4\ 3]'+2[5]'+[4\ 1]'+[3\ 2]'+[4\ 1]'+2[3]' \\ +[2\ 1]'+[3\ 2]'+[2\ 1]'+2[1]'+[4\ 1]'+2[3]'+[2\ 1]' \\ = \{6\ 3\}+\{4\ 3\}+\{5\}+\{4\ 1\}+\{3\ 2\}-\{1\} \\ = \langle 6\ 3\rangle+\langle 5\ 2\rangle+\langle 4\ 1\rangle+\langle 3\rangle+\langle 4\ 3\rangle+\langle 3\ 2\rangle+\langle 2\ 1\rangle \\ +\langle 1\rangle+\langle 5\rangle+\langle 4\ 1\rangle+\langle 3\rangle+\langle 3\ 2\rangle+\langle 2\ 1\rangle.$$

The irreducible concomitants up to degree 3 in the coefficients of the symplectic form of type $\langle 2\ 1\rangle$ are thus of the following types:

Degree 1; $\langle 2\ 1\rangle$.

Degree 2; $\langle 3\ 1\rangle$, $\langle 2\rangle$, $\langle 2\rangle$.

Degree 3; $\langle 5\rangle$, $\langle 4\ 3\rangle$, $\langle 3\ 2\rangle$, $\langle 3\ 2\rangle$, $\langle 3\rangle$, $\langle 3\rangle$, $\langle 2\ 1\rangle$, $\langle 2\ 1\rangle$.

The irreducible concomitants up to degree 3 in the coefficients of the 5-dimensional spinor of type $[\frac{3}{2}\ \frac{1}{2}]$ are thus of types:

Degree 1; $[\frac{3}{2}\ \frac{1}{2}]$.
Degree 2; $[2\ 1], [1\ 1], [1\ 1]$.
Degree 3; $[\frac{5}{2}\ \frac{5}{2}], [\frac{7}{2}\ \frac{1}{2}], [\frac{5}{2}\ \frac{1}{2}], [\frac{5}{2}\ \frac{1}{2}], [\frac{3}{2}\ \frac{3}{2}], [\frac{3}{2}\ \frac{3}{2}], [\frac{3}{2}\ \frac{1}{2}], [\frac{3}{2}\ \frac{1}{2}]$.

The spinor of type $[\frac{3}{2}\ \frac{3}{2}]$ gives a much simpler calculation, as it corresponds to the symplectic cubic.

Since
$$\langle 3\rangle = \{3\},$$
then
$$\begin{aligned}\{3\}\otimes\{2\} &= \{6\}+\{4\ 2\}\\ &= \langle 6\rangle+\langle 4\ 2\rangle+\langle 3\ 1\rangle+\langle 2\rangle,\end{aligned}$$
$$\{3\}\otimes\{3\} = \{9\}+\{7\ 2\}+\{6\ 3\}+\{5\ 2\ 2\}+\{4\ 4\ 1\}.$$

The S-function $\{5\ 2\ 2\}$ is expressed in terms of S-functions with $\leqslant 2$ parts by the method described in "Restricted Groups". Thus
$$\begin{aligned}\{5\ 2\ 2\} = \{3;\ 2\} &= \{5\}+\{4\ 1\}+\{3\ 2\}-\{3\}-\{2\ 1\}\\ &= \langle 5\rangle+\langle 4\ 1\rangle+\langle 3\ 2\rangle.\end{aligned}$$
Also
$$\{4\ 4\ 1\} = \{4\ 3\} = \langle 4\ 3\rangle+\langle 3\ 2\rangle+\langle 2\ 1\rangle+\langle 1\rangle.$$
Thus
$$\begin{aligned}\langle 3\rangle\otimes\{3\} = \langle 9\rangle&+\langle 7\ 2\rangle+\langle 6\ 1\rangle+\langle 5\rangle+\langle 6\ 3\rangle+\langle 5\ 2\rangle+\langle 4\ 1\rangle+\langle 3\rangle+\langle 5\rangle\\ &+\langle 4\ 1\rangle+\langle 3\ 2\rangle+\langle 4\ 3\rangle+\langle 3\ 2\rangle+\langle 2\ 1\rangle+\langle 1\rangle.\end{aligned}$$
Similarly
$$\begin{aligned}\langle 3\rangle\otimes\{4\} = \langle 1\ 2\rangle&+\langle 10\ 2\rangle+\langle 9\ 3\rangle+\langle 9\ 1\rangle+\langle 8\ 4\rangle+\langle 8\ 2\rangle+2\langle 8\rangle+2\langle 7\ 3\rangle\\ &+3\langle 7\ 1\rangle+\langle 6\ 6\rangle+\langle 6\ 4\rangle+5\langle 6\ 2\rangle+2\langle 6\rangle+\langle 5\ 5\rangle+3\langle 5\ 3\rangle\\ &+4\langle 5\ 1\rangle+2\langle 4\ 4\rangle+4\langle 4\ 2\rangle+5\langle 4\rangle+3\langle 3\ 3\rangle+3\langle 3\ 1\rangle\\ &+3\langle 2\ 2\rangle+\langle 1\ 1\rangle+\langle 0\rangle.\end{aligned}$$

The types of irreducible concomitants up to degree 4 are:

Degree 1; $\langle 3\rangle$.
Degree 2; $\langle 4\ 2\rangle, \langle 3\ 1\rangle, \langle 2\rangle$.
Degree 3; $\langle 6\ 3\rangle, \langle 5\ 2\rangle, \langle 5\rangle, \langle 4\ 3\rangle, 2\langle 4\ 1\rangle, 2\langle 3\ 2\rangle, \langle 3\rangle, \langle 2\ 1\rangle, \langle 1\rangle$.
Degree 4; $\langle 6\ 6\rangle, \langle 6\ 4\rangle, \langle 6\ 2\rangle, \langle 6\rangle, \langle 5\ 5\rangle, 3\langle 5\ 3\rangle, 2\langle 5\ 1\rangle, 2\langle 4\ 4\rangle,$
$4\langle 4\ 2\rangle, \langle 4\rangle, 3\langle 3\ 3\rangle, 3\langle 3\ 1\rangle, 3\langle 2\ 2\rangle, \langle 1\ 1\rangle, \langle 0\rangle$.

The types of irreducible concomitants of a 5-dimensional spinor of type $[\frac{3}{2}\ \frac{3}{2}]$ up to degree 4 in the coefficients are:

Degree 1; $[\frac{3}{2}\ \frac{3}{2}]$.
Degree 2; $[3\ 1], [2\ 1], [1\ 1]$.
Degree 3; $[\frac{9}{2}\ \frac{3}{2}], [\frac{7}{2}\ \frac{3}{2}], [\frac{5}{2}\ \frac{5}{2}], [\frac{7}{2}\ \frac{1}{2}], 2[\frac{5}{2}\ \frac{3}{2}], 2[\frac{5}{2}\ \frac{1}{2}], [\frac{3}{2}\ \frac{3}{2}], [\frac{3}{2}\ \frac{1}{2}], [\frac{1}{2}\ \frac{1}{2}]$.
Degree 4; $[6], [5\ 1], [4\ 2], [3\ 3], [5], 3[4\ 1], 2[3\ 2], 2[4], 4[3\ 1], [2\ 2],$
$3[3], 3[2\ 1], 3[2], [1], [0]$.

6. *Six variables*

If the latent roots of an orthogonal matrix in 6 variables are

$$e^{\pm i\alpha}, \quad e^{\pm i\beta}, \quad e^{\pm i\gamma},$$

then the latent roots of the complete basic spin matrix are

$$e^{\frac{1}{2}i(\pm\alpha\pm\beta\pm\gamma)}.$$

For the rotation group the complete spin representation which is of degree 8 separates into two conjugate representations each of degree 4, according to the equation

$$[\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}]' = [\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}] + [\tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}].$$

The latent roots of one which is associated with $[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]$, may be expressed as

$$e^{\frac{1}{2}i(\alpha+\beta+\gamma)}, \quad e^{\frac{1}{2}i(\alpha-\beta-\gamma)}, \quad e^{\frac{1}{2}i(\beta-\gamma-\alpha)}, \quad e^{\frac{1}{2}i(\gamma-\alpha-\beta)}.$$

Hence
$$\begin{aligned}
[\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{2\} &= e^{i(\alpha+\beta+\gamma)} + e^{i(\alpha-\beta-\gamma)} + e^{i(\beta-\gamma-\alpha)} + e^{i(\gamma-\alpha-\beta)} \\
&\quad + e^{i\alpha} + e^{i\beta} + e^{i\gamma} + e^{-i\alpha} + e^{-i\beta} + e^{-i\gamma} \\
&= [1\ 1\ 1], \\
[\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{1^2\} &= e^{i\alpha} + e^{i\beta} + e^{i\gamma} + e^{-i\alpha} + e^{-i\beta} + e^{-i\gamma} \\
&= [1], \\
[\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{1^3\} &= e^{-\frac{1}{2}i(\alpha+\beta+\gamma)} + e^{\frac{1}{2}i(\alpha+\beta-\gamma)} + e^{\frac{1}{2}i(\beta+\gamma-\alpha)} + e^{\frac{1}{2}i(\gamma+\alpha-\beta)} \\
&= [\tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}].
\end{aligned}$$

From a consideration of principal parts, it is clear that $[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]\otimes\{p, q, r\}$ contains $[\frac{1}{2}(p+q-r), \frac{1}{2}(p-q+r), \frac{1}{2}(p-q-r)]$. Since it is readily verified that the number of terms is the same in each case, equality follows, and

$$[\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{p\ q\ r\} = [\tfrac{1}{2}(p+q-r), \tfrac{1}{2}(p-q+r), \tfrac{1}{2}(p-q-r)]. \qquad (6\cdot1)$$

Again, since

$$[\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{1^2\} = [1]$$

and the quaternary linear complex has a quadratic invariant

$$\{1^2\}\otimes\{2\} = \{2^2\} + \{0\},$$

it follows that the operation $\otimes$ may be followed by an orthogonal character, and

$$[\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{1^2\}\otimes[\lambda] = [\lambda].$$

Thus
$$\{1^2\}\otimes[\tfrac{1}{2}(p+q-r), \tfrac{1}{2}(p-q+r), \tfrac{1}{2}(p-q-r)] = \{p\ q\ r\}$$
or
$$\{1^2\}\otimes[u\ v\ w] = \{u+v, u-w, v-w\}. \qquad (6\cdot2)$$

Thus is established a complete correspondence between the characters, including spin characters, of the 6-dimensional rotation group and those of the 4-dimensional unitary group. An isomorphism, which is $2:1$ because of the 2-valued nature of the spinors, between the two groups follows. A complete correspondence exists between the invariant theories of the two groups.

This isomorphism has been used by Turnbull† to obtain the concomitants of the quaternary quadratic complex from those of the orthogonal quadratic in 6 variables. It was also mentioned in "Restricted Groups", but the full implications, in particular the position of the spinors, were not discussed.

As with the orthogonal groups in 5 variables, the relative advantages of the two groups depend on the particular type considered. Thus, as in the case considered by Turnbull, the 6-dimensional orthogonal quadratic leads to a simpler calculation than the quaternary quadratic complex. On the other hand the reverse procedure is sometimes preferable, particularly in considering spinors.

A form of type $[1^3]$ under the 6-variable rotation group corresponds to a quaternary quadratic for which the irreducible concomitants are of types:

Degree 1; $\{2\}$.
Degree 2; $\{2^2\}$.
Degree 3; $\{2^3\}$.
Degree 4; $\{0\}$.

Thus the types of irreducibles for a form of type $[1^3]$ are:

Degree 1; $[1\ 1\ 1]$.
Degree 2; $[2]$.
Degree 3; $[1\ 1\ -1]$.
Degree 4; $[0]$.

This is complete for all degrees.

The extension to the full orthogonal form of type $[1\ 1\ 1]'$ is straightforward but will not be considered here.

The spinor of type $[\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}]$ corresponds to a quaternary cubic, for which the types of concomitants up to degree 6 have been found.‡ From these results the types of irreducible concomitants of the spinor may be written down:

Degree 1; $[\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}]$.
Degree 2; $[3\ 1\ 1]$.
Degree 3; $[\frac{9}{2}\ \frac{3}{2}\ \frac{3}{2}]$, $[\frac{5}{2}\ \frac{5}{2}\ \frac{1}{2}]$, $[\frac{7}{2}\ \frac{1}{2}\ -\frac{1}{2}]$.
Degree 4; $[6]$, $[4\ 3\ 1]$, $[4\ 2]$, $[2^3]$, $[3\ 1]$, $[2\ 2\ -2]$.
Degree 5; $[\frac{11}{2}\ \frac{5}{2}\ \frac{1}{2}]$, $[\frac{9}{2}\ \frac{7}{2}\ \frac{1}{2}]$, $[\frac{7}{2}\ \frac{5}{2}\ \frac{5}{2}]$, $[\frac{11}{2}\ \frac{3}{2}\ -\frac{1}{2}]$, $[\frac{9}{2}\ \frac{3}{2}\ \frac{1}{2}]$, $[\frac{7}{2}\ \frac{5}{2}\ \frac{1}{2}]$,
$[\frac{7}{2}\ \frac{3}{2}\ \frac{3}{2}]$, $[\frac{9}{2}\ \frac{3}{2}\ -\frac{3}{2}]$, $[\frac{7}{2}\ \frac{1}{2}\ \frac{1}{2}]$, $[\frac{5}{2}\ \frac{5}{2}\ -\frac{1}{2}]$, $[\frac{3}{2}\ \frac{3}{2}\ -\frac{1}{2}]$.
Degree 6; $[6\ 4\ 1]$, $[7\ 2]$, $[5\ 2\ 2]$, $[5\ 3\ 2]$, $[4\ 4\ 1]$, $[4\ 3\ 2]$, $[6\ 1]$, $[5\ 3\ -1]$,
$[5\ 2]$, $[5\ 1\ 1]$, $[4\ 3]$, $[4\ 2\ 1]$, $[3\ 3\ 1]$, $[5\ 1\ -1]$, $[4\ 2\ -1]$,
$[4\ 1]$, $[3\ 2]$, $[2\ 2\ 1]$, $[3\ 3\ -3]$, $[3\ 1\ -1]$, $[1\ 1\ 1]$.

† Turnbull (13). ‡ Littlewood (8).

The spinor of type $[\frac{3}{2}\ \frac{1}{2}\ \frac{1}{2}]$ corresponds to $\{2\ 1\}$. Since in 4 variables

$$\{2\ 1\}\otimes\{2\} = \{4\ 2\}+\{3\ 2\ 1\}+\{2^3\}+\{3\ 1^3\},$$

$$\{2\ 1\}\otimes\{3\} = \{6\ 3\}+\{5\ 3\ 1\}+\{4\ 3\ 2\}+\{5\ 2\ 1^2\}+\{5\ 2^2\}+\{4^2\ 1\}$$
$$+2\{4\ 2^2\ 1\}+\{4\ 3\ 1^2\}+\{3^2\ 2\ 1\}+\{3\ 2^3\}+\{3^3\},$$

it follows that for the 6-variable rotation group

$$[\tfrac{3}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{2\} = [3\ 1\ 1]+[2\ 1]+[1\ 1\ -1]+[1\ 1\ 1],$$

$$[\tfrac{3}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}]\otimes\{3\} = [\tfrac{9}{2}\ \tfrac{3}{2}\ \tfrac{3}{2}]+[\tfrac{7}{2}\ \tfrac{3}{2}\ \tfrac{1}{2}]+[\tfrac{5}{2}\ \tfrac{3}{2}\ -\tfrac{1}{2}]+[\tfrac{5}{2}\ \tfrac{3}{2}\ \tfrac{3}{2}]+[\tfrac{5}{2}\ \tfrac{5}{2}\ \tfrac{1}{2}]+[\tfrac{7}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}]$$
$$+2[\tfrac{3}{2}\ \tfrac{3}{2}\ \tfrac{1}{2}]+[\tfrac{5}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}]+[\tfrac{3}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}]+[\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}]+[\tfrac{3}{2}\ \tfrac{3}{2}\ -\tfrac{3}{2}].$$

The irreducible concomitants are of types:

Degree 1; $[\frac{3}{2}\ \frac{1}{2}\ \frac{1}{2}]$.

Degree 2; $[2\ 1]$, $[1\ 1\ -1]$, $[1\ 1\ 1]$.

Degree 3; $[\frac{5}{2}\ \frac{5}{2}\ \frac{1}{2}]$, $[\frac{7}{2}\ \frac{1}{2}\ -\frac{1}{2}]$, $2[\frac{3}{2}\ \frac{3}{2}\ \frac{1}{2}]$, $[\frac{5}{2}\ \frac{1}{2}\ \frac{1}{2}]$, $[\frac{3}{2}\ \frac{1}{2}\ -\frac{1}{2}]$, $[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]$, $[\frac{3}{2}\ \frac{3}{2}\ -\frac{3}{2}]$.

These last two systems are not complete; the irreducibles continue for considerably higher degrees.

7. *Seven and eight variables*

The case of eight variables is first considered. There are two conjugate basic spin representations of the rotation group in 8 variables, each of degree 8, and these each give automorphisms of the rotation group.

Let the latent roots of the orthogonal matrix in 8 variables be

$$e^{\pm i\alpha},\quad e^{\pm i\beta},\quad e^{\pm i\gamma},\quad e^{\pm i\delta}.$$

Then the latent roots of the complete basic spin matrix are

$$e^{\frac{1}{2}i(\pm\alpha\pm\beta\pm\gamma\pm\delta)}.$$

For the rotation group the spin representation separates into two conjugate representations, according to the equation

$$[\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}]' = [\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}]+[\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}]$$

the latent roots corresponding to the one being those which have an even number of negative signs, and corresponding to the other an odd number of negative signs.

Put

$$\alpha' = \tfrac{1}{2}(\alpha+\beta+\gamma+\delta),$$
$$\beta' = \tfrac{1}{2}(\alpha+\beta-\gamma-\delta),$$
$$\gamma' = \tfrac{1}{2}(\alpha+\gamma-\beta-\delta),$$
$$\delta' = \tfrac{1}{2}(\beta+\gamma-\alpha-\delta).$$

Then the latent roots corresponding to $[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]$ are

$$e^{\pm i\alpha'},\quad e^{\pm i\beta'},\quad e^{\pm i\gamma'},\quad e^{\pm i\delta'}.$$

It is convenient to denote $[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]$ by Δ, and $[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}]$ by $\bar{\Delta}$.

Clearly

$$\begin{aligned}\Delta\otimes\{2\} &= e^{i(\alpha+\beta+\gamma+\delta)}+e^{i(\alpha+\beta-\gamma-\delta)}+e^{i(\alpha+\gamma-\beta-\delta)}+e^{i(\alpha+\delta-\beta-\gamma)}\\ &\qquad +e^{-i(\alpha+\beta+\gamma+\delta)}+e^{i(\gamma+\delta-\alpha-\beta)}+e^{i(\beta+\delta-\alpha-\gamma)}+e^{i(\beta+\gamma-\alpha-\delta)}\\ &\qquad +\Sigma e^{i(\alpha+\beta)}+\Sigma e^{-i(\alpha+\beta)}+\Sigma e^{i(\alpha-\beta)}+\Sigma e^{-i(\alpha-\beta)}+4\\ &= [1\ 1\ 1\ 1]+[0].\end{aligned}$$

Hence the spin representation has a quadratic invariant. It is thus an orthogonal group and possesses a pair of conjugate basic spin representations. The latent roots of the representation corresponding to $[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]$ are

$$e^{\pm i\alpha''},\quad e^{\pm i\beta''},\quad e^{\pm i\gamma''},\quad e^{\pm i\delta''},$$

where

$$\begin{aligned}\alpha'' &= \tfrac{1}{2}(\alpha'+\beta'+\gamma'+\delta') = \tfrac{1}{2}(\alpha+\beta+\gamma-\delta),\\ \beta'' &= \tfrac{1}{2}(\alpha'+\beta'-\gamma'-\delta') = \tfrac{1}{2}(\alpha+\beta-\gamma+\delta),\\ \gamma'' &= \tfrac{1}{2}(\alpha'+\gamma'-\beta'-\delta') = \tfrac{1}{2}(\alpha-\beta+\gamma+\delta),\\ \delta'' &= \tfrac{1}{2}(\beta'+\gamma'-\alpha'-\delta') = \tfrac{1}{2}(\alpha-\beta-\gamma-\delta).\end{aligned}$$

The character of the representation is thus the character $[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}]$ of the original orthogonal group. Thus

$$\Delta\otimes\Delta = \bar{\Delta}.$$

Similarly

$$\begin{aligned}\alpha''' &= \tfrac{1}{2}(\alpha''+\beta''+\gamma''+\delta'') = \alpha,\\ \beta''' &= \tfrac{1}{2}(\alpha''+\beta''-\gamma''-\delta'') = \beta,\\ \gamma''' &= \tfrac{1}{2}(\alpha''+\gamma''-\beta''-\delta'') = \gamma,\\ \delta''' &= \tfrac{1}{2}(\beta''+\gamma''-\alpha''-\delta'') = \delta,\end{aligned}$$

and

$$\Delta\otimes\Delta\otimes\Delta = \Delta\otimes\bar{\Delta} = \bar{\Delta}\otimes\Delta = [1].$$

Now

$$\begin{aligned}\Delta\otimes[1] &= [\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}],\\ \Delta\otimes[1^2] &= [1\ 1],\\ \Delta\otimes[\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}] &= [\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}],\\ \Delta\otimes[\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ -\tfrac{1}{2}] &= [1].\end{aligned}$$

It follows from a consideration of principal parts that

$$\Delta\otimes[p\ q\ r\ s] = [u\ v\ w\ z],$$

where
$$\begin{aligned}
u &= \tfrac{1}{2}(p-q)+(q-r)+\tfrac{1}{2}(r+s)+(r-s) = \tfrac{1}{2}(p+q+r-s),\\
v &= \tfrac{1}{2}(p-q)+(q-r)+\tfrac{1}{2}(r+s) \qquad\qquad\quad = \tfrac{1}{2}(p+q-r+s),\\
w &= \tfrac{1}{2}(p-q) \qquad\qquad\; +\tfrac{1}{2}(r+s) \qquad\qquad\quad = \tfrac{1}{2}(p-q+r+s),\\
z &= \tfrac{1}{2}(p-q) \qquad\qquad\; -\tfrac{1}{2}(r+s) \qquad\qquad\quad = \tfrac{1}{2}(p-q-r-s).
\end{aligned}$$

Thus $\Delta\otimes[p\ q\ r\ s] = [\tfrac{1}{2}(p+q+r-s),\ \tfrac{1}{2}(p+q-r+s),\ \tfrac{1}{2}(p-q+r+s),\ \tfrac{1}{2}(p-q-r-s)].$

The operation $\Delta\otimes$ is therefore equivalent to the operation of the matrix

$$M = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & 1 & -1\\ 1 & 1 & -1 & 1\\ 1 & -1 & 1 & 1\\ 1 & -1 & -1 & -1 \end{bmatrix}$$

on the column vector whose 4 components are the 4 parts p, q, r, s of the partition defining the representation.

The operation $\bar{\Delta}\otimes$ corresponds to the matrix

$$M^2 = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ -1 & 1 & 1 & -1 \end{bmatrix}.$$

Every rotation group with an even number of variables possesses an automorphism which interchanges conjugate representations. For 8 variables this corresponds to the matrix

$$J = \begin{bmatrix} 1 & . & . & .\\ . & 1 & . & .\\ . & . & 1 & .\\ . & . & . & -1 \end{bmatrix}$$

operating on the column vector of the 4 parts.

Since
$$JM = M^2J,$$
$$JM^2 = MJ,$$

it is clear that the group generated by J and M is the symmetric group of order $3'$. Hence the group of *outer* automorphisms of the rotation group in 8 variables, or at least of its covering group, is the symmetric group of order $3!$.

For $\nu \leqslant 3$, the whole group of automorphisms of the rotation group in 2ν variables is the full orthogonal group, but for $\nu = 4$ it is a larger group which, it will be shown, is not expressible as a linear group in less than 24 variables.

The characters are obtained from the characters of the rotation group as follows. Let A denote the partition [1], B the partition $[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]$, C the partition $[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}]$ and X the partition [1 1]. Then every partition may be expressed in the form

$$[pA+qB+rC+sX] \equiv [p+s+\tfrac{1}{2}(q+r),\ s+\tfrac{1}{2}(q+r),\ \tfrac{1}{2}(q+r),\ \tfrac{1}{2}(q-r)]$$

where p, q, r, s are positive or zero integers.

There is a character of the whole group of automorphisms corresponding to

$$[pA+qB+rC+sX]+[pA+rB+qC+sX]+[qA+pB+rC+sX] \\ +[qA+rB+pC+sX]+[rA+pB+qC+sX]+[rA+qB+pC+sX],$$

provided that p, q, r are all distinct.

If, however, $p = q \neq r$, corresponding to

$$[pA+pB+rC+sX]+[pA+rB+pC+sX]+[rA+pB+pC+sX]$$

there are two associated characters, in the second of which the sign is changed for every element which corresponds to one of the outer automorphisms J, MJ, or JM.

For $p = q = r$, corresponding to

$$[pA+pB+pC+sX]$$

there are three characters, which are obtained from the first of them by multiplying by the characters of the symmetric group of order 3!, according to the outer automorphism to which the group element belongs, i.e.

I	J, MJ, JM	$M\ M^2$
1	1	1
2	0	-1
1	-1	1

The lowest degree of the characters, apart from those which correspond to the partition [0] and thus give only the characters of the group of outer automorphisms, is 24 which is the degree of $[1]+\Delta+\bar{\Delta}$. Hence a representation of the group as a linear group must have a minimum degree of 24.

Concerning the applications to invariant theory, the group itself cannot be simplified as the isomorphism is an automorphism. Concomitants of some forms, however, may be obtained from those of other forms whose types are easier for calculation.

Thus, since $$\Delta \otimes [2] = [1^4],$$

the concomitants of a form of type $[1^4]$ may be obtained from those of a quadratic. Following the method described in "Restricted Groups", the

irreducible concomitants up to degree 4 of an 8-variable orthogonal quadratic are found to be of the following types:

Degree 1; $[2]$.

Degree 2; $[2^2], [2], [0]$.

Degree 3; $[3\ 1], [2^3], [2^2], [2], [0]$.

Degree 4; $[3\ 2\ 1], [3\ 1], [2^4], [2^3], [2^2], [2^2], [2], [0]$.

Thus the types of irreducible concomitants up to degree 4 of a form of type $[1^4]$ are:

Degree 1; $[1^4]$.

Degree 2; $[2^2], [1^4], [0]$.

Degree 3; $[2^2\ 1^2], [3\ 1^2 - 1], [2^2], [1^4], [0]$.

Degree 4; $[3\ 2\ 1], [2^2\ 1^2], [4], [3\ 1^2 - 1], [2^2], [2^2], [1^4], [0]$.

The method may be employed to convert spinors into true representations. The case of the basic spinor of type Δ is of course trivial, as it corresponds to a simple vector of type $[1]$, for which the only irreducible concomitant is the quadratic invariant. The complete basic spinor of type $\Delta+\bar{\Delta}$ has been considered elsewhere.†

Consider the spinor of type $[\frac{3}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]$. Since

$$\Delta \otimes [\tfrac{3}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}] = [1\ 1\ 1]$$

the concomitants of the spinor can be deduced from those of a form of type $[1\ 1\ 1]$.

Then

$$\begin{aligned}
[1\ 1\ 1]\otimes\{2\} &= \{1\ 1\ 1\}\otimes\{2\} \\
&= \{2\ 2\ 2\}+\{2\ 1^4\} \\
&= [2\ 2\ 2]+[2\ 2]+[2]+[0]+[2\ 1\ 1]+[1^4], \\
\{1\ 1\ 1\}\otimes\{3\} &= \{3\ 3\ 3\}+\{3\ 2\ 2\ 1\ 1\}+\{2^3\ 1^3\}+\{3\ 1^6\} \\
&= [3\ 3\ 3]+[3\ 2\ 1]+[3\ 2\ 2]+[3\ 2\ 1\ 1]+[3\ 1\ 1]+[3\ 1\ 1]+[3] \\
&\qquad +[2\ 2\ 2\ 1]+3[2\ 2\ 1]+2[2\ 1\ 1]+2[2\ 1]+3[1\ 1\ 1]+[1], \\
\{1\ 1\ 1\}\otimes\{4\} &= \{4^3\}+\{4\ 3^2\ 1^2\}+\{4\ 2^4\}+\{4\ 2^2\ 1^4\}+\{3^3\ 1^3\}+\{3^2\ 2^2\ 1^2\} \\
&\qquad +\{3\ 2^3\ 1^3\}+\{3\ 2^2\ 1^5\}+\{2^6\} \\
&= [4^3]+[4^2\ 2]+[4^2]+[4\ 3^2]+[4\ 3\ 2\ 1]+[4\ 3\ 1]+3[4\ 2^2] \\
&\qquad +[4\ 2\ 1^2]+2[4\ 2]+2[4\ 1^2]+2[4]+[3^3\ 1]+[3^2]+3[3^3\ 2] \\
&\qquad +3[3^2\ 1^2]+4[3\ 2^2\ 1]+8[3\ 2\ 1]+7[3\ 1^3]+3[3\ 1]+2[2^4] \\
&\qquad +7[2^3]+4[2^2\ 1^2]+8[2^2]+10[2\ 1^2]+4[2].
\end{aligned}$$

† Littlewood (**10**).

If we delete reducible concomitants, the list of irreducibles up to degree 4 is as follows:

Degree 1; $[1\ 1\ 1]$.

Degree 2; $[2^2]$, $[2\ 1^2]$, $[2]$, $[1^6]$, $[0]$.

Degree 3; $[3\ 2\ 1^2]$, $[3\ 1^2]$, $[3]$, $3[2^2\ 1]$, $2[2\ 1^3]$, $2[2\ 1]$, $2[1^3]$, $[1]$.

Degree 4; $[4\ 2\ 1^2]$, $[4\ 2]$, $[4]$, $[3^2]$, $3[3^2\ 1^2]$, $[3\ 2^2\ 1]$, $6[3\ 2\ 1]$, $5[3\ 1^3]$, $3[3\ 1]$, $[2^4]$, $4[2^3]$, $4[2^2\ 1^2]$, $7[2^2]$, $8[2\ 1^2]$, $3[2]$, $3[1^4]$, $2[1^2]$.

Hence the types of irreducible concomitants up to degree 4, of a spinor of type $[\frac{3}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]$, are as follows:

Degree 1; $[\frac{3}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]$.

Degree 2; $[2^2]$, $[2\ 1^2]$, $[1^3\ -1]$, $[2]$, $[0]$.

Degree 3; $[\frac{7}{2}\ \frac{3}{2}\ \frac{1}{2}\ -\frac{1}{2}]$, $[\frac{5}{2}\ \frac{3}{2}\ \frac{3}{2}\ -\frac{1}{2}]$, $[\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ -\frac{3}{2}]$, $3[\frac{5}{2}\ \frac{3}{2}\ \frac{1}{2}\ \frac{1}{2}]$, $2[\frac{5}{2}\ \frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}]$, $2[\frac{3}{2}\ \frac{3}{2}\ \frac{1}{2}\ -\frac{1}{2}]$, $2[\frac{3}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]$, $[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}]$.

Degree 4; $[4\ 2\ 1\ -1]$, $[3\ 3\ 1\ -1]$, $[2\ 2\ 2\ -2]$, $[3^2]$, $3[4\ 2]$, $[4\ 1^2]$, $6[3\ 2\ 1]$, $5[3\ 1\ 1\ -1]$, $3[2\ 2\ 1\ -1]$, $[4]$, $4[3\ 1^3]$, $4[3\ 1]$, $7[2^2]$, $8[2\ 1^2]$, $3[1\ 1\ 1\ -1]$, $3[2]$, $2[1^2]$.

The basic spin representation of the orthogonal group in 7 variables is of degree 8. The group of the spin transformations is thus a sub-group of the corresponding group for 8 variables.

The rotation group in 7 variables can be obtained from that in 8 variables by keeping a vector of type $[1]$ invariant. To the type $[1]$ corresponds the spinor type $[(\frac{1}{2})^4]$.

Hence the method gives a representation of the 7-variable rotation group as that sub-group of the 8-variable rotation group which is obtained by keeping a spinor of type $[(\frac{1}{2})^4]$ invariant.

The existence of this sub-group is of some interest, but otherwise the isomorphism would seem to have little practical significance from the point of view of invariant theory.

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University College
Swansea

SOME PROPERTIES OF A CERTAIN DOUBLE SURFACE IN SPACE OF FOUR DIMENSIONS

By R. R. JONES

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The figure of six general points in four-space has received recent attention in a paper by Bronowski (**1**). There he discusses the system of quartic curves through the six points, and studies in particular the subsystem formed by those curves which further meet a plane; hence the configuration of sets of three associated quartics is determined. We are here concerned in detail with the manifold filled by the same subsystem and in particular with the surface which is doubly contained by that threefold.

1. *The primal V in* [4]

We maintain the notation, used in (**1**), that (q) denotes the triple infinity of normal rational curves of order four in four-space which pass through six points (A_i) $(i = 1, 2, \ldots, 6)$, fixed in general position.

Those curves of (q) which meet a general line, l say, generate a surface, F^9 as we have called it in (**2**). That F^9 is of order nine was proved there by showing that the plane representation of prime sections of F^9 is by quintic curves through (i) the fifteen points of intersection of pairs of six fixed lines, representing (A_i) which are fourfold points on the surface, and through (ii) the point L say, corresponding to l.

When the six lines are each tangent to a conic Σ, F^9 becomes a specialized surface F^9_L. We have shown that F^9 and F^9_L are the general surfaces of their type and are generated each by ∞^1 curves of (q), represented by the pencil of lines through L. We now indicate the proof, used in (**2**), that F^9_L contains doubly that curve of (q) to which l is chordal. To show this, we employed a special case of a theorem which is in fact the key result of (**2**), namely: in space of s dimensions, the primals of order $n+1$ which pass through the secunda of intersection of $n+2$ primes osculating a given normal rational curve of order s and through two of the vertices of an independent simplex

osculating this curve, necessarily pass through the remaining vertices of the simplex. By taking $n = 4$ and $s = 2$ in this result, we found that there is an involution of pairs of points set up on the tangents from L to the conic Σ by the curves of the system representing prime sections of F^9_L. Accordingly, we deduced that *each* of the tangents from L to Σ represents a curve which has l as chord and is double on F^9_L.

The double infinity of those curves of (q) which in addition to passing through the points (A_i) meet a given arbitrary plane, p say, generate a threefold, which we have called V. To find the order of V, we obtain the number of points in which it is met by a general line, say l as above. But those curves of (q) which meet l generate the novenic surface F^9, and so nine of these curves meet the plane p. That is, the generators of V meet l in nine points so that V is of order nine.

The three-space representation of prime sections of V is by quintic surfaces,* (F) say, through a point P and through (A_{ij}) $(i, j = 1, 2, \ldots, 6,$ $i \neq j, A_{ij} \equiv A_{ji})$ the fifteen lines of intersection of six planes (a_i) $(i = 1, 2, \ldots, 6)$ which necessarily osculate a twisted cubic curve, C say.

We now list, for later use, some properties of V, readily obtainable from the representation: in each case the three-space configuration occurs first:

six planes $a_1, a_2, \ldots, a_6 \rightarrow$ six sixfold points $A_1, A_2, \ldots, A_6$;

fifteen lines (A_{ij}), simple on $(F) \rightarrow$ fifteen lines (A_iA_j) which are triple on V;

twenty points such as N_{ijk}, the intersection of the lines A_{jk}, A_{ki}, A_{ij}, are double points on the surfaces (F), so that the tangent (quadric) cones at N_{ijk} to two general members of (F) have only one free intersection there $\rightarrow$ twenty planes such as $A_iA_jA_k$, and these are simple;

genus of ${}_{11}C^{10}$, the decimic curve of intersection of two general members of (F), is eleven $\rightarrow$ section-genus of V is eleven;

cubic $C \rightarrow$ a curve C^{15} of order fifteen, which has six triple points at (A_i) at each of which the curve has a unique tangent;

base point $P \rightarrow$ plane p and this is simple on V;

double infinity of lines through $P \rightarrow \infty^2$ curves of (q) which generate V; and

each of the three lines of intersection of pairs of osculating planes from P to $C \rightarrow$ the unique curve of (q), say q_1, to which p is trisecant and which is therefore triple on V.

There is an involution connecting by pairs the ∞^2 lines through P: these pairs represent those curves of (q) which form with q_1 sets of "associated"

* This is seen from (2), 159, by taking $n = 4$ and $s = 3$ in the generalized result. Conversely, it can be shown by counting the linear conditions involved in [3] and [4] that V is the general threefold of its type.

curves. We shall see that these pairs of quartics are always represented by two lines through P even when the quartics coincide and become double curves of V, meeting the plane p in two points.

In summary: *the* ∞^2 *normal rational quartic curves which pass through six fixed points in* [4] *and which meet a given plane generate a threefold of order nine, on which the base points are sixfold, the lines joining pairs of these points are triple, and each plane determined by three of the base points is simple, as is the given plane; this plane is trisecant to a unique generating curve, which is triple on the primal.*

2. *Some surfaces lying on V*

A general plane of [3] meets the curve ${}_{11}C^{10}$ in ten points and so represents a surface of order ten which lies on V and passes through each sixfold point. If the plane also passes through P the corresponding surface is of order nine: every surface of this type on V meets the plane p in a line. Further, each such surface is filled by the ∞^1 curves of (q) which meet that line in p: these are represented by the pencil of lines through P in the mapping plane. Hence the surface is of the type F^9 of § 1, but not in general of the type F^9_L since the traces of (a_i) on the plane do not necessarily all touch the same conic.

When the plane through P contains, for example, the line A_{56} the corresponding surface is the projection of either an F^9 or an F^9_L from one of the multiple points. Such projected surface we call F^5_L: in our notation F^5 is necessarily F^5_L.

By considering the (five) traces on PA_{56} of the osculating planes (a_i) in [3] we see that F^5_L may be regarded as the projection from one of its general points of a special Bordiga sextic surface, designated F^6_L by Room (**3**, 395). F^5_L is part of the section of V by the prime $A_1A_2A_3A_4$, say π. For, in [3] the four base planes a_1, a_2, a_3, a_4 form with PA_{56} a degenerate surface of the system (F). Thus the section of V by π, residual to the four (simple) planes joining triads of the "vertices" of π, is F^5_L. This surface accordingly has five triple points, four at A_1, A_2, A_3, A_4 and the fifth at the intersection of π with the (triple) line A_5A_6.

The plane representation on PA_{56} shows in addition that F^5_L is generated by ∞^1 twisted cubic curves meeting a directrix line, namely, that line in p which corresponds to the neighbourhood of P in PA_{56}. One of these cubics is double on F^5_L, being chordal to the directrix line, and is represented by either of the tangents from P to the unique conic enveloped by the traces of (a_i) on PA_{56} (for these and further details see **2**, 154–155).

To summarize: *every general plane through the point* P *in* [3] *represents on* V *in* [4] *a surface* F^9 *which is of order nine, has fourfold points at the sixfold*

points on V, and is filled by those generating curves of V which meet a line lying in the plane p. Each of the fifteen planes through P which contain one of the lines (A_{ij}) represents on V a surface F^5_L. In general these surfaces F^9 and F^5_L meet the curve C^{15} on V in three distinct points.

3. *The double surface on V*

In particular, the three planes which can be drawn through P to osculate C represent surfaces F^9 which are also F^9_L and have three-point contact with C^{15}. For, each of these planes is met by the six given planes (a_i) in lines which are tangent to the same conic. Each of the three planes therefore meets the surfaces (F) in quintic curves which pass through P and the fifteen points of intersection of six tangents to a conic. In fact, there is on each of the three osculating planes from P the representation of an F^9_L as given in § 1.

But there are in all ∞^1 planes which pass through P and meet (a_i) in tangents to a conic. For, consider the dual of the well-known result that: the locus of the vertices of the cones in the ∞^3 system of quadric surfaces which contain six general points in three-space is Weddle's quartic surface. A general plane accordingly meets the locus in a quartic curve, each of the ∞^1 points on the curve being a vertex of a quadric cone through the six points.

Dually, therefore, the envelope of the planes meeting six given planes in tangents to a conic is the surface dual to Weddle's surface. There are ∞^1 tangent planes from a general point to this dual-surface, that is, there are ∞^1 planes through P which meet (a_i) in tangents to a conic and which accordingly represent surfaces F^9_L on V.

Let w be any general plane of this system. Then there are two lines in w drawn from P tangent to the conic which is touched by the six lines of intersection of w with (a_i). We already know that these two tangents represent the same quartic curve, namely the double curve of the F^9_L corresponding to w. Thus, as w varies, the ∞^1 pairs of such tangent lines from P generate a conical surface which is the map of a double surface, S say, on V. S is generated by those curves of (q) which meet p twice.

The order of S is the number of points in which it meets a general plane t of the four-space. But t meets V in a curve of order nine whose genus we have noted is eleven. Hence this curve has $\frac{1}{2} \times 8 \times 7 - 11 = 17$ double points. These lie at the meets of t with the double surface and there only since there is no other double surface on V. Thus S is of order seventeen.

That is, *the single infinity of curves of (q) which meet twice the plane p generate a surface S which is double on the threefold V and is of order seventeen.*

The definition of w as a tangent plane to the surface dual to Weddle's surface has an interesting interpretation. Since a general line in [3] meets the Weddle surface in four points, there are dually four planes which pass through a general line and touch the dual-surface. In particular there are four planes (w) through a line r which passes through P. As we have seen, there correspond to these planes four surfaces of the type F^9_L on V whose directrix lines lie in p. Further, these four lines all pass through the point of intersection of p with that curve of (q) which is mapped by r in [3] and which is common to all four surfaces F^9_L. In other words, if c is the curve in which p meets S, there pass, through any general point of p, four joins of point-pairs of the involution, γ^1_2 say, formed on c by the pairs of points in which p meets the quartics generating S. The genus of the γ^1_2 is seen to be three, by correspondence with the genus of a plane section of the Weddle surface.

Since a general point of p determines a unique curve of (q) we may restate the above result thus: *through a given general member of the curves generating V there pass ∞^1 surfaces of the type F^9 lying in V of which four are F^9_L; the double curves of these F^9_L are generating curves of the double surface S.*

4. *A prime section of the double surface*

We now examine the intersection of the double surface with the fixed prime π, that is, $A_1A_2A_3A_4$. The curves which form this section are components of composite curves of (q) which meet the plane p twice, since no curve of (q) can meet $A_1A_2A_3A_4$ again without having part of itself in that solid.

(a) The line A_1A_2 is one such component: for if A_1A_2 meets the prime $A_3A_4A_5A_6$ in the point B_{12}, a unique cubic curve can be drawn to pass through A_3, A_4, A_5, A_6, to meet A_1A_2, and to meet p twice, namely, that cubic which passes through A_3, A_4, A_5, A_6, B_{12} and has as chord the line in which p meets the solid $A_3A_4A_5A_6$. (The cubic is in fact the double curve of an F^5_L and is represented by each of a pair of lines through P meeting A_{12}: see § 2.) Thus, this cubic and the line A_1A_2 form a reducible curve of (q), i.e. A_1A_2 lies on S. Similarly, the five other joins of the "vertices" of π are part of its section of S.

(b) Let the plane $A_1A_2A_3$ meet p in the point B_{123} and the plane $A_4A_5A_6$ in the point $D_{123} \equiv D_{456}$; and similarly for other incidences. The conic defined by A_1, A_2, A_3, B_{123}, D_{123} lies on the required intersection: for, a unique conic can be drawn through A_4, A_5, A_6, B_{456}, D_{123}, and the two conics form a composite member of the rational quartic curves which generate S. That is, the conic $A_1A_2A_3B_{123}D_{123}$ lies on the section of S by π, as do the remaining three similarly defined conics.

(c) One twisted cubic curve can be drawn to pass through A_1, A_2, A_3, A_4, to meet $A_5 A_6$ at B_{56} and to meet p twice, namely, that cubic which passes through those five points and has as chord the line in which the prime meets p. Residual to $A_5 A_6$, this cubic makes up a (reducible) curve of (q) meeting p twice, and so lies on S.

The above details enable us, in order to verify and extend our knowledge of the double surface, to apply the special Cremona transformation known as the reciprocal inversion transformation.

5. *An application of the reciprocal inversion transformation*

In § 1 we considered the quartic curves (q) which pass through the six general points (A_i) and meet a fixed plane p. We now consider the homaloidal system of primals which, based on the points (A_i) $(i = 1, 2, \ldots, 5)$ as simplex of reference, birationally transforms our [4] into another four-space, say [4]′. This transformation is defined by the appropriate equations $x_i' = k/x_i$ for the same range $i = 1, 2, \ldots, 5$, k being constant and any one of the sixteen self-corresponding points being taken as unit point.

Next we list those details (see Room, **3**, 180) which are of immediate application:

- the base point A_1, for example $\rightarrow B_2 B_3 B_4 B_5$, the prime of the base simplex in [4]′ with vertices at (B_i) $(i = 1, 2, \ldots, 5)$;
- the curves $(q) \rightarrow$ lines through B_6, say, which corresponds to A_6;
- a general plane $\rightarrow$ a special Bordiga sextic surface F_L^6;
- q_1, the unique curve of (q) to which the plane p is chordal $\rightarrow$ the unique trisecant from B_6 to the corresponding surface F_L^6;
- the general primes $\rightarrow$ Lüroth quartic primals, on which the base points $B_1, B_2, \ldots, B_5$ are triple, the lines joining pairs of these points are double, and the planes joining triads of these points are simple. Each such primal is represented in ordinary space by quartic surfaces through the ten lines of intersection of five planes.*

In [4] we have already seen that on V, the double surface S is generated by those curves of (q) which meet the plane p twice. The lines in [4]′ which are the transforms of the generating curves of S appear as the chords from

* We note from the representations, that $F_L^6[4]$ projects from one of its general points into F_L^5 and from one of its triple points into a Lüroth cubic surface, that is, a four-nodal cubic surface; similarly, V and the Lüroth quartic threefold are the projections in five-space of a threefold, of order ten with six sixfold points, from, respectively, one of its general points and one of its multiple points.

B_6 to the surface F_L^6. The order of the conical sheet formed by these chords from B_6 is (see Baker, **4**, 242) half the order of the general "chord curve" on F_L^6, namely $\frac{1}{2}[(\mu_0-1)(\mu_0-2)-2\bar{p}]$, where μ_0 = order of $F_L^6 = 6$ and $\bar{p}$ = section-genus of $F_L^6 = 3$. Thus the required order is seven.

Further, the point B_6 is the vertex of the cone of chords and accordingly A_6 has multiplicity seven on S. Hence each of the base points (A_i) on S is sevenfold.

Consider a section of S by one of the Lüroth primals of the reverse transformation, say, L_3^4 corresponding to a general prime through B_6. If the double surface is of order h and has each join $A_i A_j$ m-fold, then the section consists of seven curves of (q) corresponding to the seven chords through B_6 in the prime, together with (see § 4) the ten joins of pairs of (A_i) excluding A_6, each counted $2m$ times, and the ten conics such as that through A_1, A_2, A_3 (excluding A_6), and through the two points in which the plane $A_1 A_2 A_3$ meets the planes $A_4 A_5 A_6$ and p. Hence the curve of section has order

$$4h = 7 \times 4 + 10 \times 2m + 10 \times 2.$$

Also since A_1 is sevenfold on S but triple on L_3^4, the total intersection at A_1 is of order

$$7 \times 3 = 7 + 4 \times 2m + 6.$$

From these equations, $h = 17$ and $m = 1$. Thus, S is of order seventeen and contains simply the lines $(A_i A_j)$.

It is likewise easy to calculate the order of the curve c which S traces on p. We proceed as above.

This curve is the locus of the meets of p with those curves of (q) which meet the plane twice. Hence, the twenty points such as B_{123} in which p meets the plane $A_1 A_2 A_3$ (see § 4) are on this locus and lie by sets of four on the fifteen lines in which primes like $A_1 A_2 A_3 A_4$ meet p. As we have already shown, such a line has only two other points on the locus, namely, the two points in which it meets the unique cubic curve which passes through A_1, A_2, A_3, A_4, meets $A_5 A_6$ at B_{56} and has the line as chord. Thus if the order of c is a, and each point B_{123} is b-fold on it, we have the equation

$$a = 4b + 2.$$

In the reverse transformation, the primal L_3^4 meets c in seven pairs of points, corresponding to the intersections of a general prime through B_6 with the conical sheet of chords from B_6. Also, L_3^4 contains those points such as B_{123} whose suffix does not contain the number 6. Since there are ten such points, we have the equation

$$4a = 14 + 10b.$$

These equations yield $a = 6, \quad b = 1.$

Hence c is of order six; its intersections with the prime π are simple points. From the mode of generation of c, we deduce that the three points in which q_1 meets p are double on c.

The above transformation has thus shown: *the double surface S is of order seventeen, has sevenfold points at the multiple points of the threefold V, and contains simply the lines joining pairs of those points; S meets the plane p in a sextic curve with three double points.*

It is worthy of remark that the recalculation of these results for S, using only the curves (a), (b), (c) of § 4, shows that those curves are simple on S. From such verification, we also deduce that the section of V by the solid π is not exceptional, though π is in a special position.

6. *Other properties of the double surface*

In the three-space representation of V, let the conical surface which maps S (see § 3) and has vertex at P, be of order $\bar{x}$. This cone meets ${}_{11}C^{10}$, the curve corresponding to the section by the general plane t, in (i) the seventeen pairs of points which lie on the seventeen trisecants through P of the curve, (ii) the $\bar{x}$-fold vertex P, and (iii) the twenty points (N_{ijk}), since in [4] the planes represented by those nodes intersect S in curves which are components of simple, reducible members of (q). Thus we have the equation

$$10\bar{x} = 2 \times 17 + \bar{x} + 20.$$

Whence $\bar{x} = 6$, so that *the double surface is represented by a sextic cone.* The cone has three double generators, namely, the meets by pairs of the planes from P osculating C.

With such properties of S available, it is of interest to find the primal of lowest order in [4] which will meet V in S. Since in [3] the conical sheet representing S is a sextic, and since the surface representing a general prime section is a quintic, we begin with a surface of order thirty, which has P sixfold and the points (N_{ijk}) twelvefold, and contains the sextic cone. Residual to this cone, the surface has order twenty-four and has elevenfold points at (N_{ijk}). As a special surface of this type, we may take four times the system of planes (a_i). Such a surface represents the section of V in [4] by a primal of order six which has fourfold points at (A_i), has their joins double, and contains (once) all the twenty planes represented by (N_{ijk}). Hence the primal meets V in a (double) surface of order $\frac{1}{2}(6 \times 9 - 20) = 17$. We note also that, since there are ten of the exceptional planes through any one of (A_i), these points have multiplicity on the double surface of $\frac{1}{2}(6 \times 4 - 10) = 7$; and since there are four exceptional planes through each join $A_i A_j$ (which

is double on the sextic primal and triple on V) this line on the surface has multiplicity $\frac{1}{2}(3 \times 2 - 4) = 1$.

Thus our conclusion is: *on the threefold V, the double surface S is cut out by a sextic primal which has fourfold points at the six sixfold points of V.*

7. "*Representation*" *on a Segre cubic primal*

By the method of representing curves by points,* we can map on a Segre ten-nodal cubic primal, K say, the threefold V and the surface S.

Let s be a fixed solid through the points A_1, A_2, A_3 in [4]. Each curve of (q) meets s in one further point which we take to "represent" that curve. A double infinity of the curves (q) lie on each member of the fourfold infinity of cubic primals which have nodes at the six points (A_i). Thus each such cubic is represented, in this sense, by the trace on s of the ∞^2 curves of (q) which lie on that primal. These intersections form a cubic surface which has nodes at A_1, A_2, A_3 and simple base points at the (three) meets with s of the lines joining pairs of A_4, A_5, A_6. The system of all such cubic surfaces represents the prime sections in another four-space say $[4]'$ of a ten-nodal cubic primal, which we take as the above locus K.

We have seen that the plane p is met by a double infinity of curves of (q) forming the threefold V. These curves are represented in $[4]'$ by the points of a rational surface which has a triple point, Q_1 say, corresponding to the unique curve q_1 which is trisecant to p. Bronowski has shown (**1**, 15) that this surface is traced on K by a cubic cone which has vertex at Q_1, contains the ten nodes of K and touches its fifteen planes. Thus, *the threefold V, regarded as an aggregate of quartic curves, is represented on the Segre cubic primal K by a surface of order nine traced on K by a cubic cone with vertex at the point representing the triple curve q_1.*

Similarly, in the present special sense, the double surface S is mapped on K by the ∞^1 points corresponding to the curves of (q) which meet the sextic curve c. But c meets in eighteen points each of the six-nodal cubic primals whose intersections with s represent the prime sections of K. Thus on K, S is mapped by a curve c^{18} of order eighteen, which naturally lies on the novenic surface corresponding to p. Also, c^{18} lies on the polar quadric of Q_1 in regard to K. For, from § 9 of (**1**), we know that on K three collinear points Q_1, Q_2, Q_3, say, represent three members of (q) which are associated curves. Consider those points of K which when joined to Q_1 are coincident intersections of K with lines from Q_1. Such points, representing curves of (q) which meet p twice, clearly lie on the polar quadric of Q_1.

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* See Babbage (**5**); also, in detail, Bronowski (**1**).

Thus, *the double surface S, regarded as an aggregate of quartic curves, is represented on the Segre ten-nodal cubic primal K by a curve of order eighteen, traced on the surface representing the threefold V by the polar quadric of the point representing the triple quartic curve q_1.*

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Technical College
Hull

ON THE AVERAGE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION (II)

By M. KAC

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1. In a paper under the same title, which appeared several years ago,* I proved that the average number of real roots of the random algebraic equation

$$X_0 + X_1 x + X_2 x^2 + \ldots + X_{n-1} x^{n-1} = 0,$$

with normally distributed independent coefficients, is asymptotically

$$\frac{2}{\pi} \log n. \tag{1}$$

I have also stated that the same conclusion holds if the X's are uniformly distributed or assume only the values 1 and -1 with equal probabilities.

Upon a closer examination it turns out that the proof which I had in mind, based on the central limit theorem of the calculus of probability, is inapplicable to the discrete case, i.e. when the coefficients assume the values 1 and -1 only. However, the proof is applicable to the case of uniformly distributed coefficients and to a rather wide class of continuous distributions. This situation tends to emphasize the particular interest of the discrete case, which surprisingly enough turns out to be the most difficult.

It may be mentioned that Littlewood and Offord,† who were the first to consider the problem of the average number of real roots of algebraic equations, in all of the three cases mentioned here (i.e. normal, uniformly distributed, discrete), have found them to be essentially equivalent. It should be emphasized that our method yields an essential refinement of the Littlewood-Offord estimate in the continuous cases, but fails to yield any significant result in the discrete case. In fact, so far as the discrete case is concerned, the Littlewood-Offord estimate is the only available one.

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* *Bull. American Math. Soc.* 49 (1943), 314–320. Referred to later as paper I.

† *Proc. Cambridge Phil. Soc.* 35 (1939), 133–148.

Since the simplicity with which the result was obtained in the case of normally distributed coefficients was to some extent misleading, and since the method proved inapplicable to the discrete case, it seemed worth while to show how the proof is carried out in the case of uniformly distributed coefficients. Although the fundamental idea is the same as that used in the case of normally distributed coefficients, the actual execution is incomparably more tedious.

2. We start with a general theorem which is of independent interest.

THEOREM 1. *If $f(x)$ has a continuous derivative in (a, b) which vanishes only at a finite number of points, then the number of real roots of $f(x)$ in (a, b) is given by the formula*

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{\int_a^b \cos(\xi f(x))\,|f'(x)|\,dx\right\}d\xi. \tag{2}$$

It is to be understood that multiple roots are to be counted only once, and that, if either a or b is a root, it is to be counted as $\frac{1}{2}$.

For the proof of this theorem (which is an almost immediate consequence of Dirichlet's discontinuous integral) the reader is referred to § 3 (pp. 612–613) of my note "On the distribution of values of trigonometric sums with linearly independent frequencies".*

THEOREM 2. *Let*

$$X_0 + X_1 x + X_2 x^2 + \ldots + X_{n-1}x^{n-1} = 0 \tag{3}$$

be an algebraic equation, and let

$$X_0, X_1, X_2, \ldots, X_{n-1}$$

be independent random variables each having $\sigma(u)$ as its distribution function. Then the average number of roots of (3) falling within (a, b) is given by the formula

$$\overline{N_n(a,b)} = \frac{1}{2\pi^2}\int_{-\infty}^{+\infty}\int_a^b\int_{-\infty}^{+\infty}$$

$$\times\frac{1}{\eta^2}\left\{\prod_{k=0}^{n-1}\rho(\xi x^k) - \tfrac{1}{2}\left(\prod_{k=0}^{n-1}\rho(x^k\xi + kx^{k-1}\eta) + \prod_{k=0}^{n-1}\rho(x^k\xi - kx^{k-1}\eta)\right)\right\}d\eta\,dx\,d\xi,$$

where

$$\rho(\xi) = \int_{-\infty}^{+\infty}\exp(i\xi u)\,d\sigma(u)$$

is the characteristic function of the distribution function $\sigma(u)$.

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* *American J. of Math.* 65 (1943), 609–615.

This theorem follows from Theorem 1 by considerations entirely analogous to those given in §§ 4 and 5 of my note.*

If we assume that the coefficients of (3) are uniformly distributed in $(-1, 1)$, we have

$$\rho(t) = \frac{\sin t}{t}.$$

Thus for $n \geqslant 2$ the order of the integrations can be changed and we may write

$$\overline{N_n(a, b)} = \frac{1}{2\pi^2} \int_a^b \Phi_n(x)\, dx, \tag{4}$$

where

$$\Phi_n(x) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{\eta^2} \left\{ \prod_{k=0}^{n-1} \rho(x^k \xi) - \tfrac{1}{2} \left(\prod_{k=0}^{n-1} \rho(x^k \xi + k x^{k-1} \eta) + \prod_{k=0}^{n-1} \rho(x^k \xi - k x^{k-1} \eta) \right) \right\} d\xi\, d\eta. \tag{5}$$

Inasmuch as the total average number of real roots is easily seen to be four times the average number of real roots falling within $(0, 1)$, it is sufficient to investigate $\overline{N_n(0, 1)}$.

Remark. In the case of uniformly distributed coefficients, we have

$$\rho(t) = e^{-t^2},$$

and $\Phi_n(x)$ can be evaluated explicitly. The resulting expression for $\overline{N_n(0, 1)}$ is then identical (except for the factor 4) with formula (3) of my paper I.

3. Henceforth we shall assume the coefficients of (3) to be uniformly distributed in $(-1, 1)$, i.e.

$$\rho(t) = \frac{\sin t}{t}.$$

However, we shall use $\rho(t)$ as an abbreviation for $\sin t/t$. This will simplify the appearance of many formulae and will also indicate the extent to which our considerations can be generalized. The principal result of this paper is that

$$\overline{N_n(0, 1)} \sim \frac{1}{2\pi} \log n.$$

We simplify future considerations by first noting that, if $0 < \alpha < 1$, we have, as $n \to \infty$,

$$\overline{N_n(0, \alpha)} = O(1). \tag{6}$$

This follows immediately by observing that, as $n \to \infty$, $\Phi_n(x)$ approaches

$$\Phi_\infty(x) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{\eta^2} \left\{ \prod_{k=0}^{\infty} \rho(x^k \xi) - \tfrac{1}{2} \left(\prod_{k=0}^{\infty} \rho(x^k \xi + k x^{k-1} \eta) + \prod_{k=0}^{\infty} \rho(x^k \xi - k x^{k-1} \eta) \right) \right\} d\xi\, d\eta,$$

* I take this opportunity of correcting the following minor errors in my earlier paper: In the third formula of § 4 replace C by $2C$. In § 6, $\exp(-u^2/2)$ and $\exp(-\lambda v^2/2)$ should be replaced by $\exp(-u^2/4)$ and $\exp(-\lambda v^2/4)$, and the final formula should read

$$E(b \sqrt{n}) = 2 \sqrt{\lambda} \exp(-b^2).$$

Also on p. 610, line 16 from top, the words "is the set H" should be replaced by "is contained in the set H".

and that the convergence is uniform for $0 \leqslant x \leqslant \alpha$. Thus (4) implies that

$$\lim_{n\to\infty} \overline{N_n(0,\alpha)} = \frac{1}{2\pi^2}\int_0^\alpha \Phi_\infty(x)\,dx,$$

and this in turn implies (6).

This simple argument shows that, "on the average", the real roots in (0, 1) can be expected to cluster about 1 and that it is therefore sufficient to prove that

$$\overline{N_n(\alpha,1)} \sim \frac{1}{2\pi}\log n.$$

4. We introduce the following abbreviations which will be used in the sequel:

$$B_n = \sum_{k=0}^{n-1} x^{2k}, \quad C_n = \sum_{k=0}^{n-1} k^2 x^{2(k-1)}, \quad D_n = \sum_{k=0}^{n-1} k x^{2k-1}, \quad R_n = \frac{D_n}{\sqrt{(B_n C_n)}},$$

$$F_n(\eta) = \int_{-\infty}^{+\infty}\prod_{k=0}^{n-1}\rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)d\xi, \quad G_n(\eta) = \int_0^{\infty}\prod_{k=0}^{n-1}\rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)d\xi.$$

We obviously have $\qquad F_n(\eta) = G_n(\eta) + G_n(-\eta).$

Introducing in (5) the new variables

$$\xi' = \frac{\xi}{\sqrt{B_n}}, \quad \eta' = \frac{\eta}{\sqrt{C_n}},$$

we obtain easily

$$\Phi_n(x) = \frac{\sqrt{C_n}}{\sqrt{B_n}}\int_{-\infty}^{+\infty}\frac{F_n(0) - \frac{1}{2}(F_n(\eta) + F_n(-\eta))}{\eta^2}\,d\eta, \tag{7}$$

that is
$$\Phi_n(x) = \frac{2\sqrt{C_n}}{\sqrt{B_n}}\int_0^{\infty}\frac{2G_n(0) - G_n(\eta) - G_n(-\eta)}{\eta^2}\,d\eta. \tag{8}$$

The changes of the order of the integrations are justified by the fact that the integrands are absolutely integrable for $n \geqslant 2$. In what follows absolute constants will be denoted by the letter γ with subscripts.

5. By elementary computations we obtain

$$B_n = \frac{1-x^{2n}}{1-x^2},$$

$$C_n = \frac{(1+x^2)(1-x^{2n}) - n^2(1-x^2)^2 x^{2n-2} - 2n(1-x^2)x^{2n}}{(1-x^2)^3},$$

$$D_n = x\frac{1-x^{2n} - n(1-x^2)x^{2n-2}}{(1-x^2)^2}.$$

The following results will be needed:

(*a*) If $0<\alpha<1$ and $\epsilon>0$, we have

$$\int_\alpha^{\sqrt{(1-n^{-\epsilon})}} \frac{\sqrt{C_n}}{\sqrt{B_n}}\,dx < \epsilon\sqrt{2\log n}+\sqrt{2\log 2}. \tag{9}$$

Indeed,

$$\frac{\sqrt{C_n}}{\sqrt{B_n}} = \frac{\sqrt{(P_n(x))}}{1-x^2},$$

where

$$P_n(x) = \frac{(1+x^2)(1-x^{2n})-n^2(1-x^2)^2x^{2n-2}-2n(1-x^2)x^{2n}}{1-x^{2n}}.$$

But

$$P_n(x)<1+x^2<2,$$

and hence

$$\frac{\sqrt{(P_n(x))}}{1-x^2}<\frac{\sqrt{2}}{1-x}.$$

Thus (9) follows.

In the same way we prove that

$$\int_{\sqrt{(1-n^{-\epsilon})}}^{\sqrt{\{1-3/(n-1)\}}} \frac{\sqrt{C_n}}{\sqrt{B_n}}\,dx = O(\log n). \tag{9a}$$

Noticing that, for

$$1-\frac{3}{n-1}\leqslant x^2\leqslant 1,$$

we have

$$\frac{\sqrt{C_n}}{\sqrt{B_n}} = O(n),$$

we obtain

$$\int_{\sqrt{\{1-3/(n-1)\}}}^{1} \frac{\sqrt{C_n}}{\sqrt{B_n}}\,dx = O(1). \tag{9b}$$

Moreover

$$\int_{\sqrt{(1-n^{-\epsilon})}}^{\sqrt{\{1-3/(n-1)\}}} \frac{\sqrt{C_n}}{\sqrt{B_n}}\sqrt{(1-R_n^2)}\,dx$$
$$= \int_\alpha^{\sqrt{\{1-3/(n-1)\}}} \frac{\sqrt{C_n}}{\sqrt{B_n}}\sqrt{(1-R_n^2)}\,dx - \int_\alpha^{\sqrt{(1-n^{-\epsilon})}} \frac{\sqrt{C_n}}{\sqrt{B_n}}\sqrt{(1-R_n^2)}\,dx,$$

and since

$$\int_\alpha^{\sqrt{(1-n^{-\epsilon})}} \frac{\sqrt{C_n}}{\sqrt{B_n}}\sqrt{(1-R_n^2)}\,dx < \int_\alpha^{\sqrt{(1-n^{-\epsilon})}} \frac{\sqrt{C_n}}{\sqrt{B_n}}\,dx < \epsilon\sqrt{2\log n}+\sqrt{2\log 2},$$

we have

$$\int_\alpha^{\sqrt{\{1-3/(n-1)\}}} \frac{\sqrt{C_n}}{\sqrt{B_n}}\sqrt{(1-R_n^2)}\,dx - \epsilon\sqrt{2\log n}-\sqrt{2\log 2}$$
$$< \int_{\sqrt{(1-n^{-\epsilon})}}^{\sqrt{\{1-3/(n-1)\}}} \frac{\sqrt{C_n}}{\sqrt{B_n}}\sqrt{(1-R_n^2)}\,dx < \int_\alpha^{\sqrt{\{1-3/(n-1)\}}} \frac{\sqrt{C_n}}{\sqrt{B_n}}\sqrt{(1-R_n^2)}\,dx.$$

The reasoning of § 5 of paper I yields

$$\int_{\alpha}^{\surd\{1-3/(n-1)\}} \frac{\surd C_n}{\surd B_n} \surd(1-R_n^2)\,dx = \log n + O(1),$$

and therefore

$$\log n + O(1) - \epsilon\surd 2 \log(2n) < \int_{\surd(1-n^{-\epsilon})}^{\surd\{1-3/(n-1)\}} \frac{\surd C_n}{\surd B_n} \surd(1-R_n^2)\,dx < \log n + O(1). \tag{9c}$$

(*b*) If $1-n^{-\epsilon} < x^2 \leqslant 1$, then a constant γ_1 exists such that

$$B_n > \gamma_1 n^{\epsilon}. \tag{10}$$

This is a trivial consequence of the definition of B_n.

(*c*) If $$1-\frac{1}{n^{\epsilon}} < x^2 < 1 - \frac{3}{n-1},$$

then, for sufficiently large n, $$\frac{\surd B_n}{\surd C_n} < \frac{1}{n^{\epsilon}}. \tag{11}$$

We have by an elementary transformation

$$C_n = \frac{1-x^{2n}+x^2-x^{2n-2}[n(1-x^2)+x^2]^2}{(1-x^2)^3},$$

and hence $$\frac{\surd B_n}{\surd C_n} = \frac{1-x^2}{\surd\{1+Q_n(x)\}}, \tag{12}$$

where $$Q_n(x) = \frac{x^2-x^{2n-2}[n(1-x^2)+x^2]^2}{1-x^{2n}}.$$

Consider the function $L(y) = y^{n-2}[n(1-y)+y]^2$.

It is easy to verify that $L(y)$ is an increasing function for

$$0 < y < 1 - \frac{1}{n-1},$$

and that therefore, for $$x^2 < 1 - \frac{3}{n-1},$$

we have $$L(x^2) < L\left(1-\frac{3}{n-1}\right) = 16\left(1-\frac{3}{n-1}\right)^{n-2}.$$

Hence, for $$1-\frac{1}{n^{\epsilon}} < x^2 < 1-\frac{3}{n-1}, \tag{13}$$

we have

$$Q_n(x) = x^2\frac{1-L(x^2)}{1-x^{2n}} > \left(1-\frac{1}{n^{\epsilon}}\right)\left\{1-16\left(1-\frac{3}{n-1}\right)^{n-2}\right\}\left\{1-\left(1-\frac{1}{n^{\epsilon}}\right)^{n}\right\}^{-1}.$$

Since the right-hand side of this inequality approaches the positive number

$$1-\frac{16}{e^3},$$

as $n\to\infty$, we see that for sufficiently large n and x satisfying the inequalities (13) we have

$$Q_n(x)>0.$$

This when combined with (12) gives (11).

(*d*) For x satisfying the inequality (13) and n sufficiently large we have

$$\frac{x^k}{\sqrt{B_n}}<\frac{\gamma_2}{\sqrt{n^\epsilon}}, \tag{14}$$

$$\frac{kx^{k-1}}{\sqrt{C_n}}<\frac{\gamma_3}{\sqrt{n^\epsilon}}, \tag{15}$$

where γ_2 and γ_3 are absolute constants. In fact

$$\frac{x^k}{\sqrt{B_n}}=\frac{x^k\sqrt{(1-x^2)}}{\sqrt{(1-x^{2n})}}<\frac{1}{\sqrt{n^\epsilon}}\left\{1-\left(1-\frac{3}{n-1}\right)^{2n}\right\}^{-\frac{1}{2}}<\frac{\gamma_2}{\sqrt{n^\epsilon}},$$

and

$$\frac{kx^{k-1}}{\sqrt{C_n}}=\frac{kx^{k-1}(1-x^2)^{\frac{3}{2}}}{\sqrt{\{(1+Q_n(x))(1-x^{2n})\}}}.$$

It is easily seen that for $0\leqslant x\leqslant 1$

$$kx^{k-1}(1-x^2)<2$$

and

$$\frac{\sqrt{(1-x^2)}}{\sqrt{(1-x^{2n})}}\leqslant 1.$$

Since we have seen that for sufficiently large n, $Q_n(x)>0$ the proof of (15) follows with $\gamma_3=2$.

6. We shall need various estimates of $F_n(\eta)$ and $G_n(\eta)$ for different ranges of η and x. These estimates are presented in the form of nine lemmas, all of which are reminiscent of the proof of the central limit theorem of the calculus of probabilities. To make the reading easier we have divided the lemmas into groups, presenting each group in a separate section.

LEMMA 1.

$$\lim_{k\to\infty}\int_{-\infty}^{+\infty}\left|\rho\left(\frac{u}{\sqrt{k}}\right)\right|^k du=\int_{-\infty}^{+\infty}\exp\left(-\frac{u^2}{6}\right)du=\frac{\sqrt{6}}{\sqrt{\pi}}.$$

This result is well known, but I am unable to supply an exact reference.

It follows from Lemma 1 that a constant γ_4 exists such that, for $k\geqslant 2$, we have

$$\int_{-\infty}^{+\infty}\left|\rho\left(\frac{u}{\sqrt{k}}\right)\right|^k du<\gamma_4. \tag{16}$$

LEMMA 2. *If* $\frac{1}{2} < x \leqslant 1$ *and* $n > 3$, *then a constant* γ_5 *exists such that*

$$| G_n(\eta) | < \gamma_5.$$

Let us first assume that $\frac{1}{2} \leqslant x \leqslant 1 - \dfrac{1}{n-1}$.

Since
$$| \rho(u) | \leqslant 1 \quad \text{and} \quad \frac{1}{1-x} \leqslant n-1,$$
we have
$$| G_n(\eta) | < \int_{-\infty}^{+\infty} \prod_{k=0}^{n-1} \left| \rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)\right| d\xi \leqslant \int_{-\infty}^{+\infty} \prod_{k=0}^{m(x)} \left| \rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)\right| d\xi,$$
where $m(x) = [(1-x)^{-1}]$ and $[t]$ denotes the greatest integer not exceeding t. By Hölder's inequality we have
$$\int_{-\infty}^{+\infty} \prod_{k=0}^{m(x)} \left| \rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)\right| d\xi$$
$$\leqslant \prod_{k=0}^{m(x)} \left\{ \int_{-\infty}^{+\infty} \left| \rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)\right|^{m(x)+1} d\xi \right\}^{1/\{m(x)+1\}}, \tag{17}$$
and since
$$\int_{-\infty}^{+\infty} \left| \rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)\right|^{m(x)+1} d\xi = \frac{\sqrt{B_n}}{x^k}\int_{-\infty}^{+\infty} | \rho(u) |^{m(x)+1} du$$
we see that the right-hand side of (17) is equal to
$$\frac{\sqrt{B_n}}{\left\{\prod_{k=1}^{m(x)} x^k\right\}^{1/\{m(x)+1\}}} \int_{-\infty}^{+\infty} | \rho(u) |^{m(x)+1} du.$$
But
$$\left\{\prod_{k=1}^{m(x)} x^k\right\}^{1/\{m(x)+1\}} = x^{\frac{1}{2}m(x)} > x^{1/(2-2x)}, \tag{18}$$
and since
$$\lim_{x \to 1} x^{1/(2-2x)} = \sqrt{e^{-1}}$$
we see that a constant $\gamma_6 > 0$ exists such that for $\frac{1}{2} \leqslant x \leqslant 1$ we have
$$x^{1/(2-2x)} > \gamma_6.$$
Thus the right-hand side of (17) is less than
$$\frac{\sqrt{B_n}}{\gamma_6}\int_{-\infty}^{+\infty} | \rho(u) |^{m(x)+1} du = \frac{\sqrt{B_n}}{\gamma_6\sqrt{\{m(x)+1\}}}\int_{-\infty}^{+\infty} \left| \rho\left(\frac{u}{\sqrt{\{m(x)+1\}}}\right)\right|^{m(x)+1} du.$$
Since $x > \frac{1}{2}$, we have $m(x) > 2$ and hence, by (16),
$$\int_{-\infty}^{+\infty} \left| \rho\left(\frac{u}{\sqrt{\{m(x)+1\}}}\right)\right|^{m(x)+1} du < \gamma_4.$$

Also
$$\frac{\sqrt{B_n}}{\sqrt{\{m(x)+1\}}} = \frac{\sqrt{(1-x^{2n})}}{\sqrt{(1-x^2)\{(1-x)^{-1}+1\}}} < 1, \tag{19}$$
and we see that the conclusion of Lemma 2 holds with
$$\gamma_5 = \frac{\gamma_4}{\gamma_6}.$$
If
$$1-\frac{1}{n} < x \leqslant 1$$
the proof proceeds in exactly the same way, except that instead of $m(x)$ we use $n-1$. Instead of (18) we have
$$\left\{\prod_{k=1}^{n-1} x^k\right\}^{1/n} = x^{\frac{1}{2}(n-1)} > \left(1-\frac{1}{n}\right)^{\frac{1}{2}(n-1)} > \gamma_7 > 0,$$
and instead of (19)
$$\frac{\sqrt{B_n}}{\sqrt{n}} < 1.$$
It may, of course, be assumed that $\gamma_6 = \gamma_7$.

7. It is easily seen that numbers α, $(0<\alpha<1)$ and n_0 can be chosen such that for $n>n_0$ and $\alpha \leqslant x \leqslant 1$ we have simultaneously
$$(1\cdot 3)^{\sqrt{B_n}} > \left(\sum_{j=1}^{n-1} j x^{j-1}\right)^2, \tag{20}$$
$$B_n > 25. \tag{21}$$
We can, of course, assume that $\alpha > \frac{1}{2}$.

In what follows α will always have the meaning just described and it will always be assumed that $n > n_0$. Although we make the assumption $\alpha \leqslant x \leqslant 1$ in Lemmas 3, 4 and 5, we do not utilize the property (20). Only in the proof of Lemma 9 we make full use of (20). The reason for insisting on the assumption $\alpha \leqslant x \leqslant 1$, even in lemmas in which it is not essential, is to establish a range of values of x for which the four lemmas quoted hold simultaneously.

LEMMA 3. *For $\alpha \leqslant x \leqslant 1$ and $|\eta| < \frac{1}{10}\alpha$ we have*
$$\left|\int_{1\cdot 4\sqrt{B_n}}^{\infty} \prod_{k=0}^{n-1} \rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right) d\xi\right| < \frac{\gamma_8}{(1\cdot 3)^{\sqrt{B_n}}}. \tag{22}$$
Since $|\rho(u)| \leqslant 1$ and $[\sqrt{B_n}] \leqslant \sqrt{n} < n-1$ we have
$$\left|\int_{1\cdot 4\sqrt{B_n}}^{\infty} \prod_{k=0}^{n-1} \rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right) d\xi\right| \leqslant \int_{1\cdot 4\sqrt{B_n}}^{\infty} \prod_{k=0}^{[\sqrt{B_n}]} \left|\rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)\right| d\xi.$$
Using the stronger inequality $|\rho(u)| < 1/|u|$, we have
$$\left|\rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)\right| \leqslant \left|\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right|^{-1}$$

Now, using the facts that

$$|\eta| \leqslant \tfrac{1}{10}\alpha, \quad \frac{1}{x} > \alpha \quad \text{and} \quad \frac{k}{\sqrt{C_n}} \leqslant \frac{\sqrt{B_n}}{\sqrt{C_n}} < 1,$$

we obtain
$$\left|\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right| > x^k\left(\frac{\xi}{\sqrt{B_n}} - \frac{1}{10}\right)$$
and therefore

$$\left|\int_{1\cdot 4\sqrt{B_n}}^{\infty} \prod_{k=0}^{n-1} \rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right) d\xi\right| < \frac{1}{\prod\limits_{k=1}^{[\sqrt{B_n}]} x^k} \int_{1\cdot 4\sqrt{B_n}}^{\infty} \left(\frac{\xi}{\sqrt{B_n}} - \frac{1}{10}\right)^{-[\sqrt{B_n}]} d\xi$$
$$= \frac{1}{\prod\limits_{k=1}^{[\sqrt{B_n}]} x^k} \frac{\sqrt{B_n}}{[\sqrt{B_n}] - 1} \frac{1}{(1\cdot 3)^{[\sqrt{B_n}]-1}}.$$

We have
$$\prod_{k=1}^{[\sqrt{B_n}]} x^k > x^{B_n} > x^{1/(1-x^2)}$$

and since $x > \alpha$ it follows that $x^{1/(1-x^2)}$ is bounded from below by a positive constant.

Remembering that $\sqrt{B_n} > 5$ we see that the conclusion of Lemma 3 holds.

LEMMA 4. *If*
$$\alpha^2 < 1 - \frac{1}{n^\epsilon} < x^2 < 1 - \frac{3}{n-1} \quad \text{and} \quad |\eta| < Q$$

there exists a number $N(Q,\epsilon)$ depending on Q and ϵ, such that, for $n > N(Q,\epsilon)$, the conclusion of Lemma 3 holds.

For $k < \sqrt{B_n}$ we have
$$\frac{kx^{k-1}}{\sqrt{C_n}}|\eta| < \frac{x^k}{\alpha}\frac{\sqrt{B_n}}{\sqrt{C_n}}Q.$$

By making n sufficiently large we see by the use of (11) that the inequality

$$\frac{\sqrt{B_n}}{\sqrt{C_n}}Q < \frac{\alpha}{10}$$

can be made to hold. The rest of the proof is identical with that of Lemma 3.

LEMMA 5. *For $\alpha < x \leqslant 1$, $|\eta| < \frac{1}{10}$ and $0 \leqslant \xi \leqslant 1\cdot 4\sqrt{B_n}$, we have*

$$\left|\prod_{k=0}^{n-1} \rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)\right| < \exp\left\{-\frac{\cos 1\cdot 5}{6}(\xi^2 + 2R_n\xi\eta + \eta^2)\right\}, \tag{23}$$

where
$$R_n = \frac{1}{\sqrt{(B_n C_n)}} \sum_{k=0}^{n-1} kx^{2k-1} \leqslant 1.$$

From the assumptions of the lemma it follows that

$$\left|\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right| \leqslant 1\cdot 5.$$

Since $$|\rho(u)| = \frac{\sin|u|}{|u|} = 1-\frac{u^2}{6}\cos\vartheta, \quad 0<\vartheta<|u|,$$
we have for $|u|<1{\cdot}5$
$$|\rho(u)| < 1-\frac{u^2}{6}\cos 1{\cdot}5 < \exp\left\{-\frac{\cos 1{\cdot}5}{6}u^2\right\}.$$
Thus
$$\left|\prod_{k=0}^{n-1}\rho\left(\frac{x^k}{\sqrt{B_n}}\xi+\frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)\right| \leqslant \exp\left\{-\frac{\cos 1{\cdot}5}{6}\sum_{k=0}^{n-1}\left(\frac{x^k}{\sqrt{B_n}}\xi+\frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)^2\right\},$$
and the conclusion of Lemma 5 follows, in view of the definitions of B_n and C_n in § 4. It follows at once from Schwarz's inequality that $R_n \leqslant 1$.

LEMMA 6. *There exists an integer $N(P, Q, \epsilon)$ such that, for $n > N(P, Q, \epsilon)$ and*
$$1-\frac{1}{n^\epsilon}<x^2<1-\frac{3}{n-1}, \quad |\eta|<Q, \quad P>0,$$
we have $$P<1{\cdot}4\sqrt{B_n}$$
and $$\left|\int_P^{1\cdot4\sqrt{B_n}}\prod_{k=0}^{n-1}\rho\left(\frac{x^k}{\sqrt{B_n}}\xi+\frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)d\xi\right| < \int_{P-Q}^{\infty}\exp\left\{-\frac{\cos 1{\cdot}5}{6}\xi^2\right\}d\xi. \tag{24}$$

The use of (10) and (15) shows that a number $N(P, Q, \epsilon)$ can be chosen so that, for $n > N(P, Q, \epsilon)$, we have simultaneously
$$B_n>\gamma_1 n^\epsilon, \quad P<1{\cdot}4\sqrt{(\gamma_1 n^\epsilon)}, \quad \frac{kx^{k-1}}{\sqrt{C_n}}<\frac{\gamma_3}{\sqrt{n^\epsilon}}, \quad \frac{\gamma_3}{\sqrt{n^\epsilon}}Q<\frac{1}{10},$$
and hence $$\frac{kx^{k-1}}{\sqrt{C_n}}|\eta|<\frac{1}{10}.$$

Thus the reasoning by means of which (23) was established is applicable, and (23) is seen to hold.

Hence we have *a fortiori*
$$\left|\int_P^{1\cdot4\sqrt{B_n}}\prod_{k=0}^{n-1}\rho\left(\frac{x^k}{\sqrt{B_n}}\xi+\frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)d\xi\right| < \int_P^{\infty}\exp\left\{-\frac{\cos 1{\cdot}5}{6}(\xi^2+2R_n\xi\eta+\eta^2)\right\}d\xi.$$
Now
$$\int_P^\infty \exp\left\{-\frac{\cos 1{\cdot}5}{6}(\xi^2+2R_n\xi\eta+\eta^2)\right\}d\xi < \int_P^\infty \exp\left\{-\frac{\cos 1{\cdot}5}{6}(\xi+R_n\eta)^2\right\}d\xi$$
$$= \int_{P+R_n\eta}^\infty \exp\left\{\frac{\cos 1{\cdot}5}{6}\xi^2\right\}d\xi < \int_{P-Q}^\infty \exp\left\{\frac{\cos 1{\cdot}5}{6}\xi^2\right\}d\xi.$$
The first and last steps are justified by the inequality
$$0<R_n\leqslant 1,$$
which is an immediate consequence of Schwarz's inequality.

8. LEMMA 7. *If* $|u| < \frac{1}{2}$, *we have*

$$-\frac{u^2}{6} - \frac{|u|^3}{12} < \log \rho(u) < -\frac{u^2}{6} + \frac{|u^3|}{12}.$$

In fact, for $|u| < \frac{1}{2}\pi$ we have

$$\rho(u) = \frac{\sin u}{u} = \frac{\sin |u|}{|u|} = 1 - \frac{u^2}{6} + \frac{|u|^3}{24} \sin \vartheta, \quad 0 < \vartheta < |u|.$$

Thus
$$1 - \frac{u^2}{6} < \rho(u) < 1 - \frac{u^2}{6} + \frac{|u|^3}{12},$$

and
$$\log\left(1 - \frac{u^2}{6}\right) < \log \rho(u) < \log\left(1 - \frac{u^2}{6} + \frac{|u|^3}{12}\right) < -\frac{u^2}{6} + \frac{|u|^3}{12}.$$

On the other hand, if
$$0 < z < \frac{3 - \sqrt{5}}{2},$$

we see that
$$\frac{z^2}{1-z} < z^{\frac{3}{2}},$$

and hence that

$$1 + z + z^2 + \ldots = 1 + z + \frac{z^2}{1-z} < 1 + z + z^{\frac{3}{2}} < \exp(z + z^{\frac{3}{2}}).$$

Thus
$$\log \frac{1}{1-z} < z + z^{\frac{3}{2}}$$

and
$$\log(1-z) > -z - z^{\frac{3}{2}}.$$

If $|u| < \frac{1}{2}$, we see that
$$\frac{u^2}{6} < \frac{3 - \sqrt{5}}{2},$$

and therefore
$$\log\left(1 - \frac{u^2}{6}\right) > -\frac{u^2}{6} - \frac{|u|^3}{\sqrt{216}} > -\frac{u^2}{6} - \frac{|u|^3}{12}.$$

This completes the proof of Lemma 7.

LEMMA 8. *There exists a number* $M(P, Q, \epsilon)$ *such that, for* $n > M(P, Q, \epsilon)$ *and*

$$1 - \frac{1}{n^\epsilon} < x^2 < 1 - \frac{3}{n-1}, \quad |\xi| < P, \quad |\eta| < Q,$$

we have
$$\left| \prod_{k=0}^{n-1} \rho\left(\frac{x^k}{\sqrt{B_n}} \xi + \frac{k x^{k-1}}{\sqrt{C_n}} \eta\right) - \exp\{-\tfrac{1}{6}(\xi^2 + 2R_n \xi\eta + \eta^2)\} \right|$$
$$< \exp\left\{\frac{\gamma_9}{\sqrt{n^\epsilon}} (P+Q)^3\right\} - 1. \qquad (25)$$

The number $M(P, Q, \epsilon)$ is chosen in such a way that, for $n > M(p, Q, \epsilon)$, the inequalities (14) and (15) hold and also

$$\frac{\gamma_2}{\sqrt{n^\epsilon}} P + \frac{\gamma_3}{\sqrt{n^\epsilon}} Q < \tfrac{1}{2}.$$

Thus, for $n > M(P, Q, \epsilon)$, we have

$$\frac{x^k}{\sqrt{B_n}} |\xi| + \frac{kx^{k-1}}{\sqrt{C_n}} |\eta| < \tfrac{1}{2}$$

and an application of Lemma 7 together with the definitions of B_n, C_n, D_n, and R_n yields

$$\exp\{-\tfrac{1}{6}(\xi^2 + 2R_n\xi\eta + \eta^2)\}\exp\left\{-\frac{S_n}{12}\right\} < \prod_{k=0}^{n-1} \rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)$$
$$< \exp\{-\tfrac{1}{6}(\xi^2 + 2R_n\xi\eta + \eta^2)\}\exp\left\{\frac{S_n}{12}\right\},$$

where
$$S_n = \sum_{k=0}^{n-1}\left(\frac{x^k}{\sqrt{B_n}} |\xi| + \frac{kx^{k-1}}{\sqrt{C_n}} |\eta|\right)^3.$$

Thus

$$\left|\prod_{k=0}^{n-1} \rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right) - \exp\{-\tfrac{1}{6}(\xi^2 + 2R_n\xi\eta + \eta^2)\}\right| < \exp\left\{\frac{|S_n|}{12}\right\} - 1.$$

We now observe that (14) and (15) imply that

$$S_n < \frac{\gamma_2}{\sqrt{n^\epsilon}} |\xi|^3 + \frac{3\gamma_3}{\sqrt{n^\epsilon}}\xi^2 |\eta| + \frac{3\gamma_2}{\sqrt{n^\epsilon}}\xi |\eta|^2 + \frac{\gamma_3}{\sqrt{n^\epsilon}} |\eta|^3 < \frac{\operatorname{Max}(\gamma_2, \gamma_3)}{\sqrt{n^\epsilon}}(P+Q)^3,$$

and this completes the proof of Lemma 8.

9. LEMMA 9. *A constant γ_{10} exists such that, for $\alpha < x \leqslant 1$, $n > n_0$ and $|\eta| < \frac{1}{10}\alpha$, we have*

$$\frac{|2G_n(0) - G_n(\eta) - G_n(-\eta)|}{\eta^2} < \gamma_{10}. \tag{26}$$

From Taylor's theorem it follows immediately that

$$\frac{2G_n(0) - G_n(\eta) - G_n(-\eta)}{\eta^2} = \tfrac{1}{2}(G_n''(\vartheta_1) + G_n''(\vartheta_2)),$$

where ϑ_1 is a number between $-\eta$ and 0 and ϑ_2 a number between 0 and η.

It is enough to prove then that with the conditions of the lemma we have

$$|G_n''(\eta)| < \gamma_{10}.$$

From the definition of $G_n(\eta)$ given in § 4 we have

$$G_n''(\eta) = \sum_{j=0}^{n-1} \frac{j^2 x^{2(j-1)}}{C_n} \int_0^\infty \rho''\left(\frac{x^j}{\sqrt{B_n}}\xi + \frac{jx^{j-1}}{\sqrt{C_n}}\eta\right)\Pi'\, d\xi$$
$$+ \sum_{j,l=0}^{n-1} \frac{jlx^{j-1}x^{l-1}}{C_n} \int_0^\infty \rho'\left(\frac{x^j}{\sqrt{B_n}}\xi + \frac{jx^{j-1}}{\sqrt{C_n}}\eta\right)\rho'\left(\frac{x^l}{\sqrt{B_n}}\xi + \frac{lx^{l-1}}{\sqrt{C_n}}\eta\right)\Pi''\, d\xi,$$

where, in the double sum, $j \neq l$ and the prime and double prime on the product signs indicate respectively the omission of one term and two terms. We first note that a constant γ_{11} exists such that

$$|\rho''(u)| < \gamma_{11}.$$

Thus $$\left|\int_0^\infty \rho''\Pi'\,d\xi\right| < \gamma_{11}\int_0^\infty |\,\Pi'\,|\,d\xi < \gamma_{11}\int_{-\infty}^{+\infty} |\,\Pi'\,|\,d\xi.$$

The reasoning of Lemma 2 is applicable since an omission of one term from the product has almost no bearing on the proof. We see then that a constant γ_{12} exists such that

$$\left|\int_0^\infty \rho''\Pi'\,d\xi\right| < \gamma_{12}.$$

This fact, together with the definition of C_n, implies immediately that

$$\left|\sum_{j=0}^{n-1}\frac{j^2x^{2(j-1)}}{C_n}\int_0^\infty \rho''\Pi'\,d\xi\right| < \gamma_{12}. \tag{27}$$

The estimate of the double sum is considerably more involved. We write

$$\int_0^\infty \rho'\rho'\Pi''\,d\xi = \int_0^{1\cdot 4\sqrt{B_n}} + \int_{1\cdot 4\sqrt{B_n}}^\infty.$$

Since $|\,\rho'(u)\,| \leqslant 2$, we have

$$\left|\int_{1\cdot 4\sqrt{B_n}}^\infty \rho'\rho'\Pi''\,d\xi\right| < 4\int_{1\cdot 4\sqrt{B_n}}^\infty |\,\Pi''\,|\,d\xi.$$

We may, of course, assume that $n \geqslant 6$ so that $\sqrt{B_n} \leqslant \sqrt{n} < n-3$ and we obtain (since $|\,\rho(u)\,| \leqslant 1$)

$$\int_{1\cdot 4\sqrt{B_n}}^\infty |\,\Pi''\,|\,d\xi < \int_{1\cdot 4\sqrt{B_n}}^\infty \prod_{k<\sqrt{B_n}}'' \left|\rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)\right| d\xi.$$

The estimate of the right-hand side proceeds now exactly as in the proof of Lemma 3, except that the integral

$$\int_{1\cdot 4\sqrt{B_n}}^\infty \left(\frac{\xi}{\sqrt{B_n}} - \frac{1}{10}\right)^{-[\sqrt{B_n}]} d\xi$$

and the product $$\prod_{k=1}^{[\sqrt{B_n}]} x^k$$

encountered in proving Lemma 3 may have to be replaced by the integral

$$\int_{1\cdot 4\sqrt{B_n}}^\infty \left(\frac{\xi}{\sqrt{B_n}} - \frac{1}{10}\right)^{-[\sqrt{B_n}]+2} d\xi$$

and the product $$\prod_{k=1}^{[\sqrt{B_n}]}{}'' x^k$$

respectively.

Since for $x > \alpha$ and $n > n_0$ we have $[\sqrt{B_n}] \geqslant 5$, the final estimate is of the same form as in Lemma 3 except that γ_8 should be replaced by a different constant γ_{13}. Thus

$$\left|\int_{1\cdot 4\sqrt{B_n}}^\infty \rho'\rho'\Pi''\,d\xi\right| < \frac{4\gamma_{13}}{(1\cdot 3)^{\sqrt{B_n}}}. \tag{28}$$

In order to estimate the integral

$$\int_0^{1\cdot 4\sqrt{B_n}} \rho'\rho'\Pi''\,d\xi,$$

we first notice that, since $0<\xi<1\cdot 4\sqrt{B_n}$ and $|\eta|<\frac{1}{10}\alpha<\frac{1}{10}$, we have

$$\left|\frac{x^k}{\sqrt{B_n}}\xi+\frac{kx^{k-1}}{\sqrt{C_n}}\eta\right|\leqslant\frac{\xi}{\sqrt{B_n}}+|\eta|<1\cdot 5<\tfrac{1}{2}\pi.$$

It is easy to verify that, for $|u|<\frac{1}{2}\pi$,

$$0<\frac{|\rho'(u)|}{\rho(u)}<\gamma_{14}|u|$$

and we can write

$$|\rho'\rho'\Pi''|=\left|\frac{\rho'}{\rho}\frac{\rho'}{\rho}\prod_{k=0}^{n-1}\right|$$
$$<\gamma_{14}^2\left|\frac{x^j}{\sqrt{B_n}}\xi+\frac{jx^{j-1}}{\sqrt{C_n}}\right|\left|\frac{x^l}{\sqrt{B_n}}\xi+\frac{lx^{l-1}}{\sqrt{C_n}}\eta\right|\left|\prod_{k=0}^{n-1}\rho\left(\frac{x^k}{\sqrt{B_n}}\xi+\frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)\right|.$$

Thus by Lemma 5

$$|\rho'\rho'\Pi''|<\gamma_{14}^2\left|\frac{x^j}{\sqrt{B_n}}\xi+\frac{jx^{j-1}}{\sqrt{C_n}}\eta\right|\left|\frac{x^l}{\sqrt{B_n}}\xi+\frac{lx^{l-1}}{\sqrt{C_n}}\eta\right|$$
$$\times\exp\left\{-\frac{\cos 1\cdot 5}{6}(\xi^2+2R_n\xi\eta+\eta^2)\right\}$$

and therefore

$$\left|\sum_{j,l=0}^{n-1}\frac{jlx^{j-1}x^{l-1}}{C_n}\int_0^{1\cdot 4\sqrt{B_n}}\rho'\rho'\Pi''\,d\xi\right|$$
$$<\gamma_{14}^2\int_0^{1\cdot 4\sqrt{B_n}}(R_n|\xi|+|\eta|)^2\exp\left\{-\frac{\cos 1\cdot 5}{6}(\xi^2+2R_n\xi\eta+\eta^2)\right\}d\xi$$
$$<\gamma_{14}^2\int_{-\infty}^{+\infty}(|\xi|+0\cdot 2)^2\exp\left(-\frac{\cos 1\cdot 5}{6}\xi^2\right)d\xi=\gamma_{15}. \tag{29}$$

Use has been made of the fact that

$$\sum_{j,l=0}^{n-1}\frac{jlx^{j-1}x^{l-1}}{C_n}\left|\frac{x^j}{\sqrt{B_n}}\xi+\frac{jx^{j-1}}{\sqrt{C_n}}\eta\right|\left|\frac{x^l}{\sqrt{B_n}}\xi+\frac{lx^{l-1}}{\sqrt{C_n}}\eta\right|$$
$$<\left(\sum_{j=0}^{n-1}\left|\frac{jx^{2j-1}}{\sqrt{B_n}C_n}\xi+\frac{j^2x^{2j-2}}{C_n}\eta\right|\right)^2\leqslant(R_n|\xi|+|\eta|)^2.$$

From (28) it follows that

$$\left|\sum_{j,l=0}^{n-1}\frac{jlx^{j-1}x^{l-1}}{C_n}\int_{1\cdot 4\sqrt{B_n}}^{\infty}\rho'\rho'\Pi''\,d\xi\right|<\frac{4\gamma_{13}}{(1\cdot 3)^{\sqrt{B_n}}}\frac{\left(\sum_0^{n-1}jx^{j-1}\right)^2}{C_n}<4\gamma_{13}. \tag{30}$$

In deriving (30) use has been made of the trivial fact that $C_n > 1$ and of formula (20). This is the only place in the whole proof where formula (20) is utilized and where the restriction $x > \alpha$ is really necessary.

Combining (27), (29) and (30) we obtain

$$|G''(\eta)| < \gamma_{12} + \gamma_{15} + 4\gamma_{13}$$

and the conclusion of Lemma 9 follows with $\gamma_{10} = \gamma_{12} + \gamma_{15} + 4\gamma_{13}$.

10. We are now ready to prove the principal result of this paper, namely that

$$\overline{N(0, 1)} \sim \frac{1}{2\pi} \log n.$$

As we observed in § 3 it will suffice to prove that

$$\overline{N(\alpha, 1)} \sim \frac{1}{2\pi} \log n.$$

Let

$$\Psi_n(x) = 2 \int_0^\infty \frac{2G_n(0) - G_n(\eta) - G_n(-\eta)}{\eta^2} d\eta$$

so that

$$\overline{N(\alpha, 1)} = \frac{1}{2\pi^2} \int_\alpha^1 \frac{\sqrt{C_n}}{\sqrt{B_n}} \Psi_n(x)\, dx.$$

We now establish the following two estimates:

I. For $\alpha < x \leqslant 1$, $n > n_0$ we have

$$|\Psi_n(x)| < \gamma_{16}. \tag{31}$$

II. For

$$1 - \frac{1}{n^\epsilon} < x^2 < 1 - \frac{3}{n-1} \tag{32}$$

and $\epsilon_1 > 0$ we can find a number $K(\epsilon, \epsilon_1)$ depending on ϵ and ϵ_1 such that, for $n > K(\epsilon, \epsilon_1)$ and x satisfying (32), we have

$$|\Psi_n(x) - \pi\sqrt{(1 - R_n^2)}| < \epsilon_1. \tag{33}$$

Proof of I. We write

$$\Psi_n(x) = 2 \int_0^{\alpha/10} \frac{2G_n(0) - G_n(\eta) - G_n(-\eta)}{\eta^2} d\eta + 2 \int_{\alpha/10}^\infty \frac{2G_n(0) - G_n(\eta) - G_n(-\eta)}{\eta^2} d\eta$$

and estimate the first integral by the use of Lemma 9 and the second integral by the use of Lemma 2.

We obtain

$$|\Psi_n(x)| < 2\gamma_{10} \frac{\alpha}{10} + 8\gamma_5 \frac{10}{\alpha},$$

and hence I holds with $\gamma_{16} = \frac{1}{5}\alpha\gamma_{10} + 80\gamma_5/\alpha$.

Proof of II. Let $0 < \delta < \frac{1}{10}\alpha$. We write

$$\Psi_n(x) = 2\int_0^{\delta} \frac{2G_n(0)-G_n(\eta)-G_n(-\eta)}{\eta^2}\,d\eta + 2\int_{\delta}^{1/\delta} \frac{2G_n(0)-G_n(\eta)-G_n(-\eta)}{\eta^2}\,d\eta + 2\int_{1/\delta}^{\infty} \frac{2G_n(0)-G_n(\eta)-G_n(-\eta)}{\eta^2}\,d\eta = I_1+I_2+I_3.$$

By Lemma 9 we have
$$|I_1| < 2\delta\gamma_{10}, \tag{34}$$
and by Lemma 2
$$|I_3| < 8\delta\gamma_5. \tag{35}$$

To estimate the integral I_2 we investigate the difference

$$G_n(\eta)-H_n(\eta),$$

where
$$H_n(\eta) = \int_0^{\infty} \exp\{-\tfrac{1}{6}(\xi^2+2R_n\xi\eta+\eta^2)\}\,d\xi, \tag{36}$$

and η ranges between δ and $1/\delta$.

It follows from Lemma 8 (with $P = 2/\delta$, $Q = 1/\delta$) that for $n > M(2/\delta, 1/\delta, \epsilon)$ and $|\eta| < 1/\delta$ we have

$$\left|\int_0^{2/\delta} \prod_{k=0}^{n-1} \rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)d\xi - \int_0^{2/\delta} \exp\{-\tfrac{1}{6}(\xi^2+2R_n\xi\eta+\eta^2)\}\,d\xi\right| < \frac{2}{\delta}\left(\exp\left\{\frac{27\gamma_9}{\delta^3\sqrt{n^{\epsilon}}}\right\}-1\right). \tag{37}$$

Applying Lemma 6 with $P = 2/\delta$ and $Q = 1/\delta$ we see that for $n > N(2/\delta, 1/\delta, \epsilon)$ and x in the range (32) we have

$$1{\cdot}4\sqrt{B_n} > \frac{2}{\delta}$$

and
$$\left|\int_{2/\delta}^{1\cdot4\sqrt{B_n}} \prod_{k=0}^{n-1} \rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)d\xi\right| < \int_{1/\delta}^{\infty} \exp\left\{-\frac{\cos 1{\cdot}5}{6}\xi^2\right\}d\xi. \tag{38}$$

Now by Lemma 4 with $Q = 1/\delta$ we have for $n > N(1/\delta, \epsilon)$, x in the range (32), and $|\eta| < 1/\delta$

$$\left|\int_{1\cdot4\sqrt{B_n}}^{\infty} \prod_{k=0}^{n-1} \rho\left(\frac{x^k}{\sqrt{B_n}}\xi + \frac{kx^{k-1}}{\sqrt{C_n}}\eta\right)d\xi\right| < \frac{\gamma_{10}}{(1{\cdot}3)^{\sqrt{B_n}}} < \frac{\gamma_{10}}{(1{\cdot}3)^{\gamma_1\sqrt{n^{\epsilon}}}}. \tag{39}$$

Finally, for $|\eta| < 1/\delta$ and regardless of x and n, we have the inequality

$$\int_{2/\delta}^{\infty} \exp\{-\tfrac{1}{6}(\xi^2+2R_n\xi\eta+\eta^2)\}\,d\xi < \int_{1/\delta}^{\infty} \exp\left\{-\frac{\xi^2}{6}\right\}d\xi < \int_{1/\delta}^{\infty} \exp\left\{-\frac{\cos 1{\cdot}5}{6}\xi^2\right\}d\xi. \tag{40}$$

If now $n > \operatorname{Max}\{M(2/\delta, 1/\delta, \epsilon), N(2/\delta, 1/\delta, \epsilon), N(1/\delta, \epsilon)\} = \nu(\delta, \epsilon)$ and x satisfies (32), by combining (37), (38), (39) and (40), we see that, for $|\eta| < 1/\delta$, we have

$$|G_n(\eta) - H_n(\eta)| < \frac{2}{\delta}\left(\exp\left\{\frac{27\gamma_9}{\delta^3\sqrt{n^\epsilon}}\right\} - 1\right) + \frac{\gamma_{10}}{(1\cdot 3)^{\gamma_1\sqrt{n^\epsilon}}} + 2\int_{1/\delta}^{\infty} \exp\left\{-\frac{\cos 1\cdot 5}{6}\xi^2\right\} d\xi.$$

Thus for $n > \nu(\delta, \epsilon)$ and x in the range (32) we have

$$\left| I_2 - 2\int_{\delta}^{1/\delta} \frac{2H_n(0) - H_n(\eta) - H_n(-\eta)}{\eta^2} d\eta \right|$$
$$< \frac{8}{\delta^3}\left\{\frac{2}{\delta}\left(\exp\left\{\frac{27\gamma_9}{\delta^3\sqrt{n^\epsilon}}\right\} - 1\right) + \frac{\gamma_{10}}{(1\cdot 3)^{\gamma_1\sqrt{n^\epsilon}}} + 2\int_{1/\delta}^{\infty} \exp\left\{-\frac{\cos 1\cdot 5}{6}\xi^2\right\} d\xi\right\}. \quad (41)$$

An elementary computation shows that

$$2\int_0^{\infty} \frac{2H_n(0) - H_n(\eta) - H_n(-\eta)}{\eta^2} d\eta = \pi\sqrt{(1 - R_n^2)}.$$

It is also easily verified that

$$\left| \pi\sqrt{(1 - R_n^2)} - 2\int_{\delta}^{1/\delta} \frac{2H_n(0) - H_n(\eta) - H_n(-\eta)}{\eta^2} d\eta \right| = O(\delta) \quad (42)$$

uniformly for $0 < x \leqslant 1$ (and hence, *a fortiori*, for x in the range (32)). Combining (34), (35), (41) and (42), we see that for x in the range (32) and $n > \nu(\delta, \epsilon)$ we have

$$|\Psi_n(x) - \pi\sqrt{(1 - R_n^2)}| < 2\gamma_{10}\delta + 8\gamma_5\delta + O(\delta)$$
$$+ \frac{8}{\delta^3}\left\{\frac{2}{\delta}\left(\exp\left\{\frac{27\gamma_9}{\delta^3\sqrt{n^\epsilon}}\right\} - 1\right) + \frac{\gamma_{10}}{(1\cdot 3)^{\gamma_1\sqrt{n^\epsilon}}} + 2\int_{1/\delta}^{\infty} \exp\left\{-\frac{\cos 1\cdot 5}{6}\xi^2\right\} d\xi\right\}$$
$$= O(\delta) + \frac{16}{\delta^3}\int_{1/\delta}^{\infty} \exp\left\{-\frac{\cos 1\cdot 5}{6}\xi^2\right\} d\xi + \frac{16}{\delta^4}\left(\exp\left\{\frac{27\gamma_9}{\delta^3\sqrt{n^\epsilon}}\right\} - 1\right) + \frac{8\gamma_{10}}{\delta^3(1\cdot 3)^{\gamma_1\sqrt{n^\epsilon}}}.$$

Now, with ϵ and ϵ_1 given, we first choose $\delta(\epsilon_1)$ (depending on ϵ_1) in such a way that

$$O(\delta) + \frac{16}{\delta^3}\int_{1/\delta}^{\infty} \exp\left\{-\frac{\cos 1\cdot 5}{6}\xi^2\right\} d\xi < \tfrac{1}{2}\epsilon_1.$$

With $\delta = \delta(\epsilon_1)$ thus chosen, we find a number $\nu_1(\delta(\epsilon_1), \epsilon)$ depending on $\delta(\epsilon_1)$ (and hence on ϵ_1) such that, for $n > \nu_1(\delta(\epsilon_1), \epsilon)$, we have

$$\frac{16}{\delta^4}\left(\exp\left\{\frac{27\gamma_9}{\delta^3\sqrt{n^\epsilon}}\right\} - 1\right) + \frac{8\gamma_{10}}{\delta^3(1\cdot 3)^{\gamma_1\sqrt{n^\epsilon}}} < \tfrac{1}{2}\epsilon_1.$$

Putting $K(\epsilon, \epsilon_1) = \operatorname{Max}\{\nu(\delta(\epsilon_1), \epsilon), \nu_1(\delta(\epsilon_1), \epsilon)\}$ we complete the proof of (33).

The proof of the main result follows now very easily. In fact, we have

$$\overline{N(\alpha,1)} = \frac{1}{2\pi^2}\int_\alpha^{\sqrt{(1-n^{-\epsilon})}} \frac{\sqrt{C_n}}{\sqrt{B_n}}\Psi_n(x)\,dx + \frac{1}{2\pi^2}\int_{\sqrt{(1-n^{-\epsilon})}}^{\sqrt{\{1-3/(n-1)\}}} \frac{\sqrt{C_n}}{\sqrt{B_n}}\Psi_n(x)\,dx + \frac{1}{2\pi^2}\int_{\sqrt{\{1-3/(n-1)\}}}^{1} \frac{\sqrt{C_n}}{\sqrt{B_n}}\Psi_n(x)\,dx$$

and hence, by (31) and (33) and for $n > K(\epsilon, \epsilon_1)$,

$$\frac{1}{2\pi^2}\int_{\sqrt{(1-n^{-\epsilon})}}^{\sqrt{\{1-3/(n-1)\}}} \frac{\sqrt{C_n}}{\sqrt{B_n}}(\pi\sqrt{(1-R_n^2)}-\epsilon_1)\,dx < \overline{N(\alpha,1)} < \frac{\gamma_{16}}{2\pi^2}\int_\alpha^{\sqrt{(1-n^{-\epsilon})}} \frac{\sqrt{C_n}}{\sqrt{B_n}}\,dx + \frac{1}{2\pi^2}\int_{\sqrt{(1-n^{-\epsilon})}}^{\sqrt{\{1-3/(n-1)\}}} \frac{\sqrt{C_n}}{\sqrt{B_n}}(\pi\sqrt{(1-R_n^2)}+\epsilon_1)\,dx + \frac{\gamma_{16}}{2\pi^2}\int_{\sqrt{\{1-3/(n-1)\}}}^{1} \frac{\sqrt{C_n}}{\sqrt{B_n}}\,dx.$$

Making use of (9), (9a), (9b) and (9c) we complete the proof by observing that ϵ and ϵ_1 are quite arbitrary.

Cornell University
Ithaca, N.Y., U.S.A.

THE DIFFERENTIABLE PARAMETRIZATION OF A SURFACE

By A. J. WARD

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Let O be any point of a two-dimensional manifold in three-dimensional Euclidean space, so that some neighbourhood S of O, taken closed, is the topological image of a closed square. This neighbourhood may be parametrized in an infinite number of ways, the coordinates x, y and z being expressed as functions f_1, f_2 and f_3 of real variables u and v in the range $|u| \leqslant 1$, $|v| \leqslant 1$, where the correspondence between (u, v) and (x, y, z) is one-one. We shall say that the parametrization is *differentiable at* O if, at the corresponding values of u and v, the functions f_1, f_2 and f_3 are all totally differentiable and the Jacobians $\partial(f_i, f_j)/\partial(u, v)$ are not all zero. It is clear that, if any such differentiable parametrization exists, we may, by simple transformations of the variables, suppose that

$$\frac{\partial f_1}{\partial u} = \frac{\partial f_2}{\partial v} = 1, \quad \frac{\partial f_1}{\partial v} = \frac{\partial f_2}{\partial u} = \frac{\partial f_3}{\partial u} = \frac{\partial f_3}{\partial v} = 0.$$

The problem discussed in this paper is that of finding a necessary and sufficient condition, involving S only as a set of points, for the existence of a parametrization differentiable at a given point O. It is clear that a necessary condition is the existence of a tangent plane; that is a plane (which we may take to be $z = 0$, O being the origin) such that $z/\sqrt{(x^2+y^2+z^2)} \to 0$ uniformly as $\sqrt{(x^2+y^2+z^2)} \to 0$, for (x, y, z) lying in S. This condition, however, is easily seen not to be sufficient.* A further condition arises from the fact that, if the derivatives are normalized as above, any curve in S near O differs but little from its pattern curve in the (u, v) plane, or rather from the congruent curve obtained by putting $x = u$, $y = v$, $z = 0$. We have, in fact (if there is a differentiable parametrization), the following result.

There exists a function $\psi(r)$, which $\to 0$ as $r \to +0$, with the following property. Let P be any point of S and C a simple closed curve on S, such that P

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* Cf. the corresponding case of a curve. A. J. Ward, *Fundamenta Math.* 28 (1937), 280–288.

and all points of C lie within r of O, and let $\mathbf{P}$, $\mathbf{C}$ *be the orthogonal projections of P, C on any plane. Then there exists a deformation* $\mathbf{C}'$ *of* $\mathbf{C}$ *(lying in the same plane) in which each point moves by at most $\rho\psi(r)$, where ρ is the distance from O of the corresponding point of C, such that* $\mathbf{C}'$ *does or does not surround* $\mathbf{P}$ *according as C does or does not surround P on S. (If C passes through O, the plane of projection must not be perpendicular to OP.*)*

We call this *condition A*. It is comparatively easy to show that condition A, together with the existence of a tangent plane, is sufficient for the existence of a parametrization differentiable at O. The main result of this paper, however, is that condition A itself involves the existence of a tangent plane, and thus is by itself sufficient for a differentiable parametrization.

Two further problems naturally suggest themselves; first, to find conditions for the existence of a parametrization which is differentiable at all points of a given set, and secondly, to extend the result to manifolds of higher dimensions. It appears unlikely that the elementary methods used in this paper would be adaptable to this last problem.

1. *Notation and preliminaries.* We suppose O, S given as in the Introduction, satisfying condition A. (x, y, z) are coordinates in the space in which S lies and (u, v) the parameters of any one parametrization of S by continuous functions, assumed given.

Italic capitals P, C, ... denote points or curves on S, Clarendon letters $\mathbf{P}$, $\mathbf{C}$, ... their projections on some plane, and Greek letters such as Π, Γ, ... the corresponding points or curves in the (u, v) plane. Curves, surfaces or volumes with no immediate relation to S are denoted by Roman capitals. As usual, r, δ and ϵ denote (small) positive numbers. We shall, when there is no risk of confusion, use the symbol $\pm$ to indicate bounds of approximation; for instance, if we say that a distance is $r \pm \epsilon$ we mean that it lies between $r-\epsilon$ and $r+\epsilon$.

We shall often have to consider, as in condition A, a deformation of a projected curve $\mathbf{C}$ in which each point moves by at most $\epsilon\rho$, where ϵ is given and ρ is the distance from O of the corresponding point on S. We shall call this for brevity an $\epsilon\rho$ deformation, pointing out once and for all that in this notation ρ does not represent a constant.

If P, Q are any two points of S and A, B any two points in the space, we shall say that *P and Q may be connected within δ of AB* if there exists in S an arc joining P and Q, all points of which lie within δ of the straight seg-

* Since C is a simple closed curve, the question whether C surrounds P on S is a purely topological one. $\mathbf{C}$ and $\mathbf{C}'$ need not, however, be simple. Let ξ, η be rectangular coordinates in the plane of projection, with $\mathbf{P}$ as origin. We say that $\mathbf{C}'$ surrounds $\mathbf{P}$ if $\arg(\xi + i\eta)$ changes when (ξ, η) describes $\mathbf{C}'$ once.

ment AB. The statement that *P and Q may be connected within δ of AB produced* is interpreted similarly, with reference to the infinite straight line through A and B.

We shall often have to consider the part of S cut off by a convex volume V, which includes a point P of S but no point of the boundary of S. To simplify topological considerations we shall always suppose the (u, v) plane divided into extremely small squares.* Those squares corresponding to parts of S lying entirely inside V will form one or more closed domains, one of which, say Δ, will include Π, the pattern of P. The boundary of this domain Δ will consist of one or more simple closed curves: if there are more than one, then one, which we shall call the *outer boundary*, must enclose all the others.† The corresponding curve on S we shall call the *outer boundary of the component, determined by P, cut off from S by* V. Strictly, we should specify the subdivision of the (u, v) plane into squares, but it is clear that we can make it fine enough to ensure that all points of this outer boundary (on S) lie within a given distance δ of the boundary of V. We shall then denote this outer boundary (of the component determined by P) by $B(P; \mathrm{V}, \delta)$, and the part of S interior to $B(P; \mathrm{V}, \delta)$ by $S(P; \mathrm{V}, \delta)$. (We remark that $S(P; \mathrm{V}, \delta)$ need not lie entirely inside V.) As a particular case, if V is a sphere of centre P and radius r, which we shall denote by $\mathrm{K}(P; r)$, we shall write $B(P; r, \delta)$ and $S(P; r, \delta)$ for $B(P; \mathrm{V}, \delta)$ and $S(P; \mathrm{V}, \delta)$.

We now state three lemmas which follow almost immediately from condition A:

LEMMA 1. *If P is inside C and all points of C lie within r of O, then P lies within $r[1+\psi(2r)]$ of O, provided that $\psi(2r)<1$.*

It is sufficient to prove that no point P inside C lies exactly at distance $r[1+\psi(2r)]$ from O. For this we have only to project C on to a plane through O and P, and apply condition A.

LEMMA 2. *Under the conditions of lemma* 1, *if* **P**, **C** *are projections on to any plane, then* **P** *lies inside* **C** *or within $r\psi(2r)$ of* **C**.

This follows at once, since $OP<2r$ by lemma 1. We remark that we may put $C=B(O; r, \delta)$, and P any point of $S(O; r, \delta)$.

LEMMA 3. *Let P, Q be any two points of $S(O; r, \epsilon r)$, where $\psi(2r)<\epsilon<\frac{1}{4}$. Then* either *$P$, Q may be connected within $3\epsilon r$ of PQ produced,* or *each of P and Q may be connected, within $3\epsilon r$ of PQ produced, with a point P', Q' respectively whose distance from O is $r \pm \epsilon r$.*

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* It is easily proved that curves and domains made up from a network of squares have those topological properties which appear intuitive.

† It may have a finite number of points in common with any of the other parts of the boundary of Δ.

We take for V the common part of $K(O; r)$ and an infinite cylinder H of axis PQ and radius $2\epsilon r$. Suppose first that Q is in $S(P; V, \epsilon r)$, so that P, Q can be joined by an arc lying entirely in this domain. The projection of $B(P; V, \epsilon r)$ on to the plane through P perpendicular to PQ lies entirely within $2\epsilon r$ of P. Hence, by lemma 2, all points of the arc PQ have projections within $3\epsilon r$ of P, so that the arc itself lies within $3\epsilon r$ of PQ produced.

The alternative is that Q is outside $B(P; V, \epsilon r)$. We say that in this case there is a point P' of $B(P; V, \epsilon r)$ at distance $r \pm \epsilon r$ from O. The above argument will then apply at once to PP', and, by symmetry, also to Q and some point Q'.

If no such point P' exists, all points of $B(P; V, \epsilon r)$ must lie within $r - \epsilon r$ of O and within ϵr of the boundary of H. The projection of $B(P; V, \epsilon r)$ on to the same plane as before, will not now approach within ϵr of P. An $\epsilon\rho$ deformation cannot therefore alter its inclusion relation with P, and since P is inside B it must lie inside the projection of B. But P is also the projection of Q lying outside B, so that we have a contradiction.

2. Our first task is to show that the part of S within r of O lies within ϵr, say, of some plane, for all sufficiently small r. We take first a comparatively easy case, when (to speak roughly) S fails to fill completely any small sphere round O. Later we shall show that the other case is in fact impossible; this occupies §§ 3 and 4.

THEOREM I. *Let $\epsilon < 0{\cdot}1$ be given, and r such that $\psi(2r) < 10^{-3}\epsilon^2$. Suppose that there exists a point P of $S(O; r, \epsilon r)$ with $OP \leqslant \frac{1}{2}r$, and a line L through P, such that P cannot be connected with any point of S at distance $r \pm \epsilon r$ from O by a curve within $3\epsilon r$ of L. Then all points of $S(O; r, \epsilon r)$ lie within $\frac{1}{2}\epsilon r$ of some fixed plane, and so within ϵr of some plane through O.*

Take P as the origin and L as the z-axis. We remark first that any point $\mathbf{Q} = (x_1, y_1, 0)$ satisfying $\sqrt{(x_1^2 + y_1^2)} \leqslant \epsilon r$ must be the projection of some point (x_1, y_1, z_1) of S. For, taking V as the common part of $K(O, r)$ and the cylinder $\sqrt{(x^2 + y^2)} \leqslant 2\epsilon r$, we see by the data that $B(P; V, \epsilon r)$ contains no point at distance $r \pm \epsilon r$ from O, and hence (as in lemma 3) that P, and so $\mathbf{Q}$, must be inside the projection $\mathbf{B}$. But B can be continuously shrunk to a point in $S(P; V, \epsilon r)$, and as $\mathbf{B}$ shrinks to a point it must at some stage pass through $\mathbf{Q}$. We remark that the point Q so obtained may be connected with P within $3\epsilon r$ of L.

Consider then a point $Q = (\epsilon r, 0, z_1)$ of S. Applying lemma 3 with ϵ replaced by $\epsilon_1 = 10^{-3}\epsilon^2$, we see that P may be connected, within $3\epsilon_1 r$ of PQ produced, either with Q or with a point distant $r \pm \epsilon_1 r$ from O. In either case, writing $z_1 = k\epsilon r$, we see that there exists an arc PQ_1 say in S, which

lies within $3\epsilon_1 r$ of the segment L_1 say, namely, $|x| \leqslant \epsilon r$, $y = 0$, $z = kx$, and whose end-point Q_1 satisfies $|x| = \epsilon r$. By a similar argument there exists an arc PR_1 lying within $3\epsilon_1 r$ of the segment L_1', namely, $x = 0$, $|y| \leqslant \epsilon r$, $z = k'y$ say, whose end-point R_1 satisfies $|y| = \epsilon r$. For simplicity we shall suppose that $x(Q_1) = y(R_1) = +\epsilon r$.

Now let V_1 be the set of points, distant at most r from O, which lie within $5\epsilon_1 r$ of the plane strip $x \geqslant 0$, $y \geqslant 0$, $x+y = \epsilon r$. By our data (since $6\epsilon_1 + \epsilon < 3\epsilon$), we see that $B(Q_1; \mathrm{V}_1, \epsilon_1 r)$ cannot contain any point distant $r \pm \epsilon r$ from O. Thus $B(Q_1; \mathrm{V}_1, \epsilon_1 r)$ lies entirely within $\epsilon_1 r$ of the "walls" of V_1 parallel to the z-axis L. Its projection $\mathbf{B}$ on the plane $z = 0$ is therefore distant at least $4\epsilon_1 r$ from the segment $x \geqslant 0$, $y \geqslant 0$, $x+y = \epsilon r$, and so at least $\epsilon_1 r$ from the segment joining the projections $\mathbf{Q}_1$, $\mathbf{R}_1$. By an argument like that of lemma 3, $\mathbf{B}$ must therefore surround $\mathbf{Q}_1$, and so $\mathbf{R}_1$; whence B surrounds both Q_1 and R_1, so that Q_1 and R_1 may be connected in $S(Q_1; \mathrm{V}_1, \epsilon_1 r)$; that is (by lemma 2), by an arc lying within $6\epsilon_1 r$ of the plane $x+y = \epsilon r$.

We now have a closed, not necessarily simple, curve PQ_1R_1P. This will contain a simple closed curve $P_2Q_2R_2P_2$, or C say, where P_2Q_2 lies within $3\epsilon_1 r$ of L_1, Q_2R_2 within $6\epsilon_1 r$ of the plane $x+y = \epsilon r$, and R_2P_2 within $3\epsilon_1 r$ of L_1'. Consider now two triangular prisms V_2, V_3; namely, $x \geqslant 4\epsilon_1 r$, $y \geqslant 4\epsilon_1 r$, $x+y \leqslant \epsilon r - 7\sqrt{2}\epsilon_1 r$, and $x \geqslant 5\epsilon_1 r$, $y \geqslant 5\epsilon_1 r$, $x+y \leqslant \epsilon r - 8\sqrt{2}\epsilon_1 r$ respectively. By projecting on to the plane $z = 0$ we see that any point X of S, lying in V_2 and within r of O, is inside C. Now project on to a plane perpendicular both to the plane $x+y = 0$ and also to the plane, H say, which contains the segments L_1 and L_1'. Any point of C, and therefore any point of its projection, lies within $3\epsilon_1 r$ of H or within $6\epsilon_1 r$ of the plane $x+y = \epsilon r$. It follows that the point X, whose projection must be inside some $\epsilon_1\rho$ deformation of the projection of C, must lie within $4\epsilon_1 r$ of H.

Consider in particular the point T, say, of $S(P; \mathrm{V}, \epsilon r)$, for which $x = y = \frac{1}{4}\epsilon r$, and form $B(T; \mathrm{V}_3, \epsilon_1 r)$. By the usual argument, we see that any point X inside or on this curve is certainly in V_2 and within r of O, and hence within $4\epsilon_1 r$ of H; we note also that any point of B is distant at least $\frac{1}{4}\epsilon r - 6\epsilon_1 r$ from T. Now let X be any point of S outside B but within $2r$ of O. Project on to a plane through T perpendicular to XT. Since T is inside and X outside B, there must exist one $\epsilon_1\rho$ deformation of the projection $\mathbf{B}$ which surrounds T, and another which does not; hence $\mathbf{B}$ must pass within $\epsilon_1 r$ of T. A point of $\mathbf{B}$ within $\epsilon_1 r$ of T must be at most $5\epsilon_1 r$ from H and at least $\frac{1}{4}\epsilon r - 7\epsilon_1 r$ from the corresponding point of B. It easily follows that the angle α between XT and H must satisfy the relation $(\frac{1}{4}\epsilon - 7\epsilon_1)\sin\alpha < 9\epsilon_1$. The distance of X from H is then at most $3r\sin\alpha + 4\epsilon_1 r$, which is clearly less than $\frac{1}{2}\epsilon r$ since $\epsilon_1 = 10^{-3}\epsilon^2$. Thus the theorem is proved.

3. We next examine the consequences of supposing that the conditions of theorem I are not satisfied.

THEOREM II. *Suppose that, for given ϵ and r, $S(O; r, \epsilon r)$ does not lie entirely within ϵr of any plane through O, and that $\psi(4r) < \epsilon_1 = 10^{-9}\epsilon^4$. Take any orthogonal axes Ox, Oy, Oz. Then, for any N such that $2N\epsilon < \frac{1}{2}$, there exist $2N$ arcs $C_i''(x)$, $C_i''(y)$, $i = 1, 2, \ldots, N$, on S such that*

(i) *each arc runs from O to a point of $B(O; r, \epsilon r)$, no two arcs meeting except at O;*

(ii) *an arbitrarily small simple closed curve can be drawn on S, surrounding O, to meet these arcs in the cyclic order $C_1''(x)$, $C_1''(y)$, $C_2''(x)$, $\ldots$, $C_N''(x)$, $C_N''(y)$;*

(iii) *$C_i''(x)$ lies entirely within $(2N+1)\epsilon r$ of Ox and $C_i''(y)$ within $(2N+1)\epsilon r$ of Oy.*

We state first

LEMMA 4. *Suppose that P, Q are connected by two arcs PR_1Q, PR_2Q lying respectively within a distance δ of two perpendicular planes, say $x = 0$, $y = 0$, and within $r + \epsilon r$ of O. Then P, Q may be connected by an arc lying within $\delta + 3\epsilon r$ of each plane, if $\psi(4r) < \frac{1}{2}\epsilon < \frac{1}{8}$.*

It is clearly sufficient to consider the case when the given arcs are simple and do not meet except at their end-points. Projecting on to the plane $z = 0$, we see that all points of S within the simple closed curve $C = PR_1QR_2P$ satisfy $|x| \leqslant \delta + \epsilon r$ or $|y| \leqslant \delta + \epsilon r$. As in lemma 1, they also satisfy

$$\sqrt{(x^2 + y^2 + z^2)} \leqslant r + 2\epsilon r.$$

Now take for V the set of points within $2r$ of O satisfying $|x| \leqslant \delta + 2\epsilon r$. Then $S(P; \mathrm{V}, \epsilon r)$ will certainly include all points of PR_1Q but not in general all points of PR_2Q. It will therefore cut off, from the domain inside C, a component bounded by the arc $C_1 = PR_1Q$ and an arc C_2 consisting of portions of PR_2Q and portions of $B(P; \mathrm{V}, \epsilon r)$ lying inside C.* Now all points inside or on $B(P; \mathrm{V}, \epsilon r)$ satisfy $|x| \leqslant \delta + 3\epsilon r$, and all points inside or on C satisfy $\min(|x|, |y|) \leqslant \delta + \epsilon r$. Further, all points of B satisfy either $\sqrt{(x^2 + y^2 + z^2)} > 2r - \epsilon r$ or $|x| > \delta + \epsilon r$; for those points inside C the former is impossible and the latter implies $|y| \leqslant \delta + \epsilon r$. It follows that C_2 satisfies our requirements. We note further that if PR_1Q and PR_2Q lie in the region $a \leqslant z \leqslant b$, then all points inside or on C, and so all points of C_2, satisfy $a - \epsilon r \leqslant z \leqslant b + \epsilon r$.

We now turn to the main proof of theorem II. We may suppose $\epsilon < \frac{1}{4}$. It is clear that the conditions of theorem I, with r replaced by $2r$ and ϵ by

* In general there might be an infinite number of such portions, and also a closed set, formed by their limit points, common to C and $B(P; \mathrm{V}, \epsilon r)$. If, however, our curves are all made up of a finite number of stretches given by u = constant or v = constant, there can be only a finite number of such portions to consider.

$\epsilon' = 10^{-3}\epsilon^2$, cannot be satisfied. Hence any point P of $S(O; r, \epsilon r)$ with $OP < r$ can be connected, within $6\epsilon' r$ of any line L through P, with some point distant $2r \pm 2\epsilon' r$ from O, and so certainly with some point of $B(O; r, \epsilon r)$. In particular, we can connect O itself with points of B by two arcs $C_0(x)$, $C_0(y)$ lying within $6\epsilon' r$ of Ox, Oy respectively. If we replace $6\epsilon' r$ by $10\epsilon' r$, we may further suppose that these arcs do not meet except at O. For let Q be the last point on $C_0(x)$ where it meets $C_0(y)$. By lemma 4, there is an arc $K = OQ$ of which all points satisfy

$$\max(|x|, |y|, |z|) \leqslant 6\epsilon' r + 6\epsilon_1 r.$$

We replace the arcs OQ of $C_0(x)$, $C_0(y)$ respectively by the arc K. If this introduces closed loops (for example, if $C_0(y)$ meets K again beyond Q) we eliminate these and are left with two arcs having a certain part K_1 in common and not meeting again. We finally replace K_1 by two separate arcs very nearly coinciding with it, and thus obtain two separate curves (which we shall still call $C_0(x)$, $C_0(y)$) satisfying the required inequalities.

Let us suppose for definiteness that the end-point of $C_0(y)$ has y positive. Then let P_1 be the first point of $C_0(y)$ with $y = \epsilon r$. We may connect P_1 with a point of B by a curve $C_1(x)$ lying within $6\epsilon' r$ of the line $y = \epsilon r$, $z = z(P_1)$, and so within $16\epsilon' r$ of $y = \epsilon r$, $z = 0$. As above, we may suppose that $C_1(x)$ does not meet $C_0(y)$ again, if we modify $C_0(y)$ as may be necessary, and replace $10\epsilon' r$ for $C_0(y)$ and $16\epsilon' r$ for $C_1(x)$ by $24\epsilon' r$ in each case.* Since $24\epsilon' < \frac{1}{2}\epsilon$, the modification cannot cause $C_0(y)$ to cut $C_0(x)$, neither can $C_0(x)$ meet $C_1(x)$. Again, the part of $C_0(y)$ for which $y > \frac{3}{2}\epsilon$ is still within $10\epsilon' r$ of Oy.

Now let P_2 be the first point of the modified $C_0(y)$ satisfying $y = 2\epsilon r$. We construct similarly an arc from P_2 to B, lying within $24\epsilon' r$ of $y = 2\epsilon r$, $z = 0$, which we may again suppose not to meet $C_0(y)$ except at P_2. We proceed similarly up to P_{2N} obtaining arcs $C_1(x), C_2(x), \ldots, C_{2N}(x)$.

On each curve $C_i(x)$ we take the first point Q_i such that $|x| = i\epsilon r$ and connect it with a point of B by a curve $C_i(y)$ lying within $6\epsilon' r$ of $x = x(Q_i)$, $z = z(Q_i)$. We may, as before, suppose that $C_i(y)$ does not meet $C_i(x)$ again, if we modify $C_i(x)$ and replace the $24\epsilon' r$ for $C_i(x)$ and the $6\epsilon' r$ for $C_i(y)$ by $36\epsilon' r$. As before, this cannot introduce intersections with curves already constructed, and no two $C_i(y)$ can meet. Then $C_i(y)$ lies within $60\epsilon' r$ of the line $x = \pm i\epsilon r$, $z = 0$.

We next prove that no two curves $C_i(x)$, $C_j(y)$ $(i \neq j, j \neq 0)$ can meet. Suppose, if possible, that they do; let R, say, be the first point on $C_i(x)$ where it meets $C_j(y)$. Then $P_i P_j Q_j R P_i$ is a simple closed curve C whose four

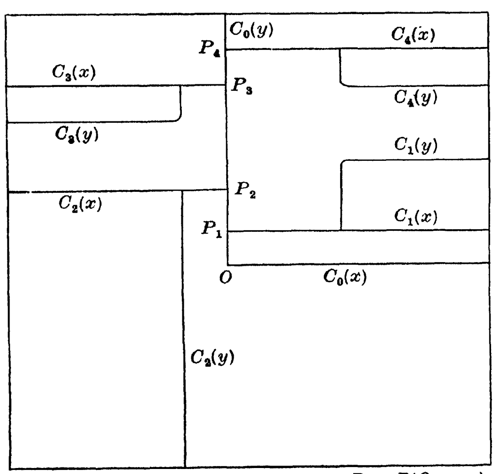
* Since $16\epsilon' + 6\epsilon_1 < 17\epsilon'$ and $17\sqrt{2} < 24$. It may be remarked that the case is a little more complicated than the preceding one, as $C_1(x)$ may meet $C_0(y)$ in points on either side of P_1.

portions lie within $60\epsilon' r$ of the straight lines $x = 0$, $y = j\epsilon r$, $x = \pm j\epsilon r$, $y = i\epsilon r$, all in the plane $z = 0$. Just as in the last part of § 2, it now follows that any point of S inside C is within $61\epsilon' r$ of the plane $z = 0$; while if P is any point of $S(O; r, \epsilon r)$ outside C, and α the angle between the plane $z = 0$ and the line joining P to the midpoint of the rectangle formed by the above straight lines, then $(\frac{1}{2}\epsilon - 60\epsilon')r \sin\alpha < 62\epsilon' r$. Thus every point of $S(O; r, \epsilon r)$ lies within $8r \,.\, 122\epsilon'/\epsilon$ of $z = 0$. Since $\epsilon' = 10^{-3}\epsilon^2$ this contradicts our assumptions.

Now the curves $C_i(x)$, together with the part of $C_0(y)$ from O to P_{2N}, divide $S(O; r, \epsilon r)$ into $(2N+1)$ sectors. From what has just been proved, each curve $C_j(y)$ $(j \neq 0)$ lies entirely in one sector. No sector can contain more than two $C_j(y)$, namely, those originating from the two $C_i(x)$ which bound the sector. Hence at least N sectors must contain a curve $C_j(y)$. We pick one curve $C_j(y)$ from each of N such sectors, and renumber them in the order in which they are met in traversing $B(O; r, \epsilon r)$, as $C'_1(y)$, $C'_2(y)$, ..., $C'_N(y)$. Between each two curves $C'_j(y)$, $C'_{j+1}(y)$ lies at least one curve $C_i(x)$, forming the boundary of the sector containing $C'_j(y)$. We pick out, for each j, one such x curve and call it $C'_{j+1}(x)$. Similarly, we have $C'_1(x)$ between $C'_N(y)$ and $C'_1(y)$.*

* For the sake of clearness we give a diagram illustrating a possible topological arrangement on S of the curves considered, taking $N = 2$.

Here, for example, $C'_1(y)$ could be either $C_1(y)$ or $C_4(y)$, and $C'_2(y)$ could be $C_3(y)$ or $C_2(y)$. If we take $C'_1(y) = C_1(y)$ and $C'_2(y) = C_3(y)$, then $C'_2(x)$ may be either $C_4(x)$ or $C_3(x)$, and $C'_1(x)$ either $C_2(x)$, $C_0(x)$ or $C_1(x)$.



$B = B(O; r, \epsilon r)$

Starting from O, we now clearly have $2N$ curves which satisfy the conditions laid down, except that they coincide for a certain distance from O. We can avoid this by replacing any portion of arc, forming part of, say, p curves, by p nearly coincident arcs. We thus have the required curves $C_i''(x)$, $C_i''(y)$.

4. We now show that, for sufficiently small r, the conditions envisaged in theorem II cannot in fact arise. That is, we can prove:

THEOREM III. *Given any $\epsilon > 0$, then, for all sufficiently small r, $S(O; r, \epsilon r)$ lies within ϵr of some plane through O.*

COROLLARY. *Given $\epsilon > 0$, then, for any sufficiently small r, any point of S which lies within r of O is within ϵr of some fixed plane through O.*

Let us assume the theorem false. We may clearly suppose $\epsilon < 0{\cdot}0001$. Let r_0 be such that $\psi(32r_0) < \epsilon_1 = 10^{-3}\epsilon^2$. Consider $B(O; r_0, \epsilon r_0)$; this curve can contain at most a finite number, say N, of non-overlapping arcs each of diameter not less than $\frac{1}{2}r_0$. We take now ϵ_2 such that $(2N+1)\epsilon_2 < \epsilon_1$, and write $\epsilon_3 = 10^{-9}\epsilon_2^4$. By our assumption, there exists r_1 so small that $\psi(4r_1) < \epsilon_3$ while $S(O; r_1, \epsilon r_1)$, and *a fortiori* $S(O; r_1, \epsilon_2 r_1)$,* is not entirely within $\epsilon_2 r_1$ of any plane through O. We take any orthogonal axes through O.

Let now $r_n = 4^{n-1}r_1$, $S_n = S(O; r_n, \epsilon r_n)$, and $B_n = B(O; r_n, \epsilon r_n)$ $(n = 1, 2, \ldots)$. We now say that, so long as $\psi(8r_n) < \epsilon_1$, S_n contains

(*a*) a point for which $|z| > \frac{1}{2}r_n$, $|x|$, $|y| < 0{\cdot}1r_n$;

(*b*) $2N$ non-intersecting arcs $C_i^n(x)$, $C_i^n(y)$ leading from O to B_n [arranged so that in traversing B_n they are met in the order $C_1^n(x)$, $C_1^n(y)$, $C_2^n(x)$, ..., $C_N^n(y)$], such that all points on $C_i^n(x)$ satisfy

$$|y| < 16\epsilon r_n, \quad |z| < 0{\cdot}04r_n + 0{\cdot}15|x|,$$

while all points on $C_i^n(y)$ satisfy

$$|x| < 16\epsilon r_n, \quad |z| < 0{\cdot}06r_n + 0{\cdot}3|y|.$$

In particular, it will follow that the end-points X_i^n, Y_i^n of $C_i^n(x)$, $C_i^n(y)$ satisfy $|x| > 0{\cdot}8r_n$, $|z| < 0{\cdot}2r_n$ and $|y| > 0{\cdot}8r_n$, $|z| < 0{\cdot}4r_n$ respectively.

The above statement is true for $n = 1$; part (*a*) follows from theorem I (according to which it is possible to connect O with a point distant $r_1 \pm \epsilon r_1$ from O by a curve within $3\epsilon r_1$ of Oz) and part (*b*) from theorem II. Assume that the statement is true for $n = k$. Then S_k, and so S_{k+1}, is not entirely within ϵr_{k+1} of any plane through O. Just as above, this proves part (*a*) for r_{k+1} by theorem I.

* We may clearly suppose that $S(O; r, \delta_1)$ includes $S(O; r, \delta_2)$ if $\delta_1 < \delta_2$. This remark will be used again without explicit reference.

Consider now any end-point $X_i^k = (x_0, y_0, z_0)$ say. Suppose first that $z_0 \geqslant 0$. Let L be the line joining X_i^k to the point $(0, y_0, 0{\cdot}1r_k)$. By theorem I, there exists an arc in S, lying within $3\epsilon r_{k+2}$ of L produced, which joins X_i^k to a point distant $r_{k+2}(1 \pm \epsilon)$ from O. The part of this arc in S_{k+1} joins X_i^k to a point of B_{k+1}, which we take for X_i^{k+1}. Let $C_i^{k+1}(x)$ be made up of $C_i^k(x)$ together with this arc $X_i^k X_i^{k+1}$, cutting out any closed portions so as to obtain a simple arc. We see at once that, throughout $C_i^{k+1}(x)$,

$$|y| < |y_0| + 3\epsilon r_{k+2}$$

$$< 16\epsilon r_k + 12\epsilon r_{k+1} = 16\epsilon r_{k+1}.$$

Now since $|x_0| \geqslant 0{\cdot}8r_k$ and $0 \leqslant z_0 \leqslant 0{\cdot}2r_k$, the z/x slope of L is numerically less than $0{\cdot}15$. Hence, at all points of $X_i^k X_i^{k+1}$, and therefore throughout $C_i^{k+1}(x)$, we have

$$|z| < 0{\cdot}1r_k + 0{\cdot}15\,(|x| + 3\epsilon r_{k+2}) + 3\epsilon r_{k+2}$$

$$< 0{\cdot}04r_{k+1} + 0{\cdot}15\,|x|.$$

The conditions laid down are therefore satisfied for $C_i^{k+1}(x)$. If $z_0 < 0$ we take L as the line joining X_i^k to $(O; y_0, -0{\cdot}1r_k)$. For $C_i^{k+1}(y)$ we work similarly, interchanging x and y, but replacing $0{\cdot}1r_k$ by $0{\cdot}2r_k$. We find again that the required conditions are satisfied.

We now have to prove that no two of our curves meet. For topological reasons, since $C_i^k(x)$ lies in a sector of S_k bounded by $C_{i-1}^k(y)$, $C_i^k(y)$, and B_k, it is sufficient to prove that $C_i^{k+1}(x)$ cannot meet $C_{i-1}^{k+1}(y)$ or $C_i^{k+1}(y)$. Consider, for example, $C_i^{k+1}(x)$ and $C_i^{k+1}(y)$. We know that $C_i^k(x)$, $C_i^k(y)$ do not meet. All points of $C_i^{k+1}(y)$ satisfy $|x| < 16\epsilon r_{k+1}$. Any point of the arc $X_i^k X_i^{k+1}$ satisfying $|x| < 16\epsilon r_{k+1}$ also satisfies the inequalities $|y| < 16\epsilon r_{k+1}$ and

$$\big||z| - 0{\cdot}1r_k\big| < 0{\cdot}15\,(|x| + 3\epsilon r_{k+2}) + 3\epsilon r_{k+2}$$

$$< 0{\cdot}03r_k$$

(since $\epsilon < 0{\cdot}0001$). Similarly, any point of $Y_i^k Y_i^{k+1}$ for which $|y| < 16\epsilon r_{k+1}$ must satisfy

$$\big||z| - 0{\cdot}2r_k\big| < 0{\cdot}04r_k.$$

It is thus clear that these two arcs cannot meet. Again, any point of $C_i^{(k)}(y)$ for which $|y| < 16\epsilon r_{k+1}$ will satisfy

$$|z| < 0{\cdot}06r_k + 0{\cdot}3\,|y| < 0{\cdot}07r_k,$$

so that $X_i^k X_i^{k+1}$ cannot meet $C_i^k(y)$. Similarly, $Y_i^k Y_i^{k+1}$ cannot meet $C_i^k(x)$. It follows that no two of the curves C_i^{k+1} can meet.

By induction, our statement is therefore true for all n such that $\psi(8r_n) < \epsilon_1$, and in particular for n such that $r_{n-1} < r_0 \leqslant r_n$. Now the paths $C_i^n(x)$, $C_i^n(y)$ meet $B(O; r_0, \epsilon r_0)$ in $2N$ points P_i, Q_i, say arranged in the order $P_1 Q_1 P_2 Q_2 \ldots$. For P_i, we have $|y| < 16\epsilon r_n < 64\epsilon r_0$ and $|z| < 0{\cdot}04 r_n + 0{\cdot}15|x| < 0{\cdot}35 r_0$, and therefore $|x| > 0{\cdot}7 r_0$, since $OP_i > r_0(1-\epsilon)$. Similarly Q_i satisfies the relation $|y| > 0{\cdot}7 r_0$. Thus each arc $P_i Q_i$ or $Q_i P_{i+1}$ is of diameter greater than $\frac{1}{2}r_0$, and so we have a contradiction. Thus, the main theorem cannot in fact be false.

To prove the corollary, we remark that if $S(O; r, \epsilon r)$ lies entirely within ϵr of, say, the plane $z = 0$ (taking axes through O), then $B(O; r, \epsilon r)$ must lie within $2\epsilon r$, at most, of the circle $x^2 + y^2 = r^2$, $z = 0$. Projecting on to the plane $z = 0$, we see that any point of S within $\frac{1}{2}r$ of O must lie inside B (provided that $\psi(r) < \frac{1}{2} - 2\epsilon$) and therefore within ϵr of $z = 0$. We obtain the corollary as stated by writing $2r$ and $\frac{1}{2}\epsilon$ for r and ϵ.

5. THEOREM IV. *There exists a tangent plane at O.*

Let $\mathrm{H}(r, \epsilon)$ denote any plane through O such that all points of $S.\mathrm{K}(O, r)$ lie within ϵr of $\mathrm{H}(r, \epsilon)$. Given ϵ, which we may suppose less than $0{\cdot}001$, we can find δ such that, for all $r \leqslant \delta$, $\mathrm{H}(r, \epsilon)$ exists and also $\psi(2r) < \epsilon$. We remark first that, arguing as in the proof of theorem I, we can show that any point of H within, say, $0{\cdot}8r$ of O must be the orthogonal projection of some point of $S.\mathrm{K}(O, r)$. Again, if P, Q are any two such points, and $\mathbf{P}$, $\mathbf{Q}$ their projections on to H, we may take as the volume V a cylinder, with generators perpendicular to H, surrounding the segment $\mathbf{PQ}$ at a distance $2\epsilon r$. It is easily seen that P and Q must both lie inside $B(P; \mathrm{V}, \epsilon r)$ and can therefore be connected by an arc lying in $S(P; \mathrm{V}, \epsilon r)$. Such an arc lies in V or within ϵr of V, and also within ϵr of H. It is therefore entirely within $4\epsilon r$ of the segment $\mathbf{PQ}$ and $5\epsilon r$ of PQ.

We now prove that if $0 < r_1 < r_0 \leqslant \delta$, then $\mathrm{H}(r_0, \epsilon)$ and $\mathrm{H}(r_1, \epsilon)$ are inclined at an angle of at most 2α, where $\sin\alpha = 40\epsilon$. This is clearly true if $r_0 \leqslant 2r_1$, for there exists a point of S distant at least $\frac{1}{2}r_1$ from O and at most ϵr_0 from either plane. Suppose that the statement is true whenever $r_0 \leqslant 2^k r_1$, and consider the case $2^k r_1 < r_0 \leqslant 2^{k+1} r_1$. Take axes through O so that $\mathrm{H}(r_1, \epsilon)$ is the plane $z = -x\tan\alpha$ and $\mathrm{H}(r_0, \epsilon)$ is $z = x\tan\theta$, where $-\alpha < \theta < \frac{1}{2}\pi - \alpha$. Using $\mathrm{H}(2^k r_1, \epsilon)$ as an intermediary we easily see that $\theta < 3\alpha$; we wish to prove $\theta < \alpha$. We write $r_i = 2^{i-1} r_1$ $(i \geqslant 1)$ and $\mathrm{H}_i = \mathrm{H}(r_i, \epsilon)$. Let P_i, Q_i be points of $S.\mathrm{K}(O, r_i)$ whose projections $\mathbf{P}_i$, $\mathbf{Q}_i$ on H_i satisfy $x = y = \frac{1}{2}r_i$ and $x = -y = \frac{1}{2}r_i$ respectively $(i = 0, 1, \ldots, k+1)$, and let R be a point of $S.\mathrm{K}(O, r_1)$ whose projection on H_1 satisfies $x = \frac{1}{4}r_1$, $y = 0$. We can connect O, P_1 and O, Q_1 by arcs lying within $4\epsilon r_1$ of $O\mathbf{P}_1$, $O\mathbf{Q}_1$ respectively. We connect P_i with P_{i+1} and Q_i with Q_{i+1} by arcs lying within $5\epsilon r_{i+1}$ of $P_i P_{i+1}$, $Q_i Q_{i+1}$ respectively and so within $6\epsilon r_{i+1}$ of $\mathbf{P}_i\mathbf{P}_{i+1}$ and $\mathbf{Q}_i\mathbf{Q}_{i+1}$. Finally,

we connect P_{k+1} with P_0, P_0 with Q_0, and Q_0 with Q_{k+1} by arcs within $6\epsilon r_0$ of $\mathbf{P}_{k+1}\mathbf{P}_0$, $\mathbf{P}_0\mathbf{Q}_0$ and $\mathbf{Q}_0\mathbf{Q}_{k+1}$. We thus obtain a closed curve

$$OP_1P_2 \ldots P_{k+1}P_0Q_0Q_{k+1} \ldots Q_2Q_1O,$$

which contains a simple closed curve C passing through or near the same points.

We now project on to the plane $z = 0$. The coordinates of R satisfy

$$|x - \tfrac{1}{4}r_1| < \epsilon r_1, \quad |y| < \epsilon r_1, \quad |z + \tfrac{1}{4}r_1 \tan\alpha| < \epsilon r_1.$$

The projections of the arcs OP_1, P_1P_2, etc., lie within $6\epsilon r_1$, $6\epsilon r_2$, ..., of straight segments which together make up the triangle of vertices $(0, 0, 0)$, $(\frac{1}{2}r_0, \frac{1}{2}r_0, 0)$ and $(\frac{1}{2}r_0, -\frac{1}{2}r_0, 0)$. It is easily seen that any $\epsilon\rho$-deformation of the projection of C must surround the projection of R, so that R is inside C on S. On the other hand, if we project on to the plane $x = 0$, the segments $O\mathbf{P}_1$, $O\mathbf{Q}_1$ project on to the segments

$$z = -y\tan\alpha, \; 0 \leqslant y \leqslant \tfrac{1}{2}r_1 \quad \text{and} \quad z = y\tan\alpha, \; 0 \geqslant y \geqslant -\tfrac{1}{2}r_1.$$

The projections of the segments $\mathbf{P}_1\mathbf{P}_2, \ldots, \mathbf{P}_k\mathbf{P}_0$ and $\mathbf{Q}_1\mathbf{Q}_2, \ldots, \mathbf{Q}_k\mathbf{Q}_0$ all satisfy $|y| \geqslant \frac{1}{2}r_1$, and the projection of $\mathbf{P}_0\mathbf{Q}_0$ satisfies $z = \frac{1}{2}r_0 \tan\theta$. It is clear that if $\theta > 0$ the projection of $O\mathbf{P}_1\mathbf{P}_2 \ldots \mathbf{P}_0\mathbf{Q}_0 \ldots \mathbf{Q}_1 O$ does not surround the projection of R, and on calculating the distances it is found that, since $\sin\alpha = 40\epsilon$, if $\theta \geqslant \alpha$ no $\epsilon\rho$-deformation of C can surround this point. Thus $\theta < \alpha$, as was to be proved.

If now $\epsilon_n \to 0$ as $n \to \infty$ in such a way that $\mathrm{H}(2^{-n}, \epsilon_n)$ exists for each n, it follows at once that $\mathrm{H}(2^{-n}, \epsilon_n)$ tends to a limiting plane H, which is clearly a tangent plane to the surface.

6. We can now prove the final result:

There exists a parametrization differentiable at O.

Let $z = 0$ be the tangent plane. By previous results we know that every point of $z = 0$ within a certain distance, say δ, of O is the projection of some point of S. Also, there exists a function $\phi(r)$, tending to zero as $r \to 0$, such that any two points P, Q of the surface, whose projections $\mathbf{P}$, $\mathbf{Q}$ on $z = 0$ lie within $2r$ of O, can be joined by an arc lying within $r\phi(r)$ of $\mathbf{PQ}$. We may suppose $\phi(r) \geqslant \psi(2r)$.

We now construct a sequence of regular polygons C_n in the plane $z = 0$, inscribed in circles of centre O and radius r_n, $n = 1, 2, \ldots$. We prescribe first that r_{n+1}/r_n lies between $\frac{3}{4}$ and 1, and increases with n, while

$$r_n - r_{n+1} > \tfrac{1}{2}(r_{n-1} - r_n) \quad \text{for} \quad n > 1.$$

We can then choose the number, k_n, of sides of C_n so that k_n is a power of 2 and the length of each side of C_n lies between $\frac{1}{2}(r_{n-1} - r_n)$ and $2(r_n - r_{n+1})$,

or for $n = 1$, between $\frac{1}{2}(r_1 - r_2)$ and $2(r_1 - r_2)$. We may suppose that, under these conditions, k_n increases with n and $k_1 \geqslant 16$. C_n and C_{n+1} will not meet, and the annulus between them can be divided into triangles whose vertices are vertices of the polygons and whose sides are of length at most $3(r_n - r_{n+1})$. There exist positive constants A, B such that, however r_n and k_n are chosen subject to the above restrictions, every side $\mathbf{PQ}$ of any triangle, with one end-point in C_n, is distant at least $A(r_n - r_{n+1})$ from any other side $\mathbf{RT}$ which it does not meet, and every angle θ of every triangle satisfies $\operatorname{cosec} \frac{1}{2}\theta < B$.

Since $\phi(r) \to 0$ as $r \to 0$, we can now finally choose the r_n, in succession, so that $A(r_n - r_{n+1}) > 4(B+1) r_n \phi(r_n)$, $r_1 \leqslant \delta$, and $r_n \to 0$ and $r_{n+1}/r_n \to 1$, $k_n \to \infty$, as $n \to \infty$. We thus get an infinite triangulation of a certain neighbourhood of O in the plane $z = 0$.

We now construct a corresponding division of S. We associate with each vertex $\mathbf{P}$ of each C_n a point P of S of which $\mathbf{P}$ is the projection. Consider now the various lines $\mathbf{PQ}$, $\mathbf{PR}$, ..., drawn from $\mathbf{P}$ in the triangulation. Let $\mathbf{P}$ be in C_m, $\mathbf{Q}$ in C_n ($n = m-1$, m, or $m+1$) and $j = \max(m, n)$. On $\mathbf{PQ}$ insert points $\mathbf{Q}_1$, $\mathbf{Q}_2$ so that $\mathbf{PQ}_1 = 2Br_m\phi(r_m)$, $\mathbf{Q}_2\mathbf{Q} = 2Br_n\phi(r_n)$, and let Q_1, Q_2 be corresponding points of S. (It is clear that $\mathbf{PQ}_1$ and $\mathbf{Q}_2\mathbf{Q}$ do not overlap.) We connect P with Q_1, Q_1 with Q_2 and Q_2 with Q by arcs lying within $r_j\phi(r_j)$ of $\mathbf{PQ}_1$, $\mathbf{Q}_1\mathbf{Q}_2$ and $\mathbf{Q}_2\mathbf{Q}$ respectively (this is possible since $\mathbf{P}$ and $\mathbf{Q}$ lie within r_{j-1} of O and $r_{j-1} < 2r_j$), and cut out any closed portions so as to get a simple arc PQ. We suppose a similar construction carried out for all other straight lines of the triangulation. In particular, let PR_1R_2R be a similarly constructed arc corresponding to $\mathbf{PR}$. Then, since the distance between the segments $\mathbf{PR}$ and $\mathbf{Q}_1\mathbf{Q}$ is at least $2Br_m\phi(r_m)\sin\frac{1}{2}\mathbf{QPR}$, whether $\mathbf{QPR}$ is acute or obtuse, the arc PR cannot meet the arc PQ except possibly in a point belonging to the part PQ_1 and similarly to PR_1. If PQ_1 and PR_1 do meet, let T say be the last point of intersection on PR_1. We may replace the part PT of PR_1 by the corresponding part of PQ_1. We may work similarly with all the arcs leaving P, so that any two may have an initial part in common but do not meet again. By replacing the common parts by two or more nearly coincident arcs, we may finally suppose that none of the arcs PQ, PR, ..., meet except at P. Since only the parts PQ_1, PR_1, ... have been altered, the modified arcs still lie within $(2B+1) r_m\phi(r_m)$ of $\mathbf{PQ}, \mathbf{PR}, \ldots$. We suppose a similar modification carried out for each vertex of the triangulation. It is then easily seen that, since $4(B+1) r_m\phi(r_m) < A(r_m - r_{m+1})$, two arcs corresponding to non-intersecting segments do not meet. Thus no two arcs meet except at a common vertex.

The arcs corresponding to the sides of C_n form, taken together, a simple closed curve C_n. Projecting on to $z = 0$ we obtain a closed curve $\mathbf{C}_n$ lying within $(2B+1) r_n\phi(r_n)$ of C_n. Thus $\mathbf{C}_n$ surrounds any vertex $\mathbf{P}$ of C_{n+1} and

is distant at least $2(B+1)r_n\phi(r_n)$, *a fortiori* at least $2r_n\psi(2r_n)$, from $\mathbf{P}$. It follows that the corresponding point P on S is inside C_n, and so C_{n+1} lies entirely inside C_n on S. The region between C_n and C_{n+1} on S is therefore divided into domains in one-one correspondence with the triangles lying in the annulus between C_n and C_{n+1}.

We now set up an arbitrary topological correspondence between each arc PQ and the corresponding segment $\mathbf{PQ}$. For each triangular area $\mathbf{PQR}$ say, there then exists a topological correspondence with the domain PQR on S, agreeing with the correspondence already set up on the bounding arcs. On combining all these, and making O correspond to itself, we have a one-one continuous correspondence between the interior of C_1 in the plane $z = 0$ and the interior of C_1 in S, for any point of S except O lies outside some C_n, and therefore falls into some domain PQR if inside C_1. Outside C_1 and C_1 the mapping can be completed in any way.

Let $(x, y, 0)$ be any point of $z = 0$, lying in some triangle $\mathbf{PQR}$ between C_n and C_{n+1} say, and (X, Y, Z) the corresponding point of S. Since all points of the arcs PQ, QR, RP are within $2r_n$ of O, the projection $(X, Y, 0)$ must lie either inside the projection of the curve $PQRP$, or at a distance at most $2r_n\psi(4r_n)$ from it; that is, either inside the triangle $\mathbf{PQR}$ or distant at most $2r_n\psi(4r_n)+(2B+1)r_n\phi(r_n)$ from it. The distance of $(x, y, 0)$ from the sides of $\mathbf{PQR}$ is less than $r_n - r_{n+1}$ and each side of $\mathbf{PQR}$ is of length at most $3(r_n - r_{n+1})$. Since $\phi(r_n)$, $\psi(4r_n)$ and $1 - r_{n+1}/r_n$ all tend to 0 as $n \to \infty$, we see that

$$X = x + o\sqrt{(x^2+y^2)}, \quad Y = y + o\sqrt{(x^2+y^2)}$$

as $x, y \to 0$; hence X, Y are differentiable functions of (x, y) at $x = y = 0$ and satisfy the equations

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y} = 1, \quad \frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x} = 0.$$

Finally, since $z = 0$ is a tangent-plane, we have

$$Z = o\sqrt{(X^2+Y^2)} = o\sqrt{(x^2+y^2)}$$

so that Z is differentiable and

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial y} = 0 \quad \text{at} \quad x = y = 0.$$

Thus we have set up a parametrization which is differentiable at O.

Emmanuel College
Cambridge

IMPROVEMENT OF A THEOREM OF LINNIK AND WALFISZ

By S. CHOWLA

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1. On the basis of the hitherto unproved "extended Riemann hypothesis", Littlewood (1) proved that there are infinitely many k such that

$$L(1) = \sum_1^\infty \frac{\chi(n)}{n} < \frac{\{1+o(1)\}}{\log\log k}\frac{\pi^2}{6}e^{-C}, \tag{1}$$

where $\chi(n)$ denotes a real primitive character $(\operatorname{mod} k)$, and C is Euler's constant.

Independently of each other, and almost simultaneously, Linnik (2), Walfisz (3) and I (4) proved the following results without assuming any hypothesis:

(I) There are infinitely many k such that

$$\sum_1^\infty \frac{\chi(n)}{n} < \frac{A}{\sqrt{(\log\log k)}},$$

where A is a certain absolute positive constant, and $\chi(n)$ is a real *primitive* character $(\operatorname{mod} k)$.

(II) There are infinitely many k such that

$$\sum_1^\infty \frac{\chi(n)}{n} < \epsilon,$$

where ϵ is an arbitrary positive number, and $\chi(n)$ is a real primitive character $(\operatorname{mod} k)$.

Of these results (II) was proved by me; the sharper result (I) is due to Linnik and Walfisz. I now find that a simple sharpening of my method used to prove (II) will prove Littlewood's result without assuming "the extended Riemann hypothesis". In fact, all we have to do is to replace the number g of my paper (4) by

$$\left[\frac{\log x}{(\log\log x)^2}\right],$$

where $[t]$ denotes the greatest integer contained in t.

As my paper (**4**) contains misprints (nor is it easily available) I develop the whole argument without any reference to this paper. We actually prove somewhat more than Littlewood's conjecture, namely, theorems 1 and 2 of § 11 (towards the end of this paper).

2. *Definitions*. p_m denotes the mth odd prime,

$$a = p_1 p_2 p_3 \cdots p_g, \tag{2}$$

where g is defined by (3) below; b is a positive integer such that $(b/p_r) = -1$ for $1 \leqq r \leqq g$, $b \equiv 5 \pmod 8$ and $1 < b < 8a$; x is a sufficiently large positive integer,

$$g = \left[\frac{\log x}{(\log\log x)^2}\right]; \tag{3}$$

(n/m) is the Jacobi symbol if m is odd and prime to n, but is 0 in all other cases (i.e. when m is even or when m and n have a common factor). We write

$$T(x) = \sum_{x<n\leqq 2x} \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{8an+b}{m}\right) \tag{4}$$

and

$$S(x) = \sum_{x<n\leqq 2x} \sum_{m=1}^{x^{\frac{3}{4}}} \frac{1}{m}\left(\frac{8an+b}{m}\right). \tag{5}$$

We observe (for further reference) the fact that we may write

$$\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{m}{8an+b}\right) = \sum_{1}^{\infty} \frac{\chi(m)}{m},$$

where $\chi(m)$ is a real primitive character $(\bmod k)$ if $8an+b$ is *quadratfrei* and $k = 8an+b$. We also note that

$$\left(\frac{m}{8an+b}\right) = \left(\frac{8an+b}{m}\right),$$

provided m is odd, which is the reciprocity law for Jacobi's symbol.

3. We first prove that $\quad T(x) = S(x) + O(x^{\frac{4}{5}}). \tag{6}$

The proof of (6) needs

LEMMA 1. *If $\chi(n)$ is a non-principal character* $(\bmod k)$, *then*

$$\sum_{n=u}^{v} \chi(n) = O(\sqrt{k}\log k).$$

This is a well-known result when $\chi(n)$ is a primitive character $(\bmod k)$; the extension (due to Davenport) to non-principal characters $\chi(n)$ is easily made.

LEMMA 2. *We have* $a < x^{\frac{1}{30}}$ $(x > x_0)$.

Proof. For $x > x_0$, $\log a < \vartheta(p_g) < 2p_g < 3g \log g < 3 \log x / \log\log x < \frac{1}{30} \log x$. Now, using lemma 1,

$$T(x) - S(x) = \sum_{x < n \leqq 2x} \sum_{m > x^{3/4}} \frac{1}{m} \left(\frac{8an+b}{m} \right)$$

$$= O\left\{ x \frac{\sqrt{(ax)} \log (ax)}{x^{3/4}} \right\} = O(x^{4/5}),$$

by lemma 2 and the fact that $\left(\dfrac{8an+b}{m} \right)$ is a character (mod $2(8an+b)$) (since the symbol is 0, by definition, whenever m is even). Thus (6) is proved.

4. We also need the following two lemmas:

LEMMA 3. *The number of quadratfrei integers* $8an+b$ $(x < n \leqq 2x)$ *is* $\{1 + o(1)\}x$.

Proof. The number $8an+b$ cannot be divisible by p_r^2 when $1 \leqq r \leqq g$. Now the number of numbers $8an+b$ $(x < n \leqq 2x)$ which are divisible by p_r^2 $(r > g)$ is clearly of the order

$$\sum_{r > g} \frac{x}{p_r^2} = O\left(\frac{x}{p_g} \right) = O\left(\frac{x}{g \log g} \right)$$

$$= o(x).$$

Hence we obtain lemma 3.

LEMMA 4. *Let* $F(y)$ *denote the number of positive integers* m *such that*

(i) $$m \leqq y,$$

(ii) $$m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_g^{\alpha_g} \quad (\alpha\text{'s} \geqq 0).$$

Then $$F(y) < \sqrt[4]{y}, \quad \textit{when} \quad y > x^{3/4},$$

where g *is as defined above.*

Proof. The number of positive integers of the form p^t (p fixed $\geqq 2$; $t = 0, 1, 2, 3, \dots$) and not exceeding y is clearly less than $2(\log y / \log p) < 3 \log y$, whenever $p \leqq p_g$ and $x > x_0$. Hence, for $x > x_0$,

$$\log F(y) < g \log (3 \log y).$$

Now $$g = \frac{\log x}{(\log\log x)^2} = O\left(\frac{\log y}{(\log\log y)^2} \right)$$

since $x < y^{4/3}$; hence $$\log F(y) = O(\log y / \log\log y), \tag{7}$$

and lemma 4 follows at once.

5. It is clear from (5) that

$$S(x) = S_1(x) + S_2(x) + S_3(x), \tag{8}$$

where, for $r = 1, 2, 3$,

$$S_r(x) = \sum_{x<n\leqq 2x} \sum_{m\leqq x^{\frac{1}{4}}} \frac{1}{m}\left(\frac{8an+b}{m}\right), \tag{9}$$

and m runs through different sets of values (described below) in the 3 sums:

(i) In $S_1(x)$, m takes all values ($\leqq x^{\frac{1}{4}}$) of the form $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_g^{\alpha_g}$ (α's $\geqq 0$), i.e. m is not divisible by any prime greater than p_g.

(ii) In $S_2(x)$, m takes only values of the form $m = m_1 m_2$, where $m_1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_g^{\alpha_g}$ (α's $\geqq 0$), while $m_2 = Q^2 M$, where $(m_2, a) = 1$, and M is *quadratfrei* and greater than 1 (so that m and m_2 cannot be perfect squares).

(iii) In $S_3(x)$, m takes only values of the form $m_1 Q^2$, where

$$m_1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_g^{\alpha_g} \ (\alpha\text{'s} \geqq 0) \quad \text{and} \quad (Q, a) = 1, \ Q > 1.$$

It is clear that these three types of m are non-overlapping and exhaust all positive integers m, and so (8) is rendered obvious.

6. In $S_1(x)$ we clearly have

$$\left(\frac{8an+b}{m}\right) = \left(\frac{b}{m}\right) = \lambda(m),$$

where $\lambda(m)$ is Liouville's function (Landau, *Handbuch der Primzahlen*, 2, (1909), 617) defined as follows:

$$\lambda(1) = 1, \quad \lambda(n) = (-1)^{\beta_1+\beta_2+\beta_3+\dots+\beta_t},$$

where $n = q_1^{\beta_1} q_2^{\beta_2} \dots q_t^{\beta_t}$, and the q's are distinct primes (β's > 0). Hence

$$S_1(x) = x \sum_{m\leqq x^{\frac{1}{4}}} \frac{\lambda(m)}{m}, \tag{10}$$

where m runs only through positive integers of the form $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_g^{\alpha_g}$ (each $\alpha \geqq 0$).

7. In $S_2(x)$ we have $m = m_1 m_2$ and so

$$\left(\frac{8an+b}{m}\right) = \left(\frac{8an+b}{m_1}\right)\left(\frac{8an+b}{m_2}\right)$$
$$= \lambda(m_1)\left(\frac{8an+b}{m_2}\right),$$

since m_1 is not divisible by any prime greater than p_g. Since m_2 is not a perfect square, we have

$$\sum \left(\frac{8an+b}{m_2}\right) = 0,$$

where n runs through a complete set of residues $(\bmod m_2)$; and hence

$$\sum_{n=u}^{v}\left(\frac{8an+b}{m_2}\right) = O(m_2) = O(m),$$

$$\sum_{n=u}^{v}\left(\frac{8an+b}{m}\right) = O(m).$$

Hence
$$S_2(x) = O\left(\sum_{m\leq x^{\frac{1}{4}}}\frac{m}{m}\right) = O(x^{\frac{1}{4}}). \tag{11}$$

8. In $S_3(x)$, m runs through numbers of the type $m_1 Q^2$ where m_1 is of the form $p_1^{\alpha_1}p_2^{\alpha_2}\dots p_g^{\alpha_g}$ while $(Q,a)=1$, $Q>1$. Hence

$$\begin{aligned}|S_3(x)| &= \left|\sum_{x<n\leq 2x}\sum_{m\leq x^{\frac{1}{4}}}\frac{1}{m}\left(\frac{8an+b}{m}\right)\right| \leq \sum_{x<n\leq 2x}\sum_{m\leq x^{\frac{1}{4}}}\frac{1}{m}\\
&= x\sum_{m\leq x^{\frac{1}{4}}}\frac{1}{m} < x\left\{\prod_{r=1}^{g}\left(1+\frac{1}{p_r}+\frac{1}{p_r^2}+\frac{1}{p_r^3}+\dots\right)\right\}\sum_{\substack{(n,a)=1\\ n>1}}\frac{1}{n^2}\\
&= O(x\log p_g)\sum_{n>p_g}\frac{1}{n^2} = O\left(\frac{x\log(g\log g)}{p_g}\right) = O\left(\frac{x}{g}\right)\\
&= O\left(\frac{x(\log\log x)^2}{\log x}\right),\end{aligned}$$

and so
$$S_3(x) = O\left(\frac{x(\log\log x)^2}{\log x}\right). \tag{12}$$

9. From (6), (8), (10), (11), (12), we get

$$\begin{aligned}T(x) &= \sum_{x<n\leq 2x}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{8an+b}{m}\right)\\
&= x\sum_{m\leq x^{\frac{1}{4}}}\frac{\lambda(m)}{m}+O\left(\frac{x(\log\log x)^2}{\log x}\right).\end{aligned} \tag{13}$$

10. We now proceed to consider the sum

$$\sum_{m\leq x^{\frac{1}{4}}}\frac{\lambda(m)}{m},$$

which occurs in (13) above. We have

$$\begin{aligned}\sum_{m\leq x^{\frac{1}{4}}}\frac{\lambda(m)}{m} &= \sum_{m}\frac{\lambda(m)}{m}-\sum_{m>x^{\frac{1}{4}}}\frac{\lambda(m)}{m}\\
&= \alpha(x)-\beta(x),\end{aligned} \tag{14}$$

where, in $\alpha(x)$, m runs over *all* positive integers of the form $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_g^{\alpha_g}$ (each $\alpha \geqq 0$); in $\beta(x)$, m runs over all such values which are, in addition, greater than $x^{\frac{3}{4}}$. Now

$$\alpha(x) = \prod_{r=1}^{g} \left(1 - \frac{1}{p_r} + \frac{1}{p_r^2} - \frac{1}{p_r^3} + \ldots\right)$$

$$= \prod_{r=1}^{g} \frac{(1-1/p_r)}{(1-1/p_r^2)} \sim \frac{2e^{-C}}{\log p_g} \frac{\pi^2}{8}$$

$$\sim \frac{\pi^2}{4} \frac{e^{-C}}{\log\log x}. \tag{15}$$

Again,

$$\left.\begin{aligned} \beta(x) &= O\Big(\sum_{m > x^{\frac{3}{4}}} \frac{1}{m}\Big), \\ \sum_{m > x^{\frac{3}{4}}} \frac{1}{m} &= \sum_{q > x^{\frac{3}{4}}} \frac{F(q) - F(q-1)}{q}, \end{aligned}\right\} \tag{16}$$

where q runs over all positive integers ($> x^{\frac{3}{4}}$), and $F(q)$ is as in lemma 4. By partial summation, from (16), using $F(y) < y^{\frac{1}{4}}$, we have

$$\beta(x) = O((x^{\frac{3}{4}})^{\frac{1}{4}} (x^{\frac{3}{4}})^{-1}) = O(x^{-\frac{9}{16}}) \tag{17}$$

by lemma 4.

From (14), (15), (16), (17) we get

$$\sum_{m \leqq x^{\frac{3}{4}}} \frac{\lambda(m)}{m} \sim \frac{\pi^2 e^{-C}}{4 \log\log x}. \tag{18}$$

11. From (13) and (18) we get

$$\sum_{x < n \leqq 2x} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{8an+b}{m}\right) \sim \frac{\pi^2}{4} \frac{e^{-C} x}{\log\log x}. \tag{19}$$

Now, from the reciprocity theorem for Jacobi's symbol, we have

$$\left(\frac{8an+b}{m}\right) = \left(\frac{m}{8an+b}\right) \tag{20}$$

if m is odd; and, by definition,

$$\left(\frac{8an+b}{m}\right) = 0$$

if m is even. It now follows, from (19) and lemma 3, that there exists an integer n with $x < n \leqq 2x$ and such that $8an+b$ is *quadratfrei*, and further

$$\sum_{m \text{ odd}} \frac{1}{m} \left(\frac{m}{8an+b}\right) > \frac{\pi^2}{4} e^{-C} \frac{\{1+o(1)\}}{\log\log x}. \tag{21}$$

Now
$$\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{m}{8an+b}\right) = \left(1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\dots\right)\sum_{m\text{ odd}} \frac{1}{m}\left(\frac{m}{8an+b}\right)$$
$$= \frac{2}{3}\sum_{m\text{ odd}} \frac{1}{m}\left(\frac{m}{8an+b}\right), \tag{22}$$
since $\left(\frac{2}{8an+b}\right) = -1$ [using $b \equiv 5 \pmod 8$]. From (21) and (22) we get the following result:

THEOREM 1. *For $x > x_0$, there exists a positive integer n satisfying*

(i) $x < n \leqq 2x$,

(ii) $8an+b$ *is quadratfrei, and*

(iii)
$$\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{m}{8an+b}\right) > \frac{\pi^2}{6} e^{-C} \frac{\{1+o(1)\}}{\log\log(8an+b)},$$

since $\log\log(8an+b) \sim \log\log x$.

Again, since $\left(\frac{m}{8an+b}\right)$ is a real primitive character (mod $(8an+b)$), when m runs through all positive integral values (because $8an+b$ is *quadratfrei*) we can write theorem 1 as follows:

THEOREM 2. *There exist infinitely many k such that*
$$L(1) = \sum_{1}^{\infty} \frac{\chi(n)}{n} > \{1+o(1)\}\frac{\pi^2}{6} e^{-C} \frac{1}{\log\log k},$$
where $\chi(n)$ denotes a real primitive character (mod k). *In fact such a k exists between x and $2x$ for all large x.*

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Government College
Lahore (India)

THE GEOMETRY OF THE BINARY (3, 1) FORM

By J. A. TODD

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1. The object of this paper is to interpret geometrically the 20 concomitants of the binary (3, 1) form which, as I have recently shown,* constitute the complete irreducible system for the form.

A binary (m, n) form $a_x^m \alpha_y^n$ in two sets of binary variables (x_0, x_1) and (y_0, y_1) may be interpreted geometrically in the following manner.† If we put

$$X = x_0 y_0, \quad Y = x_0 y_1, \quad Z = x_1 y_0, \quad T = x_1 y_1, \tag{1}$$

and interpret (X, Y, Z, T) as homogeneous coordinates in space, then the equations (1) are the parametric equations of the quadric surface Q whose equation is

$$XT - YZ = 0. \tag{2}$$

Those points of Q for which the ratio $x_0 : x_1$ is fixed lie on a generator of one system, (x) say, on Q, while those points for which the ratio $y_0 : y_1$ is fixed lie on a generator of the opposite system (y). If $f \equiv a_x^m \alpha_y^n$ is a binary (m, n) form, then the equation $f = 0$ determines an (m, n) correspondence between the two systems of generators on Q, and the locus of points of intersection of corresponding generators is a curve on Q, of order $m+n$, meeting the generators of the systems (x), (y) in n, m points respectively. That is, the points represented by the vanishing of an (m, n) form lie on an (n, m) curve on Q.

In the case of a (3, 1) form, with which we are immediately concerned, the curve C which represents the form is a rational quartic curve having the generators (y) as trisecants and meeting each of the generators (x) in a single point. The geometrical properties of such a curve have been intensively studied,‡ and we shall recall, in the next section, the facts which are relevant to the present account.

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* Todd, *Proc. Camb. Phil. Soc.* 42 (1946), 196–205.

† This representation has been applied to the (2, 2) form by Turnbull, *Proc. Roy. Soc. Edinburgh*, 44 (1924), 23–50.

‡ A detailed account is given by Telling, *The rational quartic curve in space of three and four dimensions* (Cambridge Tracts in Mathematics and Mathematical Physics, no. 34, 1936), where proofs of the statements made below will be found.

2. A non-singular rational quartic curve C in space lies on a unique quadric surface Q, and has one system (y) of generators of Q as trisecant lines. The coordinates of a variable point on C are expressible as homogeneous binary quartic polynomials in a pair of parameters (t_0, t_1), and the four points of C which lie in a plane are represented by the vanishing of a quartic form in (t_0, t_1) which, as the plane varies, remains apolar to a fixed quartic ϕ. The four points of C for which $\phi = 0$ are the points of contact of C with its four stationary osculating planes. The two invariants I and J of the binary quartic ϕ vanish, respectively, if the parameters of these four points form an equianharmonic or a harmonic set. The four points of C represented by the vanishing of the hessian H of ϕ are the double points of the g_3^1 (involution of triads of points) traced on C by its trisecants, that is, they are the points of contact of C with the four tangents which meet C again. The six points on C represented by the vanishing of the sextic covariant of ϕ form three mutually harmonic pairs of points; the three chords joining these pairs of points are concurrent in a point O, and their three polar lines with respect to Q lie in the polar plane π of O with respect to Q. These three chords, and their polar lines, form the edges of a tetrahedron with respect to which Q is self-polar, and the curve C is invariant under the three harmonic inversions with respect to pairs of opposite edges of the tetrahedron.

The planes which meet C in equianharmonic tetrads of points envelop a quadric Σ, and the planes which meet C in harmonic tetrads envelop a Steiner surface k which has O as a triple point and the three chords of C which pass through O as double lines; and the curve C is an asymptotic line on this surface.

3. It is well-known that any general pencil of binary cubics can be regarded as the pencil of first polars of a binary quartic.* Hence, by a linear transformation of the variables (y_0, y_1) the (3, 1) form $a_x^3\alpha_y$ can be written in the form

$$y_0\frac{\partial\psi}{\partial x_0} + y_1\frac{\partial\psi}{\partial x_1},$$

* I find no reference for this, though the result is familiar. A simple proof is as follows. Let r and s be two binary cubics in (x_0, x_1). Then the partial derivatives r_i, s_i of r and s with respect to x_i $(i = 0, 1)$ are quadratics. Since any four binary quadratics are linearly dependent there exist constants λ_i, μ_i such that

$$\lambda_0 r_0 + \lambda_1 s_0 = \mu_0 r_1 + \mu_1 s_1.$$

Thus, if $R = \lambda_0 r + \lambda_1 s$, $S = \mu_0 r + \mu_1 s$, $\partial R/\partial x_0 = \partial S/\partial x_1$. Hence there is a binary quartic ψ such that $\partial\psi/\partial x_0 = S$, $\partial\psi/\partial x_1 = R$. The pencil of first polars of ψ is the pencil defined by R and S, and hence these polars all depend linearly on r and s.

where ψ is a binary quartic in (x_0, x_1). In general, by a linear transformation of the variables (x_0, x_1), we can arrange that ψ takes the form

$$ax_0^4 + 6bx_0^2x_1^2 + ax_1^4,$$

and so we may take our binary (3, 1) form to be

$$f \equiv ax_0^3y_0 + 3bx_0^2x_1y_1 + 3bx_0x_1^2y_0 + ax_1^3y_1. \tag{3}$$

This is, essentially, a canonical form* for the binary (3, 1) form. The concomitants of the form (3) have been given explicitly at the end of the author's paper already quoted. It will not be necessary to reproduce the entire list here, but it is desirable to give a résumé of the 20 forms of the complete system. They are given in the following table, in which an (m, n) form is indicated by the symbol (m, n) following its definition in terms of transvectants:

Degree	Forms
1	f (3, 1).
2	$F \equiv (f,f)^{1,1}$ (4, 0), $h \equiv (f,f)^{2,0}$ (2, 2), $I_2 \equiv (f,f)^{3,1}$ (0, 0).
3	$j \equiv (f, F)^{1,0}$ (5, 1), $t \equiv (f, h)^{1,0}$ (3, 3), $p \equiv (f, F)^{3,0}$ (1, 1).
4	$(f,p)^{0,1}$ (4, 0), $(f,p)^{1,0}$ (2, 2), $\Delta \equiv (h,h)^{2,0}$ (0, 4).
5	$(h,p)^{0,1}$ (3, 1), $(h,p)^{1,0}$ (1, 3).
6	$(j,p)^{0,1}$ (6, 0), $(t,p)^{0,1}$ (4, 2), $(t,p)^{1,0}$ (2, 4), $I_6 \equiv (p,p)^{1,1}$ (0, 0).
7	$(f,p^2)^{2,0}$ (1, 3).
8	$(h,p^2)^{2,0}$ (0, 4).
9	$(t,p^2)^{2,0}$ (1, 5).
12	$(t,p^3)^{3,0}$ (0, 6).

It will be observed that the bilinear form p occupies a prominent position in the scheme, since 12 of the 20 concomitants are expressed as transvectants of a power of p with some other form.

4. We now represent the (3, 1) form (3) by means of a (1, 3) curve on the quadric $XT - YZ = 0$ in the manner explained in § 1. From (3) we see that, when $f = 0$,

$$\frac{y_0}{y_1} = -\frac{x_1(3bx_0^2 + ax_1^2)}{x_0(ax_0^2 + 3bx_1^2)}.$$

From this it is easily seen that, if $f = 0$, the point

$$(X, Y, Z, T) = (x_0y_0, x_0y_1, x_1y_0, x_1y_1)$$

* Precisely, it becomes canonical if we divide by a and put $b/a = m$. The use of the homogeneous form (3), however, has the advantage that the degree of any concomitant can be seen by inspection.

lies on the curve C whose parametric equations are

$$\left.\begin{aligned} X &= -(3bt_0^3t_1+at_0t_1^3), & Z &= -(3bt_0^2t_1^2+at_1^4), \\ Y &= at_0^4+3bt_0^2t_1^2, & T &= at_0^3t_1+3bt_0t_1^3, \end{aligned}\right\} \tag{4}$$

where
$$t_0:t_1 = x_0:x_1. \tag{5}$$

The binary quartics X, Y, Z, T defined in (4), and any linear combination of them, are all apolar to the quartic

$$\phi \equiv bt_0^4-2at_0^2t_1^2+bt_1^4, \tag{6}$$

and accordingly the four stationary osculating planes of C have their points of osculation given by $\phi = 0$. The invariants I and J of this quartic are

$$I = b^2+\tfrac{1}{3}a^2, \quad J = -\tfrac{1}{3}a(b^2-\tfrac{1}{9}a^2),$$

and the discriminant of ϕ is

$$I^3-27J^2 = b^2(a^2-b^2)^2.$$

Now the invariants I_2 and I_6 of the form f are given by

$$I_2 = 2(a^2+3b^2), \quad I_6 = 18b^2(a^2-b^2)^2.$$

Hence I_2 vanishes if the fundamental quartic ϕ is equianharmonic, and I_6 vanishes if ϕ has a repeated factor.* When this is the case, the curve C has a stationary tangent line.†

The hessian of ϕ is

$$H \equiv -\tfrac{1}{3}[abt_0^4+(a^2-3b^2)\,t_0^2t_1^2+abt_1^4].$$

Since the $(4, 0)$ form F is given by

$$F \equiv 2[abx_0^4+(a^2-3b^2)\,x_0^2x_1^2+abx_1^4],$$

we see from (5) that $F = 0$ represents the four generators (x) of Q whose intersections with C are given by $H = 0$, and we have seen in § 2 that these points are the double points of the g_3^1 cut on C by its trisecants.

The sextic covariant of ϕ is a multiple of $t_0t_1(t_0^4-t_1^4)$, and the three pairs of mutually harmonic points on C represented by the vanishing of this covariant are given by the pairs of values $\infty, 0$; $1, -1$; $i, -i$ of t_0/t_1. The coordinates of these three pairs of points are thus

$$(0, 1, 0, 0),\ (0, 0, 1, 0);\quad (-1, 1, -1, 1),\ (1, 1, -1, -1);$$
$$(i, -1, 1, i),\ (-i, -1, 1, -i),$$

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* The canonical form is no longer valid in this case; in fact the form (6) has a repeated factor only when it is a perfect square.

† Telling, *loc. cit. ante*, 69.

and the three lines joining the pairs are

$$X = T = 0, \quad X+T = Y+Z = 0, \quad X-T = Y+Z = 0,$$

which meet in the point O whose coordinates are $(0, 1, -1, 0)$. The polar plane π of O with respect to Q is thus the plane $Y-Z = 0$, and this meets Q in points of a conic expressed by the vanishing of the form

$$x_0 y_1 - x_1 y_0.$$

Reference to the list of concomitants of the form (3) in the author's paper already quoted shows that

$$p \equiv 3b(a^2-b^2)(-x_0y_1+x_1y_0).$$

Hence the section of Q by the plane π is given by $p = 0$. We observe, moreover, that since $I_6 \equiv (p,p)^{1,1}$ the vanishing of I_6 is the condition that p should be the product of linear factors in (x_0, x_1) and (y_0, y_1) respectively, that is, it is the condition that π touches Q.

Moreover, if g is any (m, n) form, the transvectant $(g, p^\lambda)^{\lambda, 0}$ (for $\lambda \leqslant m$) is simply a multiple of

$$\left(y_0 \frac{\partial}{\partial x_0} + y_1 \frac{\partial}{\partial x_1}\right)^\lambda g.$$

It therefore admits the following interpretation. On any generator of the system (y), the form g determines a set of m points. The generator meets π in a point P for which $x_0 : x_1 = y_0 : y_1$. The form $(g, p^\lambda)^{\lambda, 0}$ determines, on this particular generator, the λth polar group of P with respect to the m points on the generator given by $g = 0$. Hence $(g, p^\lambda)^{\lambda, 0}$ represents a curve on Q which is the locus of these polar groups as the generator varies in the system (y). A precisely similar interpretation of the form $(g, p^\lambda)^{0, \lambda}$ $(\lambda \leqslant n)$ is obtained by interchanging the parts played by the two systems of generators.

Thus all the forms of our system will have been accounted for if we interpret those forms of the system whose definition does not involve p. Since f itself, the form F, and the invariants have already been considered, the forms which remain are h, t, j and Δ.

Now h, t, Δ are respectively the hessian, cubic covariant, and discriminant of f regarded merely as a binary cubic in (x_0, x_1). Thus, on any generator of the system (y), the two points given by $h = 0$ are the hessian pair of the three intersections of the generator with C, and the equation $h = 0$ determines a $(2, 2)$ curve on the quadric which is the locus of these points as the generator (y) varies. The interpretation of the form t is exactly similar. And the form Δ, equated to zero, clearly represents the four generators of the system (y) which touch C. The form j is the Jacobian of F and f, regarded as binary forms in (x_0, x_1); and can thus be interpreted in a similar way.

5. All the 20 forms of our system are thus accounted for, though the geometrical significance of some of the forms is not very simple. This is perhaps natural since, particularly in the case of forms of high degree, the form selected as a member of the complete system of f may be one among many other forms of the same degree and orders in the variables which differ from each other by reducible forms, and the actual selection of a particular member of such a set of equivalent forms has been made with a view to algebraic convenience and has no clear geometrical significance.

One feature of the system deserves notice. An inspection of the list of forms shows that, in the complete system of f, (m, n) and (n, m) forms are equally numerous, a fact which is not *a priori* evident since f itself is not symmetrical with respect to its two sets of variables. The explanation of this fact is the following. The quadric Q is unaltered by the harmonic inversion with respect to O and its polar plane π, and this transformation interchanges the two systems of generators on Q in such a way that corresponding generators meet on π. Thus any (m, n) form represents a curve C_1 on Q which, under this transformation, gives rise to a curve C_2 represented by an (n, m) form, one of these forms (to within a constant factor) being obtained from the other by interchange of (x_0, x_1) with (y_0, y_1), and the two curves being in perspective from O. Reference to the table of explicit forms shows that many of the concomitants in our list are paired off in this way, and it is not difficult in those cases where this symmetry with respect to the variables does not exist to obtain it by replacing one or other of the forms by an equivalent form differing from it only by reducible terms. To take just one instance; there are two $(4, 0)$ forms F and $(f, p)^{0,1}$, and two $(0, 4)$ forms Δ and $(h, p^2)^{2,0}$. Of these, F and $(h, p^2)^{2,0}$ are obtained from each other by interchange of the two sets of variables, but $(f, p)^{0,1}$ and Δ are not related in this manner. But it is easily seen that if $(f, p)^{0,1}$ is replaced by the equivalent form $-\frac{1}{3}(f, p)^{0,1} - \frac{1}{4}I_2 F$ the desired symmetry appears.

6. We conclude by considering the curves cut on Q by the quadric Σ and the Steiner surface k, enveloped by the planes which meet C in equianharmonic or harmonic sets. It is convenient to make the change of coordinate system in space defined by

$$\xi = Y + Z, \quad \eta = X - T, \quad \zeta = X + T, \quad \tau = Y - Z. \tag{7}$$

The parametric equations of C are then

$$\begin{aligned} \xi &= a(t_0^4 - t_1^4), & \eta &= -(a+3b)(t_0^3 t_1 + t_0 t_1^3), \\ \zeta &= (a-3b)(t_0^3 t_1 - t_0 t_1^3), & \tau &= a t_0^4 + 6b t_0^2 t_1^2 + a t_1^4, \end{aligned} \tag{8}$$

and the plane $l\xi+m\eta+n\zeta+p\tau = 0$ meets C in the four points given by the vanishing of the form

$$a(l+p)\,t_0^4+[-(a+3b)\,m+(a-3b)\,n]\,t_0^3t_1 \\ +6bpt_0^2t_1^2+[-(a+3b)\,m-(a-3b)\,n]\,t_0t_1^3+a(-l+p)\,t_1^4. \quad (9)$$

The condition that (9) be equianharmonic is that its invariant of the second degree vanishes, a condition which reduces to

$$-4a^2l^2-(a+3b)^2m^2+(a-3b)^2n^2+4(a^2+3b^2)\,p^2 = 0,$$

which is the tangential equation of Σ. The point equation of Σ is therefore

$$(a^2-9b^2)^2\,[a^2\tau^2-(a^2+3b^2)\,\xi^2]+4a^2(a^2+3b^2) \\ \times[(a+3b)^2\zeta^2-(a-3b)^2\eta^2] = 0. \quad (10)$$

On putting

$$\xi = x_0y_1+x_1y_0, \quad \eta = x_0y_0-x_1y_1, \quad \zeta = x_0y_0+x_1y_1, \quad \tau = x_0y_1-x_1y_0 \quad (11)$$

this reduces to

$$-3b^2(a^2-9b^2)^2\,(x_0^2y_1^2+x_1^2y_0^2)+48a^3b(a^2+3b^2)\,(x_0^2y_0^2+x_1^2y_1^2) \\ +6(a^2-b^2)\,(2a^4+45a^2b^2+81b^4)\,x_0x_1y_0y_1. \quad (12)$$

This must be a covariant $(2, 2)$ form associated with f (to within a factor involving a and b). The only $(2, 2)$ forms in the system are h, $(f, p)^{1,0}$ and p^2, of respective degrees 2, 4 and 6. And it is easily verified, on trying to express (12) as a linear combination of these forms, that the coefficients in the linear combination are in fact proportional to invariants of f, and that the intersection of Q and Σ is given by

$$I_2^2h+4I_2(f,p)^{1,0}-2p^2 = 0. \quad (13)$$

The condition that (9) be a harmonic quartic is expressed by the vanishing of its catalecticant. After some reduction, this gives

$$2a(a^2-9b^2)\,lmn+p[8a^2bl^2+(a+3b)^2\,(a-b)\,m^2+(a-3b)^2\,(a+b)\,n^2 \\ -8b(a^2-b^2)\,p^2] = 0, \quad (14)$$

which is the tangential equation of the surface k. The point equation of this surface is

$$K \equiv 8a^2b\eta^2\zeta^2+(a+3b)^2\,(a-b)\,\zeta^2\xi^2+(a-3b)^2\,(a+b)\,\xi^2\eta^2 \\ +2a(a^2-9b^2)\,\xi\eta\zeta\tau = 0. \quad (15)$$

The simplest proof of this is a verification as follows. The surface (15) admits the parametric representation

$$\left.\begin{aligned} \xi &= 2auv, & \eta &= -(a+3b)\,uw, \\ \zeta &= (a-3b)\,vw, & \tau &= (a+b)\,u^2+(a-b)\,v^2+2bw^2, \end{aligned}\right\} \quad (16)$$

and the coordinates (l, m, n, p) of the tangent plane to the surface at the point whose parameters are u, v, w are given by substituting from (16) in $\partial K/\partial\xi$, $\ldots$, $\partial K/\partial\tau$. After a little reduction and removal of a common factor $2a(a^2-9b^2)uvw$ we find

$$\begin{aligned} l &= (a^2-9b^2)[(a+b)u^2+(a-b)v^2-2bw^2]w, \\ m &= 2a(a-3b)[-(a+b)u^2+(a-b)v^2-2bw^2]v, \\ n &= 2a(a+3b)[-(a+b)u^2+(a-b)v^2+2bw^2]u, \\ p &= -2a(a^2-9b^2)uvw; \end{aligned}$$

and these values of l, m, n, p are found to satisfy (14) identically in u, v, w.

When we substitute from (11) in (15), we find that the intersection of K with Q is given by the vanishing of the (4, 4) form

$$\begin{aligned} 8a^2b(x_0^2y_0^2-x_1^2y_1^2)^2+(a+3b)^2(a-b)(x_0y_0+x_1y_1)^2(x_0y_1+x_1y_0)^2 \\ +(a-3b)^2(a+b)(x_0y_1+x_1y_0)^2(x_0y_0-x_1y_1)^2 \\ +2a(a^2-9b^2)(x_0^2y_1^2-x_1^2y_0^2)(x_0^2y_0^2-x_1^2y_1^2). \end{aligned}$$

This form contains f as a factor (as it should, since C lies on K), and the remaining factor proves to be a multiple of

$$x_0[2aby_0^3+(a^2-3b^2)y_0y_1^2]+x_1[(a^2-3b^2)y_0^2y_1+2aby_1^3]$$

which, from the table of explicit forms, we identify with $(h,p)^{1,0}$. Hence the residual intersection of K and Q is given by

$$(h,p)^{1,0}=0.$$

Trinity College
Cambridge

THREE-DIMENSIONAL SPACES OF RECURRENT CURVATURE

By H. S. Ruse

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1. A Riemannian V_n (of fundamental tensor g_{ij}) for which the covariant derivative of the curvature tensor satisfies the relation

$$R_{hijk,p} = \kappa_p R_{hijk} \tag{1·1}$$

at all points, for some vector-field κ_p, will be called a *space of recurrent curvature*, or, briefly, a *recurrent space*, and will be denoted by K_n. Such spaces have hitherto been called by the non-descriptive name *kappa spaces* (Ruse (**6**)). The new name is chosen to emphasize the property that defines them, namely, that the curvature tensor recurs, so to speak, under covariant differentiation. (1·1) will be called the *recurrence-condition*.

Every V_2 is a K_2 (Ruse (**5**)), the vector κ_p being given in a V_2 by

$$\kappa_p = \frac{\partial}{\partial x^p}\log|K|, \tag{1·2}$$

where K is the Gaussian curvature R_{1212}/g. If, therefore,

$$ds_2^2 = \sum_{1,2} a_{\alpha\beta}dx^\alpha dx^\beta$$

defines the metric of a V_2, and if dl is the line-element of an E_{n-2} (i.e. of a *flat* V_{n-2}) of variables $x^3, x^4, \ldots, x^n$, then the product-space defined by

$$ds^2 = ds_2^2 + dl^2 \tag{1·3}$$

is a K_n with κ_p a function of x^1, x^2 only. This is readily verifiable, the Riemann tensor for (1·3) having essentially only the one component R_{1212}. A K_n defined in this way will be called a *flat extension* of the V_2. More generally, the product $V_r . E_{n-r}$ of any V_r and an E_{n-r} will be called a flat extension of V_r.

Spaces K_4 that also have the property of being simply harmonic have appeared in the papers quoted.

Recurrent spaces for which $\kappa_p \equiv 0$ are symmetric in Cartan's sense (Cartan (**1**, **2**)). Their properties are well known.

The present paper, which is confined to the case $n = 3$, is intended as a further step towards determining the nature of recurrent spaces in general. It is assumed throughout (except for a brief further reference to symmetric spaces in § 9) that κ_p is not identically zero, which excludes, incidentally, the case when V_3 is of constant (non-zero) curvature. It is also assumed that V_3 is not flat. In this paper, therefore, a K_3 will mean a non-flat V_3 in which (1·1) holds for some κ_p not identically zero.

The main conclusion of the paper, reached in § 6, is that *a necessary and sufficient condition for a V_3 to be a K_3 is that it should admit a field of parallel vectors, which may be either non-null or null.*

If the curvature-scalar R is non-zero, then multiplication of (1·1) by $g^{hk}g^{ij}$ at once gives

$$\kappa_p = \frac{\partial}{\partial x^p}\log|R|. \tag{1·4}$$

It will be seen below that κ_p is likewise the gradient of a scalar even when R is everywhere zero.

2. Hereafter, then, it is assumed that $n = 3$ and that the V_3 under discussion is a K_3. The Bianchi identity gives, by (1·1),

$$R_{hijk}\kappa_p + R_{hikp}\kappa_j + R_{hipj}\kappa_k = 0. \tag{2·1}$$

This may be written $\qquad \epsilon^{mns}R_{himn}\kappa_s = 0,$

ϵ^{mns} being the dualizing tensor of components $(\pm 1/\sqrt{g}, 0)$, and so imaginary if $g < 0$. Multiplying this by $\frac{1}{4}\epsilon^{hir}$, we get

$$Q^{rs}\kappa_s = 0, \tag{2·2}$$

where $\qquad Q^{rs} \equiv \frac{1}{4}\epsilon^{rhi}\epsilon^{sjk}R_{hijk}$

is the dual of the Riemann tensor for a V_3.

In the projective space S_2 associated with any given point (x^i) of K_3, the tensor Q^{rs} represents the *Riemann conic-envelope* (Ruse (4), § 3) of equation $Q^{rs}p_r p_s = 0$ in current line-coordinates p_i, and (2·2) states that it is degenerate, consisting of a pair of points, l^i, m^i, say, on the line κ_i. In K_3 we can therefore find vectors l^i, m^i such that

$$Q^{ij}p_i p_j \equiv (l^i p_i)(m^j p_j),$$

and therefore such that $\qquad Q^{ij} \equiv \frac{1}{2}(l^i m^j + m^i l^j).$

Hence $$R_{hijk} \equiv \frac{1}{2}(l_{hi}m_{jk} + m_{hi}l_{jk}), \tag{2·3}$$

where $$\left.\begin{aligned} l_{ij} &= \epsilon_{ijk}l^k, & m_{ij} &= \epsilon_{ijk}m^k, \\ l^i &= \tfrac{1}{2}\epsilon^{ijk}l_{jk}, & m^i &= \tfrac{1}{2}\epsilon^{ijk}m_{jk}.\end{aligned}\right\} \tag{2·4}$$

In S_2, l_{ij} and m_{ij} are dual coordinates of the points l^i, m^i. These points lie on the line κ_i. Let λ_i be any line through l^i other than κ_i. Then, with a proper normalization* of the components of λ_i, we have

and similarly
$$\left.\begin{aligned} l_{ij} &= \lambda_i\kappa_j - \kappa_i\lambda_j, \\ m_{ij} &= \mu_i\kappa_j - \kappa_i\mu_j, \end{aligned}\right\} \tag{2·5}$$
μ_i being a line through m^i.

The geometrical configuration in S_2 therefore consists of three lines forming a triangle LMN. L is the point l^i (or l_{ij}) and M the point m^i (or m_{ij}). LM is the line κ_i, and LN, MN the lines λ_i, μ_i respectively. Let the coordinates of N be n^i (or, dually, n_{ij}). Then since it is the intersection of the lines λ_i, μ_i, we have, with a proper normalization of its components,
$$n_{ij} = \lambda_i\mu_j - \mu_i\lambda_j. \tag{2·6}$$
The plane S_2 also contains the *fundamental conic* of point- and tangential-equations $g_{ij}X^iX^j = 0$ and $g^{ij}p_ip_j = 0$ respectively. The raising and lowering of suffixes by means of the fundamental tensor corresponds to taking polars with respect to the conic.

The various incidences are expressed analytically by
$$(\lambda l) = 0 = (\lambda n) = (\kappa l) = (\kappa m) = (\mu m) = (\mu n), \tag{2·7}$$
where (λl) denotes $\lambda_i l^i$, and so on.

3. Differentiating (1·1) covariantly, we obtain
$$R_{hijk,p,q} = (\kappa_{p,q} + \kappa_p\kappa_q)\,R_{hijk}.$$
Interchanging p, q and using the permutation-formula for covariant derivatives (Eisenhart (3), 30), we obtain a relation that may be written
$$R_{rijk}R^r_{.hpq} - R_{rhjk}R^r_{.ipq} + R_{rkhi}R^r_{.jpq} - R_{rjhi}R^r_{.kpq} = (\kappa_{p,q} - \kappa_{q,p})\,R_{hijk}. \tag{3·1}$$
The second term on the left-hand side is minus the first with h, i interchanged, and the last two are the same as the first two with the pairs (hi), (jk) interchanged. From this it quickly follows, by substituting from (2·3) in (3·1), that
$$\begin{aligned} \{(\kappa\mu)\,l_{hi} - (\kappa\lambda)\,m_{hi} - (\kappa\kappa)\,n_{hi}\}\,(m_{jk}l_{pq} - l_{jk}m_{pq}) + \{(hi), (jk) \text{ interchanged}\} \\ = 2(\kappa_{p,q} - \kappa_{q,p})\,(l_{hi}m_{jk} + m_{hi}l_{jk}), \end{aligned} \tag{3·2}$$
where $(\kappa\mu) = \kappa_i\mu^i$, etc. Taking the dual in the manner indicated in (2·4), we get
$$\begin{aligned} \{(\kappa\mu)\,l^i - (\kappa\lambda)\,m^i - (\kappa\kappa)\,n^i\}\,(m^jl^k - l^jm^k) \\ + \{(\kappa\mu)\,l^j - (\kappa\lambda)\,m^j - (\kappa\kappa)\,n^j\}\,(m^il^k - l^im^k) \\ = 2\nu^k(l^im^j + m^il^j), \end{aligned} \tag{3·3}$$
where
$$\nu^k = \tfrac{1}{2}\epsilon^{kpq}(\kappa_{p,q} - \kappa_{q,p}).$$

* I.e. choice of the factor of proportionality.

Let σ_i be an arbitrary line in S_2 through the point l^i, so that $(\sigma l) = 0$. Multiply (3·3) by $\sigma_i \sigma_j$, summing for i, j. We then get

$$2\{-(\kappa\lambda)(\sigma m) - (\kappa\kappa)(\sigma n)\}(\sigma m) l^k = 0,$$

and hence* either (a) $(\sigma m) = 0$

or (b) $(\kappa\lambda)(\sigma m) + (\kappa\kappa)(\sigma n) = 0$

for all lines σ such that $(\sigma l) = 0$.

In case (a) we have $\sigma_i m^i = 0$ whenever $\sigma_i l^i = 0$, and therefore

$$l^i = \theta m^i, \quad \text{i.e.} \quad l_{ij} = \theta m_{ij},$$

θ being a scalar. The points l^i, m^i in S_2 coincide. Substituting in (2·3) and absorbing the scalar $\sqrt{\theta}$ into m_{ij}, we deduce that the Riemann tensor must have the form

$$R_{hijk} = m_{hi} m_{jk}, \tag{3·4}$$

where m_{hi} is a bivector.

In case (b) it follows similarly that

$$(\kappa\lambda) m^i + (\kappa\kappa) n^i = \phi l^i, \tag{3·5}$$

ϕ being a scalar. If $(\kappa\kappa) = 0$, then we deduce once again that the points l^i, m^i coincide, and (3·4) follows. If $(\kappa\kappa) \neq 0$, then, multiplying (3·5) by κ_i and using (2·7), we get

$$(\kappa\kappa)(\kappa n) = 0,$$

and therefore $(\kappa n) = 0$. So the point n^i lies on the line κ_i, and the points l^i, m^i are coincident yet again. Hence in all cases the Riemann tensor reduces to the form (3·4). Also from (3·2), with $l_{ij} = m_{ij}$, we now obtain

$$0 = 4(\kappa_{p,q} - \kappa_{q,p}) m_{hi} m_{jk},$$

and therefore, because m_{ij} is not zero except in the excluded case of a flat space,

$$\kappa_{p,q} - \kappa_{q,p} = 0.$$

Hence κ_p *is the gradient of a scalar* κ. It will still be denoted by κ_p, though consistency of notation now really requires it to be denoted by $\kappa_{,p}$.

The figure in S_2 now consists of two lines κ_i, μ_i meeting in the point M of coordinates m^i (dually m_{ij}), together with the fundamental conic. The triangle LMN has shrunk into the point M.

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* The possibility $l^k = 0$ is excluded, because, by (2·3), this would lead to the case of a flat space.

4. Thus in every K_3, the Riemann tensor has the form

$$R_{hijk} = m_{hi} m_{jk}, \tag{4·1}$$

where

$$m_{ij} = \mu_i \kappa_j - \kappa_i \mu_j. \tag{4·2}$$

If we substitute from (4·1) in the recurrence-condition (1·1) and multiply the resulting equation by s^{kl}, where s^{kl} is any bivector such that $s^{kl} m_{kl} \neq 0$, we quickly obtain (cf. Ruse (**6**), equations (3·9), (3·10))

$$m_{ij,p} = \tfrac{1}{2} \kappa_p m_{ij}. \tag{4·3}$$

Let

$$r_{ij} = e^{-\frac{1}{2}\kappa} m_{ij}, \tag{4·4}$$

so that, by (4·1),

$$R_{hijk} = e^{\kappa} r_{hi} r_{jk}. \tag{4·5}$$

In S_2, r_{ij} represents the same point M as m_{ij}, and we write as usual

$$r^i = \tfrac{1}{2} \epsilon^{ijk} r_{jk}. \tag{4·6}$$

By (4·4) and (4·3),

$$\begin{aligned} r_{ij,p} &= e^{-\frac{1}{2}\kappa}(m_{ij,p} - \tfrac{1}{2}\kappa_p m_{ij}) \\ &= 0, \end{aligned}$$

and hence, by (4·6),

$$r^i_{\ ,p} = 0. \tag{4·7}$$

Thus *any K_3 admits a field of parallel vectors r^i, the Riemann tensor being expressed in terms of the dual of r^i by* (4·5).

5. If r^i is not a null vector-field, then the metric of K_3 is reducible to the form

$$ds^2 = \sum_{1,2} a_{\alpha\beta} dx^\alpha dx^\beta \pm (dx^3)^2, \tag{5·1}$$

where the $a_{\alpha\beta}$ are independent of x^3 (Eisenhart (**3**), 71). This, by (1·3), is already known to define the metric of a recurrent space, and so we have a partial converse to the above theorem, namely: *If a V_3 admits a field of non-null parallel vectors, it is a K_3.* Such a K_3 is a flat extension of a K_2, being the product of a K_2 and an E_1.

6. Multiply (4·5) by $g^{hk} g^{ij}$. We get

$$\begin{aligned} R &= -e^{\kappa} r^{jk} r_{jk} \\ &= -e^{\kappa} \epsilon^{jkp} \epsilon_{jkq} r_p r^q \\ &= -2e^{\kappa} r_p r^p. \end{aligned}$$

Therefore $R = 0$ if $r_p r^p = 0$, and conversely. Thus *the curvature-scalar R for a K_3 is zero if, and only if, r^i is a null vector-field.*

Assume this now to be the case. Then

$$r^i r_i \equiv g^{ij} r_i r_j = 0. \tag{6·1}$$

Also by (4·7), $$r_{i,p} = 0 = r_{p,i},$$
and r_i is therefore the gradient of a scalar r. Thus
$$r_i = \frac{\partial r}{\partial x^i}. \tag{6·2}$$

From (4·7) it also follows that the equation
$$r^i \frac{\partial f}{\partial x^i} = 0$$
for f is completely integrable and so admits two independent solutions, of which one, by (6·1), may be taken to be r itself. Let u be another solution, independent of r. Then
$$r^i \frac{\partial u}{\partial x^i} = 0. \tag{6·3}$$

Further, let v be a solution of the equation
$$r^i \frac{\partial v}{\partial x^i} = 1. \tag{6·4}$$
v is then determinate but for the addition of an arbitrary function of r and u. Moreover, the functions r, u, v are independent because r and u were chosen so, and $v = F(r, u)$ implies $r^i \frac{\partial v}{\partial x^i} = 0$, in contradiction to (6·4). Therefore coordinates may be chosen so that
$$x^1 = u, \quad x^2 = r, \quad x^3 = v.$$
In terms of these, $$r_i = \delta_i^2, \tag{6·5}$$
by (6·2). Therefore, by (6·1), (6·3), (6·4),
$$g^{21} = 0 = g^{22}, \quad g^{23} = 1, \tag{6·6}$$
whence, taking the normalized cofactors,
$$g_{31} = 0 = g_{33}, \quad g_{23} = 1. \tag{6·7}$$
Also the determinant g has the value
$$g = -g_{11}. \tag{6·8}$$
Moreover, $$0 = r_{i,j} \quad \text{by (4·7)}$$
$$= -\begin{Bmatrix} 2 \\ ij \end{Bmatrix} \quad \text{by (6·5)},$$
whence, by (6·6), $$[ij, 3] = 0,$$
and therefore, by (6·7), $$\frac{\partial g_{ij}}{\partial x^3} = 0.$$

Hence the g_{ij} are functions of x^1, x^2 only. Combining this with (6·7), we have: *The metric of any V_3 that admits a field of parallel null vectors is reducible to the form*

$$ds^2 = \sum_{1,2} a_{\alpha\beta} dx^\alpha dx^\beta + 2dx^2 dx^3, \tag{6·9}$$

*where the $a_{\alpha\beta}$ are functions of x^1, x^2 only.**

Conversely, any V_3 of the form (6·9) is a K_3. For, as is easy to show, the only non-zero components of the curvature tensor of (6·9) are those related to R_{1212}, while

$$R_{1212,p} = \frac{\partial R_{1212}}{\partial x^p} - \frac{R_{1212}}{g_{11}} \frac{\partial g_{11}}{\partial x^p}$$

$$= R_{1212} \frac{\partial}{\partial x^p} \log \left| \frac{R_{1212}}{g} \right|,$$

by (6·8). Thus if we write

$$\kappa_p = \frac{\partial}{\partial x^p} \log \left| \frac{R_{1212}}{g} \right| \tag{6·10}$$

(cf. (1·2)), we have

$$R_{1212,p} = \kappa_p R_{1212}.$$

Further, for all components other than those related to R_{1212},

$$R_{hijk,p} = 0 = \kappa_p R_{hijk},$$

so the curvature tensor of (6·9) satisfies the recurrence-condition.

Therefore every V_3 of the form (6·9) is a K_3. Combining this result with the theorems stated in §§ 4, 5, we obtain the general theorem enunciated in § 1.

The K_3 defined by (6·9) is a *null extension* (Walker (7)) of the V_2 (i.e. K_2) defined by

$$ds_2^2 = \sum_{1,2} a_{\alpha\beta} dx^\alpha dx^\beta,$$

and it has therefore been shown that *every K_3 is either a flat extension or a null extension of a K_2.*

7. Apart from certain changes of notation and presentation, the main contents of § 6 are due to Prof. A. G. Walker, to whom I am greatly indebted. By a rather longer and less elegant method, I had shown that the metric of any V_3 which admits a field of parallel null vectors is reducible to the form

$$ds^2 = e dx^2 + \psi(x, y) dy^2 + 2dy dz \quad (e = \pm 1). \tag{7·1}$$

For this,

$$g = -e, \quad R_{1212} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2}. \tag{7·2}$$

It is easy to see that (6·9) may be reduced by a transformation of coordinates to this even simpler form.

* This is a special case of a general theorem for a V_n due to A. G. Walker (7). [*Note added in proof.* Cf. Eisenhart, *Annals of Math.* 39 (1938), 316–321.]

It may here be noted that, if E_{n-3} is a flat space of $n-3$ dimensions, and K_3 is the space defined by (6·9) or (7·1), then the product-space $K_3 . E_{n-3}$, which is a flat extension of K_3, is a K_n.

8. The vector κ_p appearing in (6·10) may be expressed in invariant form as follows. By (4·5),

$$R_{klmn} = e^{\kappa} r_{kl} r_{mn},$$

whence, by (4·6), $$e^{\kappa} r^i r^j = \tfrac{1}{4} \epsilon^{ikl} \epsilon^{jmn} R_{klmn}.$$

Multiplying this by $\dfrac{\partial v}{\partial x^i} \dfrac{\partial v}{\partial x^j}$, using (6·4), we get

$$e^{\kappa} = t^{kl} t^{mn} R_{klmn},$$

where $$t^{kl} = \epsilon^{ikl} \frac{\partial v}{\partial x^i}.$$

Thus $$\kappa_p = \frac{\partial}{\partial x^p} \log | t^{kl} t^{mn} R_{klmn} |.$$

This result is also due to A. G. Walker, who obtained it by a slightly different method.

9. The arguments in §§ 2, 3, 4 by which any K_3 was shown to admit a parallel vector-field, and so to be reducible to one of the forms (5·1), (6·9) (or (7·1)), clearly break down if the space is symmetric ($\kappa_p \equiv 0$). Nevertheless, since all spaces of these forms satisfy the recurrence-condition, a particular non-flat symmetric space is obtainable from (7·1) by making $\kappa_p \equiv 0$. This, by (6·10) and (7·2), requires that $\psi(x, y)$ should be of the form

$$\psi(x, y) \equiv ax^2 + x\phi(y) + \chi(y) \quad (a \text{ constant}),$$

making $R_{1212} = -a$. In the particular case when $\phi(y)$ and $\chi(y)$ are constants, there is no loss of generality in taking $\phi = 0 = \chi$ and $a = e'$ $(= \pm 1)$, since this selection of values merely implies a linear transformation of the variables x, y, z, and we therefore obtain

$$ds^2 = e dx^2 + e' x^2 dy^2 + 2dy dz \tag{9·1}$$

as a special (symmetric) case of (7·1). It is of interest to note that, with $e = 1 = e'$, this is the symmetric V_3 given as an example by Cartan ((**1**), 217).

10. In one of the earlier papers already quoted (**6**), I showed that the metric of every *simply harmonic* K_4 is reducible to the form

$$ds^2 = \alpha dx^2 + 2\gamma dx dy + \beta dy^2 + 2dx dz + 2dy dt, \tag{10·1}$$

where α, β are functions of x, y only, and, conversely, that every V_4 of this form is a simply harmonic K_4. The method of deriving this result rather obscured one point to which my attention has been called by Prof. Walker,

to whom I am again indebted. The point in question is that *a necessary and sufficient condition for a V_4 to be a simply harmonic K_4 is that it should admit two parallel vector-fields, the vectors at any point being null and perpendicular.* The necessity of this condition was clearly brought out. The sufficiency, which was less clearly demonstrated, depends on the fact that the possession by a V_4 of two such parallel vector-fields is alone sufficient to ensure its reducibility to the form (10·1). In Walker's terminology, the V_4, or rather K_4, is a null extension of the V_2 defined by

$$ds_2^2 = \alpha\, dx^2 + 2\gamma\, dx\, dy + \beta\, dy^2.$$

No reference has been made in the earlier sections of this paper to the possibility that the K_3 might be simply harmonic. This is because any simply harmonic V_n is an Einstein space of zero scalar curvature (i.e. such that $R_{ij} = 0$), so that, inasmuch as any Einstein V_3 is of constant curvature, a simply harmonic V_3 is everywhere of zero curvature and therefore flat.

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The University
Leeds 2

ON THE TOTAL RELATIVE STRENGTH OF THE HÖLDER AND CESÀRO METHODS

By S. K. BASU

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1. *Introduction*

The (C, r) and (H, r) methods of defining a limit of a sequence, where r is a positive integer, have been shown by various authors* to be equivalent† in the case where the limit is finite. Hausdorff‡ extended the definition of the Hölder means to negative and non-integral orders and established the equivalence of the (C, α) and (H, α) methods for $\alpha > -1$. It is known that,§ for $r = 2, 3, 4, \ldots$, the (H, r) method *totally includes*|| the (C, r) method, but the (C, r) method includes the (H, r) method only *non-totally*. Recently Dr Bosanquet¶ has shown that the (H, r) method $(r = 2, 3, 4, \ldots)$ is *not totally included* in any of the (C, s) methods $(s \geqslant r)$. For $r = 1$, the two methods are identical and as such each totally includes the other. The question then arises as to what one can say of this problem of total relative strength for $1 < r < 2$, or for $-1 < r < 1$.** The question as to whether the (H, r) method totally includes the (C, r) method for non-integral $r > 2$, also seems to remain open. The present paper is an attempt to answer these questions. We have followed here Hausdorff's definitions of the Hölder and Cesàro means.

The author takes this opportunity of thanking Dr Bosanquet for the indication of the problem and for valuable suggestions and criticism throughout.

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* For references see (4), 19.

† I.e. whenever a sequence is summable by either of the methods, it is also summable by the other and to the same limit.

‡ (1).

§ (5), 453, or (7), 245.

|| A method of summation A is said to *include* another method of summation B *totally*, if every sequence summable by B to l is also summable by A to l $(-\infty \leqslant l \leqslant +\infty)$. If, however, this only holds for $-\infty < l < +\infty$, then A *includes* B *non-totally*. In case A and B totally include each other, they are said to be *totally equivalent* (see (7), *loc. cit.*).

¶ (9), 12.

** The trivial case for $r = 0$ being excluded.

2. *Hausdorff transformation and some of its familiar properties*

The *Hausdorff mean* y_n of a *real* sequence (x_n) is defined by*

$$y_n = \sum_{m=0}^{n} \binom{n}{m} \Delta^{n-m}\mu_m . x_m \quad (n = 0, 1, 2, \ldots), \tag{1}$$

where $$\Delta^0\mu_m = \mu_m, \quad \Delta^n\mu_m = \sum_{p=0}^{n} (-1)^p \binom{n}{p} \mu_{m+p}, \quad (n \geqslant 1).$$

We shall be mainly concerned with cases where the *factor sequence* (μ_n) is a *moment sequence*, given by

$$\mu_n = \int_0^1 u^n \, d\chi(u) \quad (n = 0, 1, 2, \ldots), \tag{2}$$

$\chi(u)$, the *mass function*, being a function of bounded variation in $0 \leqslant u \leqslant 1$, and the integral taken in the *Riemann-Stieltjes sense*†. The corresponding *moment function* $\mu(t)$ is then defined by

$$\mu(t) = \int_0^1 u^t \, d\chi(u) \quad (t \geqslant 0), \tag{3}$$

so that $$\mu(n) = \mu_n \quad (n = 0, 1, 2, \ldots).$$

Putting $$\lambda_{n,m} = \binom{n}{m} \Delta^{n-m}\mu_m \quad (0 \leqslant m \leqslant n),$$

(1) reduces to the form $$y_n = \sum_{m=0}^{n} \lambda_{n,m} x_m. \tag{4}$$

This is a linear transformation by means of the *triangular* matrix

$$\lambda = \| \lambda_{n,m} \| \quad (m, n = 0, 1, 2, \ldots).$$

A *Hausdorff matrix* λ is always of the form

$$\lambda = \rho\mu\rho^{-1},$$

where ρ is the triangular matrix with the elements

$$p_{n,m} = (-1)^m \binom{n}{m}, \ddagger$$

and μ is a *diagonal* matrix given by

$$\mu = \| \mu_{n,m} \|,$$

where $$\mu_{n,m} = \begin{cases} 0 & (m \neq n), \\ \mu_n & (m = n). \end{cases}$$

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* (1), 79. If we are concerned with the series $\sum_{\nu=0}^{\infty} a_\nu$, we shall take $x_n = \sum_{\nu=0}^{n} a_\nu$.

† The function u^0 is defined for $u = 0$ so as to be continuous.

‡ Evidently $\rho\rho = E$, the identity matrix, so that $\rho = \rho^{-1}$, i.e. ρ is its own reciprocal.

We shall here restrict ourselves to cases in which

$$\mu_n \neq 0 \quad (n = 0, 1, 2, \ldots),$$

so that the reciprocal matrix μ^{-1} always exists.* All Hausdorff matrices are permutable with one another. If λ, λ' are the matrices corresponding to μ, μ', then

$$\lambda\lambda' = \rho\mu\mu'\rho^{-1}, \quad \frac{\lambda}{\lambda'} = \rho\frac{\mu}{\mu'}\rho^{-1}.\dagger \tag{5}$$

In particular, if μ, μ' are reciprocal to each other, that is, $\mu_n\mu'_n = 1$ (μ'_n being the elements of the diagonal matrix μ'), so also are λ, λ' and conversely. By taking different suitable diagonal matrices μ, many important linear transformations are included in (1).

If, for every sequence (x_n), $x_n \to l$ implies $y_n \to l$ ($|l| < \infty$), the transformation (4) or (1) or the matrix λ is said to be *regular*.‡ The corresponding factor sequence (μ_n) is then called a *regular sequence*.

Necessary and sufficient conditions that λ may be regular are: §

$$\text{(i) } M_n = \sum_{m=0}^{n} |\lambda_{n,m}| \leqslant M, \quad \text{(ii) } \sum_{m=0}^{n} \lambda_{n,m} = \mu_0 = 1, \quad \text{(iii) } \lambda_{n,0} \to 0.$$

When for every (x_n) the *convergence* of (x_n) implies that of (y_n) (the limits not necessarily being equal), λ is called a *c-matrix* (convergence-preserving matrix) and the corresponding sequence (μ_n) is called a *c-sequence*. For λ to be a c-matrix (i) alone is necessary and sufficient. || Also the class of c-sequences is identical with the class of moment sequences.¶

3. *Hausdorff definitions of Cesàro and Hölder means*

Let C_n^α and H_n^α denote respectively the Cesàro and the Hölder means of order $\alpha > -1$ for the sequence (s_n), and C^α and H^α the corresponding matrices. Then, as in (1),

$$C_n^\alpha = \sum_{m=0}^{n} \binom{n}{m} \Delta^{n-m}\mu_m . s_m,$$

and

$$H_n^\alpha = \sum_{m=0}^{n} \binom{n}{m} \Delta^{n-m}\mu'_m . s_m, **$$

* (11), 416.

† The quotient $\lambda\lambda'^{-1} = \lambda'^{-1}\lambda$ will be written as λ/λ'.

‡ The term "matrix λ" or simply "λ" will henceforth be used for "method of summability which corresponds to the matrix λ".

§ (1), 79–81. Condition (i) implies that $\lambda_{n,m} \to 0$, $m = 1, 2, \ldots$.

|| (1), 81.

¶ (1), 84 and 98.

** It may be easily shown that the Hausdorff definition of C_n^α for $\alpha > -1$ is identical with the definitions of Cesàro, Knopp and Chapman, while that of H_n^α is identical with the definition of Hölder for $\alpha = 0, 1, 2, \ldots$. For the older definitions see (10), 66–67, 70.

where $$\mu_n = \frac{1}{A_n^\alpha},^* \quad A_n^\alpha = \binom{n+\alpha}{n},$$

and $$\mu'_n = \frac{1}{(n+1)^\alpha}.^\dagger$$

The following theorems will be proved:

THEOREM 1. *For* $-1<\alpha<0$, $C_n^\alpha \to +\infty$‡ *implies* $H_n^\alpha \to +\infty$; *but the converse is false.*

THEOREM 2. *If* $0<\alpha<1$, *then* $H_n^\alpha \to +\infty$ *implies* $C_n^\alpha \to +\infty$; *but the converse is false.*

THEOREM 3. *For* $\alpha>1$, $C_n^\alpha \to +\infty$ *implies* $H_n^\alpha \to +\infty$; *but the converse is false.*

THEOREM 4. *If* $-1<\alpha<\beta<0$, *then* $H_n^\alpha \to +\infty$ *does not imply* $C_n^\beta \to +\infty$.

THEOREM 5. *With* $0<\alpha<\beta<1$, $C_n^\alpha \to +\infty$ *does not imply* $H_n^\beta \to +\infty$.

THEOREM 6. *For* $1<\alpha<\beta$, $H_n^\alpha \to +\infty$ *does not imply* $C_n^\beta \to +\infty$.

4. *Totally monotone sequence and totally regular Hausdorff matrix*

The transformation (4), when regular, is said to be *totally regular*§ if, for every sequence (x_n), $x_n \to +\infty$ implies $y_n \to +\infty$. When (4) is totally regular, we shall call λ a *totally regular matrix* and the corresponding sequence (μ_n) a *totally regular sequence.*

We require some properties of a *totally monotone function* and its relation to the corresponding totally monotone sequence.

A sequence (μ_n) is said to be *totally monotone*‖ (really totally monotone decreasing), if
$$\Delta^n \mu_m \geqslant 0 \quad (m, n = 0, 1, 2, \ldots).$$

A function $\mu(t)$ is said to be *totally monotone* for $t>\tau$,‖ if it is differentiable any number of times for $t>\tau$ and
$$(-1)^n \mu^{(n)}(t) \geqslant 0 \quad (n = 0, 1, 2, \ldots); \tag{6}$$
it is said to be totally monotone for $t \geqslant \tau$, if it is totally monotone for $t>\tau$ and $\mu(\tau) \geqslant \mu(\tau+0)$. By the repeated application of the mean-value theorem we have for $t>\tau$,
$$\Delta^n \mu(t) = (-1)^n \mu^{(n)}(\xi),\P \tag{7}$$

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* (1), 82. † (1), 83.

‡ It evidently suffices to consider the case with $+\infty$, since a change of sign of every s_n produces a change of sign of every C_n^α.

§ (7), 232. ‖ (3), 188–189.

¶ Here also, as for the sequence (μ_n), $\Delta\mu(t) = \mu(t) - \mu(t+1)$.

where ξ is some number between t and $t+n$. It then follows that, if $\mu(t)$ is totally monotone for $t \geqslant \tau$, so that $\mu(\tau) \geqslant \mu(\tau+0)$, the sequence

$$\mu(\tau), \quad \mu(\tau+1), \quad \mu(\tau+2), \quad \ldots$$

is also totally monotone. In particular, for $t \geqslant 0$, a totally monotone function $\mu(t)$ gives a totally monotone sequence $\mu_n = \mu(n)$.

Again, for a given c-sequence (μ_n) there is a uniquely determined function $\chi(u)$ of bounded variation in $0 \leqslant u \leqslant 1$, satisfying (2).* In particular, if (μ_n) is totally monotone, the corresponding function $\chi(u)$ is a uniquely defined, bounded, non-decreasing function of u in $0 \leqslant u \leqslant 1$.* Therefore, whenever a moment sequence (μ_n) is given, the corresponding function $\mu(t)$ is *uniquely defined* by (3). Moreover, when (μ_n) is totally monotone, as $\mu(0) \geqslant \mu(+0)$ and $\mu(t)$ is differentiable any number of times for $t > 0$,† with

$$(-1)^n \mu^{(n)}(t) = \int_0^1 u^t \log(1/u)^n \, d\chi(u) \geqslant 0,$$

it follows that $\mu(t)$ is also totally monotone for $t \geqslant 0$, with

$$\mu(n) = \mu_n \quad (n = 0, 1, 2, \ldots).$$

The moment function $\mu(t)$ is said to be *regular* when (μ_n) is regular and *totally regular* when (μ_n) is totally regular.

We shall now prove the following lemma:

LEMMA 1. *A regular matrix λ is totally regular, if and only if*

either (i) *(μ_n) is a totally monotone sequence,*
or (ii) *$\mu(t)$ is a totally monotone function for $t \geqslant 0$.*

The two conditions are equivalent, in the sense that one implies the other.

Proof. The proof of the lemma follows immediately from our previous remark and from the fact that λ, when regular, is totally regular, if and only if

$$\lambda_{n,m} \geqslant 0 \ddagger \quad (m, n = 0, 1, 2, \ldots),$$

i.e.

$$\Delta^{n-m} \mu_m \geqslant 0 \quad (m, n = 0, 1, 2, \ldots;\ m \leqslant n),$$

i.e. (μ_n) is a totally monotone sequence.

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* These are the results in connexion with the well-known moment problem of Hausdorff (see (1), 98–102, or (2), 222–232). Actually, $\chi(u)$ is uniquely defined at all points of continuity except for an additive constant. It is then made unique in $0 \leqslant u \leqslant 1$ through suitable normalization.

† In fact, if t is complex, $\mu(t)$ is regular for $R(t) > 0$.

‡ (7), 243. It may be noted here that for a general regular transformation $y_n = \sum_{m=0}^{n} a_{n,m} x_m$, the condition for total regularity is that $a_{n,m} \geqslant 0$ for $m \geqslant K$ and all n. In the case of a Hausdorff transformation we have $K = 0$.

If we put
$$\mu(t) = e^{\rho(t)},$$
then the condition (6) is also satisfied for $t > 0$, if
$$(-1)^n \rho^{(n)}(t) \geqslant 0^* \quad (n = 1, 2, 3, \ldots). \tag{8}$$
Before taking up the proofs of the theorems we shall prove one more lemma which is of fundamental importance for our purpose.

LEMMA 2.† *Two regular Hausdorff transformations with non-vanishing moment sequences cannot be totally equivalent, unless they are identical.*

Proof. As in (1), let the two transformations be
$$y_n = \sum_{m=0}^{n} \binom{n}{m} \Delta^{n-m}\mu_m . x_m,$$
and
$$y'_n = \sum_{m=0}^{n} \binom{n}{m} \Delta^{n-m}\mu'_m . x_m \quad (n = 0, 1, 2, \ldots),$$
where
$$\mu_m \neq 0, \quad \mu'_m \neq 0 \quad (m = 0, 1, 2, \ldots).$$
Then by (5), we have
$$\text{(i)}\ y'_n = \sum_{m=0}^{n} \binom{n}{m} \Delta^{n-m}\left(\frac{\mu'_m}{\mu_m}\right) y_m,$$
and
$$\text{(ii)}\ y_n = \sum_{m=0}^{n} \binom{n}{m} \Delta^{n-m}\left(\frac{\mu_m}{\mu'_m}\right) y'_m.$$
Suppose now that the two transformations are totally equivalent, so that $y_n \to l$ implies $y'_n \to l$ $(-\infty \leqslant l \leqslant +\infty)$, and conversely. Then (i) and (ii) must both be totally regular. Therefore, by lemma 1,
$$\Delta\left(\frac{\mu'_m}{\mu_m}\right) \geqslant 0, \quad \Delta\left(\frac{\mu_m}{\mu'_m}\right) \geqslant 0 \quad (m = 0, 1, 2, \ldots).$$
Hence
$$\frac{\mu_{m+1}}{\mu'_{m+1}} = \frac{\mu_m}{\mu'_m} \quad (m = 0, 1, 2, \ldots);$$
and so
$$\frac{\mu_m}{\mu'_m} = \frac{\mu_0}{\mu'_0} = 1,$$
since the transformations are regular. Therefore,
$$\mu_m = \mu'_m \quad (m = 0, 1, 2, \ldots),$$
so that the two transformations are identical.

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* (1), 95.

† This important observation was made by Dr Bosanquet in one of his class lectures on Divergent Series.

5. *Proof of theorems* 1, 2 *and* 3

The proofs of these theorems are almost similar and so we shall give a joint proof for them. We require the following lemma.

LEMMA 3. Let
$$M_r(T) = r^{-1}\{T^{-n} - (T+r)^{-n}\} \quad (r \neq 0).$$
Then, if $T \geqslant 1$, $n > 0$ *and* $-1 < r < s$ $(r, s \neq 0)$,
$$M_r(T) > M_s(T).$$

Proof. We have
$$M_r(T) = \frac{n}{r}\int_0^r \frac{dx}{(T+x)^{n+1}} = n\int_0^1 \frac{dy}{(T+ry)^{n+1}},$$
and the integrand is a steadily decreasing function of $r > -1$, since $T \geqslant 1$.

Proof of the theorems. To prove theorems 1, 2 and 3, let
$$\sigma(t) = \log\Gamma(t+\alpha+1) - \log\Gamma(\alpha+1) - \log\Gamma(t+1) - \alpha\log(t+1),$$
and define $\rho(t)$ so that
$$\rho(t) = \sigma(t) \quad (-1 < \alpha < 0), \qquad = -\sigma(t) \quad (0 < \alpha < 1), \qquad = \sigma(t) \quad (\alpha > 1). \tag{9}$$
For the proofs of theorems 1 and 3 we consider the matrix regular H^α/C^α and for theorem 2 the matrix regular C^α/H^α. Thus in theorems 1 and 3 we take
$$\mu_n = \frac{A_n^\alpha}{(n+1)^\alpha} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\,\Gamma(n+1)}\,\frac{1}{(n+1)^\alpha},$$
$$\mu(t) = \frac{\Gamma(t+\alpha+1)}{\Gamma(\alpha+1)\,\Gamma(t+1)}\,\frac{1}{(t+1)^\alpha} = e^{\sigma(t)} = e^{\rho(t)},*$$
while in theorem 2
$$\mu_n = \frac{(n+1)^\alpha}{A_n^\alpha}, \quad \mu(t) = e^{-\sigma(t)} = e^{\rho(t)}.$$
Now for $t > 0$,
$$\sigma'(t) = \frac{\Gamma'(t+\alpha+1)}{\Gamma(t+\alpha+1)} - \frac{\Gamma'(t+1)}{\Gamma(t+1)} - \frac{\alpha}{t+1} = \sum_{m=0}^{\infty}\left(\frac{1}{m+t+1} - \frac{1}{m+t+\alpha+1}\right) - \frac{\alpha}{t+1},\dagger$$
and hence for $n = 1, 2, 3, \ldots$,
$$\frac{(-1)^n\sigma^{(n)}(t)}{(n-1)!} = \alpha(t+1)^{-n} - \sum_{m=0}^{\infty}\{(m+t+1)^{-n} - (m+t+\alpha+1)^{-n}\}$$
$$= \alpha\sum_{m=0}^{\infty}\{M_1(m+t+1) - M_\alpha(m+t+1)\},$$

---

* The function $\mu(t)$ is seen, by Carlson's theorem, (8), 186, to be the right function, since it is regular and bounded for $R(t) > 0$ and equal to the moment constant μ_n for $t = n$, $n = 0, 1, \ldots$.

† See, for example, (8), 149.

since $(t+1)^{-n} = \sum_{m=0}^{\infty} \{(m+t+1)^{-n} - (m+t+2)^{-n}\}$, the term-by-term differentiations being justified by the uniform convergence of the differentiated series. It then follows from lemma 3 that for $t > 0$,

$$\frac{(-1)^n \sigma^{(n)}(t)}{(n-1)!} > 0 \quad (-1 < \alpha < 0), \qquad < 0 \quad (0 < \alpha < 1), \qquad > 0 \qquad (\alpha > 1).$$

Therefore, if $\alpha > -1$, by (8) and (9), since $\sigma(+0) = \sigma(0)$, $\mu(t)$ is totally monotone for $t \geqslant 0$. In theorems 1 and 3, since the matrix H^α/C^α is regular, it is totally regular by lemma 1; and so C^α/H^α cannot be totally regular by lemma 2. Hence $C_n^\alpha \to +\infty$ implies $H_n^\alpha \to +\infty$; but $H_n^\alpha \to +\infty$ does not imply $C_n^\alpha \to +\infty$. For theorem 2 the matrix C^α/H^α is totally regular and so H^α/C^α is not totally regular. The result now follows.

6. *Negative results*

The remaining three theorems deal with negative results. In each case, therefore, we have to show that the corresponding matrix is not totally regular; and this will be true, by lemma 1, if and only if $\mu(t)$, the corresponding moment function, is not totally monotone for $t \geqslant 0$. Again, to show that $\mu(t)$ is not totally monotone, in each of the above three cases, it will be sufficient to consider the sign of $\mu^{(n)}(t)$ for small values of t. We shall suppose then in what follows that $0 < t < 1$; and we shall see that for small t, $\mu(t)$ does not satisfy the test (6) for a totally monotone function.

Proof of theorem 4. Here we consider the regular matrix C^β/H^α for which

$$\mu_n = \frac{(n+1)^\alpha}{A_n^\beta} = \frac{\Gamma(\beta+1)\,\Gamma(n+1)}{\Gamma(n+\beta+1)}(n+1)^\alpha \quad (-1 < \alpha < \beta < 0).$$

Put $\alpha = -p,\ \beta = -q;$ then $0 < q < p < 1.$

We have now

$$\mu(t) = \frac{\Gamma(1-q)\,\Gamma(t+1)}{\Gamma(t-q+1)}(t+1)^{-p} = \mu_1(t)\,\mu_2(t), \quad \text{say},$$

where
$$\mu_1(t) = \frac{\Gamma(1-q)\,\Gamma(t+1)}{\Gamma(t-q+1)} = t\,\frac{\Gamma(t)\,\Gamma(1-q)}{\Gamma(t-q+1)} = t\int_0^1 u^{t-1}(1-u)^{-q}\,du$$
$$= t\int_0^1 \Big\{u^{t-1} + qu^t + \frac{q(q+1)}{2!}u^{t+1} + \dots$$
$$+ \frac{q(q+1)\dots(q+n-1)}{n!}u^{t+n-1} + \dots\Big\}\,du;$$

that is,
$$\mu_1(t) = 1 + \sum_{n=1}^{\infty} \frac{q(q+1)\dots(q+n-1)}{n!}\,\frac{t}{t+n}, \tag{10}$$

the term-by-term integration being justified, since all the terms are $\geqslant 0$.* As $0<t<1$, expanding $t/(t+n)$ in a power series in t, we have from (10),

$$\mu_1(t) = 1+\sum_{n=1}^{\infty}\frac{q(q+1)\dots(q+n-1)}{n!}\sum_{k=0}^{\infty}(-1)^k\left(\frac{t}{n}\right)^{k+1} \tag{11}$$
$$= 1+\sum_{k=0}^{\infty}(-1)^k t^{k+1}\sum_{n=1}^{\infty}\frac{q(q+1)\dots(q+n-1)}{n!}\frac{1}{n^{k+1}}.$$

We have to justify the inversion in the order of summation. Denoting the double series in (11) by $\sum_n\sum_k v_{n,k}$, we have

$$\sum_n\sum_k |v_{n,k}| = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{q(q+1)\dots(q+n-1)}{n!}\left(\frac{t}{n}\right)^{k+1}$$
$$= \sum_{n=1}^{\infty}\frac{q(q+1)\dots(q+n-1)}{n!}\frac{t}{n-t}.$$

Since $0<t<1$, $$\frac{t}{n-t}<\frac{1}{n-1}\quad (n\geqslant 2),$$

and the series $\sum_{n=2}^{\infty}\frac{q(q+1)\dots(q+n-1)}{n!}\frac{1}{n-1}$ evidently converges as $q<1$. Therefore $\sum_n\sum_k v_{n,k}$ converges absolutely for $0<t<1$. The inversion is thus justified.

Putting

$$c_{k+1}(q) = \sum_{n=1}^{\infty}\frac{q(q+1)\dots(q+n-1)}{n!}\frac{1}{n^{k+1}}\quad (k=0,1,2,\dots),$$

we have then $\mu_1(t) = 1+\sum_{k=0}^{\infty}(-1)^k c_{k+1}(q)\, t^{k+1}\quad (0<t<1).$

Again, $$\mu_2(t) = (1+t)^{-p}$$
$$= 1+\sum_{k=1}^{\infty}(-1)^k\frac{p(p+1)\dots(p+k-1)}{k!}t^k.$$

Now both the power series for $\mu_1(t)$ and $\mu_2(t)$ are absolutely convergent in $0<t<1$. The product of $\mu_1(t)$ and $\mu_2(t)$, therefore, exists as a power series, absolutely convergent in $0<t<1$; as such the term-by-term differentiation of the product series is admissible, as often as we please, in $0<t<1$.

Since $$\mu(t) = \mu_1(t)\mu_2(t),$$

the coefficient of t^k in the power series for $\mu(t)$

$$= (-1)^k\left[\frac{p(p+1)\dots(p+k-1)}{k!}-\sum_{r=1}^{k}\frac{p(p+1)\dots(p+k-r-1)}{(k-r)!}c_r(q)\right]\dagger$$
$$= (-1)^k\psi_k(p,q),\quad \text{say}.$$

* See, for example, (8), 347.

† The coefficient of $c_k(q)$ is to be taken as 1.

Hence, if $\mu(t)$ be differentiated successively, for small values of t, the sign of $\mu^{(k)}(t)$ will be the same as that of $(-1)^k \psi_k(p,q)$. Therefore, for $\mu(t)$ to be totally monotone for $t > 0$, we must have, by (6),

$$\psi_k(p,q) \geqslant 0 \quad (k = 0, 1, 2, \ldots).$$

Then $\mu(t)$ cannot be totally monotone if, for some k,

$$\psi_k(p,q) < 0;$$

that is, $$\sum_{r=1}^{k} \frac{p(p+1)\ldots(p+k-r-1)}{(k-r)!} c_r(q) > \frac{p(p+1)\ldots(p+k-1)}{k!}.$$

Since $0 < p < 1$,

$$\frac{p(p+1)\ldots(p+k-2)}{(k-1)!} > \frac{p(p+1)\ldots(p+k-1)}{k!} \quad (k = 2, 3, 4, \ldots).$$

Hence to see that $\mu(t)$ is not totally monotone, it is sufficient to prove that

$$\frac{p(p+1)\ldots(p+k-2)}{(k-1)!} \sum_{r=1}^{k} c_r(q) > \frac{p(p+1)\ldots(p+k-1)}{k!},$$

or, indeed $$\sum_{r=1}^{k} c_r(q) > 1, \tag{12}$$

for some k.

Now, from the definition of $c_{k+1}(q)$ it follows that

$$c_k(q) > c_{k+1}(q) \quad (k = 1, 2, 3, \ldots),$$

and since the first term of the series for $c_k(q)$ is q,

$$c_1(q) > c_2(q) > \ldots > c_{k-1}(q) > c_k(q) > q.$$

Hence (12) certainly holds if $kq > 1$, that is,

$$k > 1/q \quad (0 < q < 1).$$

Thus, for any given q, values of k can be found so that (12) holds. Therefore C^β/H^α cannot be a totally regular matrix. This proves theorem 4.

7. *Proof of theorem* 5

The matrix for consideration is now H^β/C^α and

$$\mu_n = \frac{A_n^\alpha}{(n+1)^\beta} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\,\Gamma(n+1)} \frac{1}{(n+1)^\beta} \quad (0 < \alpha < \beta < 1).$$

Then $$\mu(t) = \frac{\Gamma(t+\alpha+1)}{\Gamma(\alpha+1)\,\Gamma(t+1)} \frac{1}{(t+1)^\beta} = \frac{\mu_1(t)\,\mu_2(t)}{\Gamma(1+\alpha)\,\Gamma(1-\alpha)},$$

where $$\mu_1(t) = \frac{\Gamma(t+\alpha+1)\,\Gamma(1-\alpha)}{\Gamma(t+2)}, \quad \mu_2(t) = (1+t)^{1-\beta}.$$

Now
$$\mu_1(t) = \int_0^1 u^{t+\alpha}(1-u)^{-\alpha}\,du$$
$$= \int_0^1 \left(\sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!} u^{t+\alpha+n} \right) du^*$$
$$= \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!} \frac{1}{t+\alpha+n+1};$$
the inversion is justified since all the terms are positive.†

Expanding $\dfrac{1}{t+\alpha+n+1}$ in powers of t, we have
$$\mu_1(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!} \frac{t^k}{(\alpha+n+1)^{k+1}}$$
$$= \sum_{k=0}^{\infty} (-1)^k t^k \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!} \frac{1}{(\alpha+n+1)^{k+1}}.$$
We can justify the inversion as before, for $0<t<1$.‡ We now write
$$\mu_1(t) = \sum_{k=0}^{\infty} (-1)^k d_{k+1}(\alpha)\, t^k \quad (0<t<1),$$
where
$$d_{k+1}(\alpha) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!} \frac{1}{(\alpha+n+1)^{k+1}}.$$
Again,
$$\mu_2(t) = (1+t)^{1-\beta}$$
$$= 1+(1-\beta)\,t - \frac{(1-\beta)\beta}{2!} t^2 + \ldots$$
$$+ (-1)^{k-1} \frac{(1-\beta)\beta(\beta+1)\ldots(\beta+k-2)}{k!} t^k + \ldots \quad (0<t<1).$$

The two power series for $\mu_1(t)$ and $\mu_2(t)$ being absolutely convergent in $0<t<1$, their product series is also so in $0<t<1$; and thus this last series can be differentiated term-by-term any number of times in $0<t<1$. Since
$$\mu(t) = \frac{\mu_1(t)\,\mu_2(t)}{\Gamma(1+\alpha)\,\Gamma(1-\alpha)},$$
the coefficient of t^k in the power series for $\mu(t)$
$$= \frac{(-1)^k}{\Gamma(1+\alpha)\,\Gamma(1-\alpha)} \left[d_{k+1}(\alpha) - (1-\beta) \sum_{r=1}^{k} \frac{\beta(\beta+1)\ldots(\beta+r-2)}{r!} d_{k-r+1}(\alpha) \right] \S$$
$$= (-1)^k \phi_{k+1}(\alpha,\beta), \quad \text{say.}$$

* The term corresponding to $n=0$ in the sum under the sign of integration is to be taken as $u^{t+\alpha}$.

† (8), 347.

‡ Actually, for $0<t<1+\alpha$; but for our purpose we have already supposed that $0<t<1$.

§ The term in this series corresponding to $r=1$ is to be taken as $d_k(\alpha)$.

Then for $\mu(t)$ to be totally monotone, for small values of t, we must have, by (6),

$$\phi_{k+1}(\alpha,\beta) \geqslant 0 \quad (k = 0, 1, 2, \ldots).$$

Therefore $\mu(t)$ cannot be totally monotone if, for some k,

$$\phi_{k+1}(\alpha,\beta) < 0;$$

that is,
$$(1-\beta)\sum_{r=1}^{k} \frac{\beta(\beta+1)\ldots(\beta+r-2)}{r!} d_{k-r+1}(\alpha) > d_{k+1}(\alpha). \tag{13}$$

Now, since
$$0 < \left(\frac{1+\alpha}{\alpha+n+1}\right)^{k-r+1} < 1 \quad (n = 1, 2, 3, \ldots;\ r < k+1),$$

$$(1+\alpha)^r d_r(\alpha) > (1+\alpha)^{k+1} d_{k+1}(\alpha) \quad (r = 1, 2, 3, \ldots),$$

or
$$\frac{d_r(\alpha)}{d_{k+1}(\alpha)} > (1+\alpha)^{k-r+1}.$$

Hence (13) will be true if, for some k,

$$(1-\beta)\sum_{r=1}^{k} \frac{\beta(\beta+1)\ldots(\beta+r-2)}{r!}(1+\alpha)^r > 1. \tag{14}$$

As the series in (14) is a part of the divergent series

$$\sum_{r=1}^{\infty} \frac{(1-\beta)\beta(\beta+1)\ldots(\beta+r-2)}{r!}(1+\alpha)^r,$$

(14) certainly holds for $k > K$, with some fixed K. Therefore, H^β/C^α cannot be a totally regular matrix. Theorem 5 now follows.

8. *Proof of theorem* 6

For the proof of the last theorem we require three more lemmas. We write

$$\Delta_s^m v_{s,n} = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} v_{s+\nu,n};$$

so that, if
$$v_{s,n} = 1/s^n \quad (n, s = 1, 2, 3, \ldots),$$

then
$$\Delta_s^m \frac{1}{s^n} = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} \frac{1}{(s+\nu)^n}. \tag{15}$$

LEMMA 4. (i) $(1/s^p)$ *is a totally monotone sequence for* $p > 0$, *with*

$$\Delta_s^m \frac{1}{s^p} > 0 \quad (m, s = 1, 2, 3, \ldots).$$

(ii)
$$\Delta_s^m \frac{1}{s^p} > \Delta_s^m \frac{1}{s^{p+1}}, \quad \textit{for } p > m \quad (s = 2, 3, 4, \ldots).$$

Proof. (i) Let
$$f(x) = 1/x^p \quad (x > 0).$$

Then $$f^{(m)}(x) = (-1)^m \frac{p(p+1)\dots(p+m-1)}{x^{m+p}};$$

so that, by (7), $$\Delta^m f(x) > 0, \quad \text{for} \quad x > 0.$$

Therefore $$f(s) = 1/s^p \quad (s = 1, 2, 3, \dots)$$

is a totally monotone sequence with $\Delta_s^m(1/s^p) > 0$.

(ii) Let $$\phi(x) = \frac{1}{x^p} - \frac{1}{x^{p+1}} \quad (x > 0).$$

Then $$\phi^{(m)}(x) = (-1)^m \frac{(p+1)\dots(p+m-1)}{x^{p+m}} \left(p - \frac{p+m}{x}\right).$$

Hence by (7), $$\Delta^m \phi(x) > 0, \quad \text{for} \quad x \geqslant 2,$$

if $$p - \frac{p+m}{x} > 0, \quad \text{for} \quad x \geqslant 2,$$

i.e. if $$p > m.$$

Thus $$\Delta^m \phi(s) > 0, \quad \text{for} \quad p > m \quad (s = 2, 3, 4, \dots),$$

i.e. $$\Delta_s^m \frac{1}{s^p} > \Delta_s^m \frac{1}{s^{p+1}}, \quad \text{for} \quad p > m \quad (s = 2, 3, 4, \dots).$$

LEMMA 5. $$\lim_{p \to \infty} \frac{\Delta_s^m(1/s^p)}{\Delta_s^m(1/s^{p+1})} = s.$$

Proof. For, by (15),

$$\frac{\Delta_s^m \frac{1}{s^p}}{\Delta_s^m \frac{1}{s^{p+1}}} = \frac{\sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} \frac{1}{(s+\nu)^p}}{\sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} \frac{1}{(s+\nu)^{p+1}}}$$

$$= \frac{\frac{1}{s^p}\left[1 - \binom{m}{1}\left(\frac{s}{s+1}\right)^p + \binom{m}{2}\left(\frac{s}{s+2}\right)^p - \dots + (-1)^m \binom{m}{m}\left(\frac{s}{s+m}\right)^p\right]}{\frac{1}{s^{p+1}}\left[1 - \binom{m}{1}\left(\frac{s}{s+1}\right)^{p+1} + \binom{m}{2}\left(\frac{s}{s+2}\right)^{p+1} - \dots + (-1)^m \binom{m}{m}\left(\frac{s}{s+m}\right)^{p+1}\right]}$$

$$\to s, \quad \text{as} \quad p \to \infty.$$

LEMMA 6.* *If a series is summable* (H, α) *to* $+\infty$, *then it is also summable* $(H, \alpha+\beta)$ *to* $+\infty$, *where* $\alpha+\beta > \alpha > -1$.

Proof. Consider the matrix $H^{\alpha+\beta}/H^\alpha$ for which the factor sequence is given by
$$\mu_n = \frac{(n+1)^{-\alpha-\beta}}{(n+1)^{-\alpha}} = \frac{1}{(n+1)^\beta}.$$

* The corresponding result for the (C, α) method is well known (see (7), 245). The case for the (H, α) method, when α, β are positive integers, is also known (see (6), 31).

Since $\beta > 0$, (μ_n) is totally monotone by lemma 4 (i). As the matrix $H^{\alpha+\beta}/H^{\alpha}$ is regular, it is therefore totally regular by lemma 1. Whence the result follows.

Proof of theorem. To prove theorem 6 we have to show that the regular matrix C^{β}/H^{α} is not totally regular for $1 < \alpha < \beta$. However, instead of using the matrix C^{β}/H^{α}, we shall first consider one in which β is replaced by an integer.

Let k be any positive integer $\geqslant 2$, and $1 < \alpha \leqslant 2$ $(\alpha \neq k)$. Then for the matrix C^k/H^{α},

$$\mu_n = \frac{(n+1)^{\alpha}}{A_n^k}.$$

Put $\alpha = 1+\eta$, $0 < \eta \leqslant 1$, so that

$$\mu_n = k!\,\frac{(n+1)^{\eta}}{(n+2)\,(n+3)\ldots(n+k)}.$$

Then
$$\mu(t) = k!\,\frac{(t+1)^{\eta}}{(t+2)\,(t+3)\ldots(t+k)};$$

or, splitting up the expression for $\mu(t)$ into partial fractions, we have

$$\mu(t) = k!(t+1)^{\eta}\sum_{r=2}^{k}(-1)^{r-2}\frac{1}{(r-2)!\,(k-r)!}\frac{1}{t+r}. \tag{16}$$

Now
$$\frac{(1+t)^{\eta}}{r+t} = \frac{1}{r}(1+t)^{\eta}\left(1+\frac{t}{r}\right)^{-1}.$$

If the right-hand side be expanded in ascending powers of t, we find that the coefficient of t^n in $\dfrac{(1+t)^{\eta}}{r+t}$ is

$$(-1)^n\left[\frac{1}{r^{n+1}}-\eta\frac{1}{r^n}-\frac{\eta(1-\eta)}{2!}\frac{1}{r^{n-1}}-\ldots-\frac{\eta(1-\eta)\ldots(n-1-\eta)}{n!}\frac{1}{r}\right].$$

Then multiplying by $(-1)^{r-2}k!/\{(r-2)!\,(k-r)!\}$, for $r = 2, 3, \ldots, k$, and adding, we see, from (16), that the coefficient of t^n in the expansion of $\mu(t)$ is

$$\begin{aligned}&(-1)^n\sum_{r=2}^{k}(-1)^{r-2}\frac{k!}{(r-2)!\,(k-r)!}\\ &\qquad\times\left[\frac{1}{r^{n+1}}-\eta\frac{1}{r^n}-\frac{\eta(1-\eta)}{2!}\frac{1}{r^{n-1}}-\ldots-\frac{\eta(1-\eta)\ldots(n-1-\eta)}{n!}\frac{1}{r}\right]\\ &\qquad= (-1)^n\theta_n(\eta),\quad\text{say.}\end{aligned}$$

The above process of multiplication and subsequent differentiations are all justified by the absolute convergence of the power series involved for $0 < t < 1$. In order that $\mu(t)$ may be totally monotone we must have then for small values of t,

$$\theta_n(\eta) \geqslant 0 \quad (n = 0, 1, 2, \ldots).$$

Therefore $\mu(t)$ is not a totally monotone function if, for some n,

$$\theta_n(\eta) < 0,$$

that is, if

$$\eta \sum_{r=2}^{k} (-1)^{r-2} \binom{k-2}{r-2} \frac{1}{r^n} + \frac{\eta(1-\eta)}{2!} \sum_{r=2}^{k} (-1)^{r-2} \binom{k-2}{r-2} \frac{1}{r^{n-1}} + \dots$$
$$+ \frac{\eta(1-\eta)\dots(n-1-\eta)}{n!} \sum_{r=2}^{k} (-1)^{r-2} \binom{k-2}{r-2} \frac{1}{r} > \sum_{r=2}^{k} (-1)^{r-2} \binom{k-2}{r-2} \frac{1}{r^{n+1}}. \tag{17}$$

Now
$$\sum_{r=2}^{k} (-1)^{r-2} \binom{k-2}{r-2} \frac{1}{r^n} = \sum_{\nu=0}^{k-2} (-1)^{\nu} \binom{k-2}{\nu} \frac{1}{(2+\nu)^n}$$
$$= \Delta_s^{k-2} \frac{1}{s^n} \quad \text{with } s = 2, \quad \text{by} \quad (15),$$
$$= T_n, \quad \text{say} \quad (n = 1, 2, 3, \dots).$$

Then (17) may be written as

$$\eta T_n + \frac{\eta(1-\eta)}{2!} T_{n-1} + \dots + \frac{\eta(1-\eta)\dots(n-1-\eta)}{n!} T_1 > T_{n+1}. \tag{18}$$

Taking $m = k-2$ and $s = 2$ in lemma 4 (i), we have

$$\Delta_s^{k-2} \frac{1}{s^p} > 0, \quad \text{where} \quad s = 2,$$

i.e.
$$T_p > 0 \quad (p = 1, 2, 3, \dots).$$

Hence (18) is true, if

$$\eta T_n + \frac{\eta(1-\eta)}{2!} T_{n-1} + \dots + \frac{\eta(1-\eta)\dots(n-k+1-\eta)}{(n-k+2)!} T_{k-1} > T_{n+1}. \tag{19}$$

Again, by lemma 4 (ii) with $m = k-2$ and $s = 2$,

$$T_p > T_{p+1}, \quad \text{for} \quad p > k-2;$$

hence
$$T_p/T_{p+1} > 1 \quad (p > k-2).$$

Also, putting $m = k-2$, $s = 2$ in lemma 5,

$$T_p/T_{p+1} \to 2, \quad \text{as} \quad p \to \infty.$$

A number c, independent of n, can therefore be found such that

$$1 < c \leqslant \text{least of } T_p/T_{p+1}, \quad \text{for} \quad p = k-1, k, \dots, n.$$

Then
$$T_n \geqslant cT_{n+1},$$
$$T_{n-1} \geqslant cT_n \geqslant c^2 T_{n+1},$$
$$\dots\dots\dots\dots\dots\dots$$
$$T_{k-1} \geqslant c^{n-k+2} T_{n+1}.$$

So it follows from (19) that $\mu(t)$ is not totally monotone if, for some n,

$$\eta c+\frac{\eta(1-\eta)}{2!}c^2+\ldots+\frac{\eta(1-\eta)\ldots(n-k+1-\eta)}{(n-k+2)!}c^{n-k+2}>1. \tag{20}$$

Now, since the series in (20) is a part of the divergent series

$$\eta c+\sum_{n=2}^{\infty}\frac{\eta(1-\eta)\ldots(n-1-\eta)}{n!}c^n,$$

as $c>1$, (20) certainly holds for $n>N$, for some N. Hence the matrix C^k/H^α is not totally regular, i.e. $H_n^\alpha\to+\infty$ does not imply $C_n^k\to+\infty$.

Next suppose $1<\alpha\leqslant 2$, $\beta>\alpha$. Suppose, if possible, that $H_n^\alpha\to+\infty$ implies $C_n^\beta\to+\infty$. Then, k being an integer $\geqslant\beta$, as the (C,β) method is totally included in the (C,k) method, $C_n^\beta\to+\infty$ implies $C_n^k\to+\infty$. Hence $H_n^\alpha\to+\infty$ implies $C_n^k\to+\infty$ which is contrary to what has just been proved. So for $1<\alpha\leqslant 2$ and $\beta>\alpha$, $H_n^\alpha\to+\infty$ does not imply $C_n^\beta\to+\infty$.

Lastly, suppose $2<\alpha<\beta$. Suppose, if possible, that $H_n^\alpha\to+\infty$ implies $C_n^\beta\to+\infty$. Then since $\alpha>2$ and the (H,γ) method is totally included in the (H,γ') method for $\gamma'>\gamma>-1$, by lemma 6, $H_n^2\to+\infty$ implies $H_n^\alpha\to+\infty$. Hence $H_n^2\to+\infty$ also implies $C_n^\beta\to+\infty$, which contradicts our previous result. Therefore, for $1<\alpha<\beta$, $H_n^\alpha\to+\infty$ does not imply $C_n^\beta\to+\infty$.

This completes the proof of theorem 6.

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University College
London

SOME REMARKS ON THE CONSTRUCTION OF CONTINUOUS NON-DIFFERENTIABLE FUNCTIONS

By F. A. BEHREND

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Weierstrass's function
$$f(x) = \sum_{\nu=0}^{\infty} a^\nu \cos b^\nu \pi x, \tag{1}$$
as well as a number of other examples of continuous functions without derivative at any point, are of the form
$$f(x) = \sum_{\nu=0}^{\infty} a^\nu g(b^\nu x) \quad (0<a<1,\ ab \geqslant 1).\dagger \tag{2}$$

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† Besides (i) $\cos \pi x$, the following may be mentioned: (ii) $\sin \pi x$, (iii) $|\sin \pi x|$, and their polygonal analogues

(iv) $$\phi(x) = \begin{cases} \tfrac{1}{2} - x, & \text{for } 0 \leqslant x \leqslant 1, \\ -\tfrac{3}{2} + x, & \text{for } 1 \leqslant x \leqslant 2, \end{cases} \quad \phi(x+2) = \phi(x),$$

(v) $$\chi(x) = \begin{cases} x, & \text{for } 0 \leqslant x \leqslant \tfrac{1}{2}, \\ 1 - x, & \text{for } \tfrac{1}{2} \leqslant x \leqslant \tfrac{3}{2}, \\ -2 + x, & \text{for } \tfrac{3}{2} \leqslant x \leqslant 2, \end{cases} \quad \chi(x+2) = \chi(x),$$

(vi) $$\psi(x) = \begin{cases} x, & \text{for } 0 \leqslant x \leqslant \tfrac{1}{2}, \\ 1 - x, & \text{for } \tfrac{1}{2} \leqslant x \leqslant 1, \end{cases} \quad \psi(x+1) = \psi(x).$$

(iv) and (vi) are essentially identical, since $\psi(x) = \tfrac{1}{4} - \tfrac{1}{2}\phi(2x)$.

Weierstrass's original example, viz. (i) with odd integral b, is reproduced by Titchmarsh (**10**). Improvements of Weierstrass's result were given by Bromwich (**4**); his results, as well as similar results for (ii), (iii) and (iv), for integral b, are found in Knopp (**6**) and reproduced in Hobson (**7**). Cellérier (**2**) treats (ii) for integral b and $ab = 1$; this example also is reproduced by Hobson. The cases (i) and (ii), with arbitrary b, were studied by Hardy (**5**); see also Dini (**3**). Examples of (vi) are found in van der Waerden (**9**) ($a = \tfrac{1}{10}$, $b = 10$), Landau (**12**) ($a = \tfrac{1}{4}$, $b = 4$), and Hildebrandt (**11**) ($a = \tfrac{1}{2}$, $b = 2$): van der Waerden's example is reproduced by Titchmarsh (**10**). Carathéodory (**8**) treats (iv) for odd b.

The usual study of these functions can be simplified by the following remark which, though rather obvious, does not seem to have been stated explicitly in the literature: $f(x)$ can be written as

$$f(x) = f_n(x) + r_n(x), \tag{3}$$

where
$$f_n(x) = \sum_{\nu=0}^{n-1} a^\nu g(b^\nu x), \tag{4}$$

and
$$r_n(x) = a^n f(b^n x), \tag{5}$$

i.e. the curve $y = r_n(x)$ is obtained from $y = f(x)$ by reducing the ordinates in the ratio $1 : a^n$ and the abscissae in the ratio $b^n : 1$. If, therefore, $f(x)$ is at all crinkly in the large, $r_n(x)$ will be crinkly in detail, and indeed—if $ab > 1$—more crinkly, since the gradients of the chords of $r_n(x)$ will be $(ab)^n$ times the gradients of the corresponding chords of $f(x)$. If, on the other hand, $g(x)$ is not too crinkly itself, e.g. satisfies a Lipschitz condition, the gradients of chords of $f_n(x)$ will increase like $(ab)^{n-1}$ at most, and we shall expect that for sufficiently large ab the contribution of $r_n(x)$ will outweigh that of $f_n(x)$ and that $f(x)$ will be non-differentiable.

It will be shown in the following that this is actually the case for a quite general class of functions $g(x)$ [Theorems I, II]. An advantage of our method is that, though elementary, it does not require the restriction of b to integral values usually adhered to in the elementary treatment. As an illustration we apply the method to Weierstrass's original example; we show how the known elementary estimates and better ones can be obtained; it will be proved, for example, that (1) nowhere possesses a derivative finite or infinite if $ab \geqslant 7{\cdot}05$ for arbitrary real b [see (30)] or, more generally, if

$$ab > 1 + \frac{3+2b_1}{2\cos \pi b_1}\pi(1-a), \quad b_1 = \frac{1}{b-1}, \quad (b > 3)$$

[see (31)]. Hardy's result (5) that (1) does not possess a finite derivative for $ab \geqslant 1$ cannot be obtained in this elementary way, but it can be proved for integral b and for real $b \geqslant \frac{20}{3}$; only the case where b is near to 1 seems really difficult and deep. Corresponding results for the functions mentioned in our first footnote are listed in later footnotes and after Theorem III. Theorem III establishes a rather curious property of a certain class of piecewise linear functions $g(x)$; the corresponding $f(x)$ is, for $ab \geqslant 1$, either non-differentiable or itself piecewise linear.

We distinguish two cases according to whether the non-existence of a finite or infinite differential coefficient or merely that of a finite differential coefficient is required.

1. *Functions without derivative finite or infinite*

As a sufficient criterion we use the following: the continuous bounded function $f(x)$ does not possess a differential coefficient finite or infinite at ξ if to every arbitrarily large $C>0$ correspond two values x, x' such that

$$\Delta(f;\, x, \xi) = \frac{f(x)-f(\xi)}{x-\xi} > C, \tag{6}$$

$$\Delta(f;\, x', \xi) = \frac{f(x')-f(\xi)}{x'-\xi} < -C.\dagger \tag{7}$$

Consider a function $f(x)$ defined by (2); assuming $g(x)$ to be defined for all x, continuous and bounded, $f(x)$ will be continuous and bounded. Suppose, further, that $g(x)$ satisfies a Lipschitz condition

$$|g(x)-g(\xi)| \leqslant M\,|x-\xi| \tag{8}$$

for all x and ξ. Then

$$|\Delta(f_n;\, x, \xi)| = \left|\sum_{\nu=0}^{n-1} a^\nu \frac{g(b^\nu x)-g(b^\nu \xi)}{x-\xi}\right| < M\frac{a^n b^n}{ab-1}, \tag{9}$$

if $ab>1$, for all x, ξ. Suppose that, on the other hand, a lower measure $K>0$ for the "crinkliness" of $f(x)$ can be found, i.e. that for every ξ values x, x' can be found such that

$$\Delta(f;\, x, \xi) \geqslant K, \quad \Delta(f;\, x', \xi) \leqslant -K. \tag{10}$$

The function $r_n(x) = a^n f(b^n x)$ will then have this property: to every ξ correspond x and x' such that

$$\Delta(r_n;\, x, \xi) \geqslant a^n b^n K, \quad \Delta(r_n;\, x', \xi) \leqslant -a^n b^n K. \tag{11}$$

From (9) and (11),

$$\Delta(f;\, x, \xi) \geqslant a^n b^n K - M\frac{a^n b^n}{ab-1} = a^n b^n\left(K - \frac{M}{ab-1}\right), \tag{12}$$

i.e. $\Delta(f;\, x, \xi)$ becomes arbitrarily large positive, and similarly $\Delta(f;\, x', \xi)$ arbitrarily large negative; and $f(x)$ *nowhere possesses a derivative, finite or infinite, if*

$$K - \frac{M}{ab-1} > 0 \quad (ab>1), \tag{13}$$

i.e. if

$$ab > 1 + M/K. \tag{14}$$

It should be noted that K may, and in general will, depend on a, b, and that (14) cannot necessarily be satisfied by giving ab sufficiently large values. For the determination of K a rough study only of $f(x)$ will be needed. It is

† For automatically ξ will be a limit point of such x, x'.

usually easy to determine the value of $f(x)$ for special values of the argument; suppose now that, for every ξ, two such special arguments x_1, x_2 exist such that $x_1 \leqslant \xi \leqslant x_2$, $x_1 < x_2$, and

$$\Delta(f;\, x_1, x_2) \geqslant K; \tag{15}$$

then

$$\Delta(f;\, x, \xi) \geqslant K,$$

where $x = x_1$ or x_2.† Similarly, if x_1', x_2' can be found with $x_1' \leqslant \xi \leqslant x_2'$, $x_1' < x_2'$, and

$$\Delta(f;\, x_1', x_2') \leqslant -K, \tag{16}$$

then

$$\Delta(f;\, x', \xi) \leqslant -K,$$

for $x' = x_1'$ or x_2'. Geometrically this means that, to every ξ, two chords "overlapping" ξ of gradients $\geqslant K$ and $\leqslant -K$ have to be found.‡

To illustrate the procedure consider Weierstrass's original example (1). Here $g(x) = \cos \pi x$ and $M = \pi$. For the determination of K observe that (15) only has to be satisfied; (16) follows from $f(x) = f(-x)$.

First let b be an integer. Since $f(x)$ has the period 2, we can restrict ξ to the interval $0 \leqslant \xi \leqslant 2$. Choose $x_1 = -1$, $x_2 = 2$, then

$$f(-1) = \begin{cases} -a_1 \\ -2+a_1 \end{cases}, \quad f(2) = a_1, \quad \Delta(f;\, -1, 2) = \begin{cases} \frac{2}{3}a_1 \\ \frac{2}{3} \end{cases} = K \quad \text{if } b \text{ is } \begin{cases} \text{odd,} \\ \text{even,} \end{cases}$$

where $a_1 = 1/(1-a)$; i.e. (14) reads

$$ab > 1 + \tfrac{3}{2}\pi(1-a), \quad \text{for odd } b, \tag{17}$$

$$ab > 1 + \tfrac{3}{2}\pi, \quad \text{for even } b. \tag{18}$$

If, in the case of even b, we choose $x_1 = -\frac{2}{3}$, $f(x_1) = -\frac{1}{2}a_1$, we get

$$\Delta(f;\, x_1, x_2) = \tfrac{9}{16}a_1 = K$$

and

$$ab > 1 + \tfrac{16}{9}\pi(1-a). \tag{19}$$

(17) and (18) are well-known estimates (Bromwich (**4**), Knopp (**6**)), while (19) appears to be new; it is better than (18) for $a > \frac{5}{32}$, but asymptotically, i.e. for large b, the lower bound for a given by (18) is better than that given by (19). Slight asymptotical improvements on (17), (18) can be obtained, but they go only just to show that the constant $\frac{3}{2}\pi$ is not the best possible estimate. Thus if μ is the solution of

$$\cot \pi\mu = (\tfrac{3}{2} - \mu)\pi$$

between 0 and $\frac{1}{2}$ (approximately 0·0697), and k is even and $|b\mu - k| \leqslant 1$, then, taking

$$x_1 = -1 + k/b, \quad x_2 = 2 - k/b,$$

† For $\Delta(f;\, x_1, x_2)$ is a mean value between $\Delta(f;\, x_1, \xi)$ and $\Delta(f;\, x_2, \xi)$; see, for example, Bromwich (**4**) or Knopp (**6**).

‡ Knopp (**6**).

we obtain a result of the form (17) or (18), the constant $\frac{3}{2}\pi$ being replaced by

$$\pi \Bigg/ \left(\frac{\cos \pi\mu}{\frac{3}{2}-\mu} + O\left(\frac{1}{b^2}\right)\right) \simeq \frac{3\pi}{2} \Bigg/ \left(1{\cdot}023 + O\left(\frac{1}{b^2}\right)\right).$$

I do not know whether this is the best possible asymptotic estimate.*

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* Although it seems likely to me that the above result is the best possible asymptotic estimate, I have not been able to prove anything beyond Hardy's theorem that, for $b = 4m+1$, $x = \frac{1}{2}$, $f(x)$ possesses a definite infinite differential coefficient if $ab \geqslant 1$ and $a(b+1) < 2$ (except that the $<$ sign can be replaced by $\leqslant$): see Hardy (5), 313.
Some estimates about the other functions mentioned in the first footnote are listed in the following table:

$g(x)$	M	b	x_1	x_2	$f(x_1)$	$f(x_2)$	$\Delta(f; x_1, x_2) = K$	Condition (14) $ab>$	Known estimates $ab>$
$\sin \pi x$	π	$2m$	$-\frac{1}{2}$	$\frac{5}{2}$	-1	1	$\frac{2}{3}$	$1+\frac{3}{2}\pi$	$1+\frac{3}{2}\pi$ (6, 7)
			$-\frac{1}{4}$	$\frac{9}{4}$	$-\left(\frac{1}{\sqrt{2}}+a\right)$	$\frac{1}{\sqrt{2}}+a$	$\frac{2}{5}(\sqrt{2}+2a)$	$1+\frac{5\pi}{2\sqrt{2}+4a}$	
		$4m+1$	$-\frac{1}{2}$	$\frac{5}{2}$	$-\frac{1}{1-a}$	$\frac{1}{1-a}$	$\frac{2}{3(1-a)}$	$1+\frac{3}{2}\pi(1-a)$	$1+\frac{3}{2}\pi$ (6, 7)
		$4m+3$	$-\frac{1}{2}$	$\frac{5}{2}$	$-\frac{1}{1+a}$	$\frac{1}{1+a}$	$\frac{2}{3(1+a)}$	$1+\frac{3}{2}\pi(1+a)$	—
$\chi(x)$	1	$2m$	$-\frac{1}{2}$	$\frac{5}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	4	4 (6, 7)†
			$-\frac{1}{4}$	$\frac{9}{4}$	$-(\frac{1}{4}+\frac{1}{2}a)$	$\frac{1}{4}+\frac{1}{2}a$	$\frac{1+2a}{5}$	$1+\frac{5}{1+2a}$	
		$4m+1$	$-\frac{1}{2}$	$\frac{5}{2}$	$-\frac{1}{2(1-a)}$	$\frac{1}{2(1-a)}$	$\frac{1}{3(1-a)}$	$1+3(1-a)$	4 (6, 7)
		$4m+3$	$-\frac{1}{2}$	$\frac{5}{2}$	$-\frac{1}{2(1+a)}$	$\frac{1}{2(1+a)}$	$\frac{1}{3(1+a)}$	$1+3(1+a)$	—
$\lvert\sin \pi x\rvert$	π	$2m$	0	$\frac{3}{2}$	0	1	$\frac{2}{3}$	$1+\frac{3}{2}\pi$	$1+\frac{3}{2}\pi$ (6)
			0	$\frac{4}{3}$	0	$\frac{\sqrt{3}}{2(1-a)}$	$\frac{3\sqrt{3}}{8(1-a)}$	$1+\frac{8\pi}{3\sqrt{3}}(1-a$	
		$2m+1$	0	$\frac{3}{2}$	0	$\frac{1}{1-a}$	$\frac{2}{3(1-a)}$	$1+\frac{3}{2}\pi(1-a)$	$1+\frac{3}{2}\pi$ (6, 7)
$\psi(x)$‡	1	$2m$	0	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{3}$	4	4 (6, 7)
			0	$\frac{4}{3}$	0	$\frac{1}{3(1-a)}$	$\frac{1}{4(1-a)}$	$1+4(1-a)$	
		$2m+1$	0	$\frac{3}{2}$	0	$\frac{1}{2(1-a)}$	$\frac{1}{3(1-a)}$	$1+3(1-a)$	4 (6, 7)†

† The condition $ab > 1$ given by Hobson (7) is erroneous; as in Hardy ((5), 313) (see the above remark), it can be shown that $f(x)$ has a definite infinite derivative at certain special points when $ab \geqq 1$ and $a(b+2) < 3$, at $x = 0$ if $g(x)$ is $\chi(x)$, at $x = \frac{1}{4}$ if $g(x)$ is $\psi(x)$.
‡ By the remark in the first footnote the same result holds for $\phi(x)$ as for $\psi(x)$.

Now let b be arbitrary, but >3. Choose ϵ so that

$$b_1 \leqslant \epsilon < \tfrac{1}{2}, \left(b_1 = \frac{1}{b-1}\right) \tag{20}$$

and put
$$\delta = \cos \pi\epsilon. \tag{21}$$

Then, $\cos \pi x \geqslant \delta > 0$ in any interval

$$2m - \epsilon \leqslant x \leqslant 2m + \epsilon \quad (m \text{ integral}). \tag{22}$$

We shall call such an interval a δ-peak of $\cos \pi x$. In every interval of length $\geqslant 2 + 2\epsilon$ lies at least one complete δ-peak. It follows that every interval of length $\geqslant (2+2\epsilon)/b^n$ contains at least one complete $(a^n\delta)$-peak of $a^n \cos b^n \pi x$. Now the length of an $(a^{n-1}\delta)$-peak of $a^{n-1} \cos b^{n-1}\pi x$ is

$$\frac{2\epsilon}{b^{n-1}} = \frac{2\epsilon}{b^n} b \geqslant \frac{2\epsilon}{b^n}\left(1 + \frac{1}{\epsilon}\right) = \frac{2+2\epsilon}{b^n}; \tag{23}$$

consequently each $(a^{n-1}\delta)$-peak of $a^{n-1} \cos b^{n-1}\pi x$ contains a complete $(a^n\delta)$-peak of $a^n \cos b^n \pi x$. Starting with any peak (22) of $\cos \pi x$ we can construct successively, for $n = 1, 2, 3, \ldots$, a chain of peaks of $a^n \cos b^n \pi x$, each of which is contained in the previous one; all these intervals will have a point, x_{2m} say, in common for which

$$f(x_{2m}) \geqslant \delta \sum_{\nu=0}^{\infty} a^\nu = \frac{\delta}{1-a}, \quad |2m - x_{2m}| \leqslant \epsilon. \tag{24}$$

Similarly for every $2m+1$, x_{2m+1} can be found such that

$$f(x_{2m+1}) \leqslant \frac{-\delta}{1-a}, \quad |2m+1-x_{2m+1}| \leqslant \epsilon. \tag{25}$$

Since $x_k \leqslant k + \epsilon < k + \frac{1}{2} < k + 1 - \epsilon \leqslant x_{k+1}$, every ξ lies in an interval (x_{2m+1}, x_{2m+4}); hence

$$\Delta(f; x_{2m+1}, x_{2m+4}) \geqslant \frac{2\delta}{(1-a)(x_{2m+4} - x_{2m+1})} \geqslant \frac{2\delta}{(3+2\epsilon)(1-a)} = K, \tag{26}$$

and we have this result: *Weierstrass's function* (1) *does not possess a differential coefficient, finite or infinite, if for some* ϵ

$$0 < \epsilon < \tfrac{1}{2}, \tag{27}$$

$$b \geqslant 1 + \epsilon^{-1}, \tag{28}$$

$$ab > 1 + \frac{3+2\epsilon}{2 \cos \pi\epsilon} \pi(1-a). \tag{29}$$

These conditions can be satisfied, for example, if

$$ab \geqslant 7{\cdot}05. \tag{30}$$

For, with $\epsilon = \frac{1}{6}$, $\delta = \cos\frac{1}{6}\pi = \frac{1}{2}\sqrt{3}$, we have

$$b > ab \geqslant 7{\cdot}05 > 7 = 1+\epsilon^{-1},$$

and
$$ab \geqslant 7{\cdot}05 > 1+\frac{3+\frac{1}{3}}{\sqrt{3}}\pi > 1+\frac{3+2\epsilon}{2\cos\pi\epsilon}(1-a).$$

More generally, if ϵ is chosen as b_1, *the condition obtained is*

$$ab > 1+\frac{3+2b_1}{2\cos\pi b_1}\pi(1-a) \quad (b>3). \tag{31}$$

(31) is less restricted than Dini's

$$ab > 1+\tfrac{3}{2}\pi\frac{1-a}{1-3a} \quad (a<\tfrac{1}{3}),$$

(*loc. cit.* (**3**)), and stronger in every case where Dini's estimate is applicable. It is not contained in Hardy's results (*loc. cit.* (**5**)), since they are concerned with the non-existence of a finite derivative only.

It is obvious that the same procedure can be applied if $\cos\pi x$ is replaced by any of the functions mentioned in the first footnote, or, indeed, by any continuous periodic function $g(x)$ satisfying a Lipschitz condition (8).† In fact, the periodicity of $g(x)$ is not required, and may be replaced by the following property of "pseudo-periodicity": two constants A, B, $A>B$ exist such that in every interval of length p two points α, β can be found for which $g(\alpha)\geqslant A$, $g(\beta)\leqslant B$. Then peaks round all α's and valleys round all β's can be constructed as follows. Choose ρ such that

$$0<\rho<\tfrac{1}{2}(A-B), \tag{32}$$

and
$$\epsilon = \rho/M. \tag{33}$$

Then

$$|g(x)-g(\alpha)| \leqslant M\epsilon = \rho, \quad g(x)\geqslant g(\alpha)-\rho \geqslant A-\rho = \delta \quad (|x-\alpha|\leqslant\epsilon),$$

$$|g(x)-g(\beta)| \leqslant M\epsilon = \rho, \quad g(x)\leqslant g(\beta)+\rho \leqslant B+\rho = \delta' \text{ (say)} \quad (|x-\beta|\leqslant\epsilon);$$

and since $\delta-\delta' = A-B-2\rho>0$, by (32), peaks and valleys do not overlap.

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† The following results may be mentioned:

(ii) $g(x)=\sin\pi x$; evidently the same estimates will hold as for $g(x)=\cos\pi x$.

(iii) $g(x)=|\sin\pi x|$; here the "valleys" round $x=m$ will show a different behaviour from the "peaks" round $x=(m+\frac{1}{2})$; modifying the procedure accordingly, (31) will be replaced by

$$ab>1+\frac{\frac{3}{2}+b_1}{\cos\frac{1}{2}\pi b_1-\sin\frac{1}{2}\pi b_1}\pi(1-a) \quad (b>3),$$

which is satisfied for $ab\geqslant 7{\cdot}91$.

(iv), (v), (vi) $g(x)=\phi(x)$, $\chi(x)$, or $\psi(x)$. The procedure will be the same in all three cases and exactly analogous to that for $\cos\pi x$; the estimate obtained is

$$ab>1+\frac{3+2b_1}{1-2b_1}(1-a) \quad (b>3),$$

which is satisfied for $ab\geqslant 6{\cdot}6$.

It follows that every interval of length $\geqslant p+2\epsilon$ contains a complete peak and a complete valley of $g(x)$. Choosing

$$b \geqslant (p+2\epsilon)/2\epsilon, \tag{34}$$

and constructing a sequence of nested peak (or valley) intervals of $a^n g(b^n x)$ as above for $a^n \cos b^n \pi x$, we find for every α, β points x_α, x_β for which

$$f(x_\alpha) \geqslant \delta a_1 \quad (|\alpha - x_\alpha| \leqslant \epsilon), \tag{35}$$

$$f(x_\beta) \leqslant \delta' a_1 \quad (|\beta - x_\beta| \leqslant \epsilon), \tag{36}$$

and in every interval of length $\geqslant p+2\epsilon$ at least one pair x_α, x_β can be found. Choosing now, for any ξ, the largest $x_\beta \leqslant \xi$ and the smallest $x_\alpha \geqslant \xi$, we have

$$x_\alpha - x_\beta = x_\alpha - \xi + \xi - x_\beta \leqslant 2(p+2\epsilon) \tag{37}$$

and

$$\Delta(f; x_\alpha, x_\beta) = \frac{f(x_\alpha)-f(x_\beta)}{x_\alpha - x_\beta} \geqslant \frac{\delta - \delta'}{2(p+2\epsilon)(1-a)} = \frac{A-B-2\rho}{2(p+2\epsilon)(1-a)} = K; \tag{38}$$

and, similarly, for appropriate $x_{\alpha'}$, $x_{\beta'}$,

$$\Delta(f; x_{\alpha'}, x_{\beta'}) \leqslant -K. \tag{39}$$

Condition (14) then reads

$$ab > 1 + M\frac{2(p+2\epsilon)}{A-B-2\rho}(1-a) = 1 + \frac{2(Mp+2\rho)}{A-B-2\rho}(1-a). \tag{40}$$

In order to satisfy (34) choose

$$b = \frac{p+2\epsilon}{2\epsilon}, \quad \text{i.e.} \quad 2\rho = 2\epsilon M = \frac{pM}{b-1}. \tag{41}$$

Then (40) becomes $\qquad ab > 1 + \dfrac{2Mpbb_1}{A-B-Mpb_1}(1-a). \qquad (42)$

Here b is restricted by (32),

$$\frac{pM}{b-1} = 2\rho < A-B, \quad \text{i.e.} \quad b > 1 + \frac{pM}{A-B}. \tag{43}$$

Both (42) and (43) are certainly satisfied if

$$ab > 1 + \frac{2Mpbb_1}{A-B-Mpb_1}, \tag{44}$$

i.e. for sufficiently large ab. Thus we have

THEOREM I. *If $g(x)$ is defined for all x, continuous, bounded, and "pseudo-periodical", and satisfies a Lipschitz condition, then*

$$f(x) = \sum_{\nu=0}^{\infty} a^\nu g(b^\nu x) \quad (0 < a < 1)$$

does not possess a derivative, finite or infinite, at any point if ab is chosen sufficiently large, e.g. according to (44).

2. *Functions without finite derivative*

As a criterion we use the following: $f(x)$ does not possess a finite differential coefficient at ξ if a constant $C > 0$ exists such that in any arbitrarily small neighbourhood of ξ two values x, x' can be found for which

$$\Delta(f;\ \xi;\ x, x') = \frac{f(x)-f(\xi)}{x-\xi} - \frac{f(x')-f(\xi)}{x'-\xi} \geqslant C. \tag{45}$$

Again let
$$f(x) = \sum_{\nu=0}^{\infty} a^\nu g(b^\nu x) = f_n(x) + r_n(x).$$

Suppose that $g(x)$ satisfies, instead of (8), a "Lipschitz condition of the second order", viz.

$$|\,\Delta(g;\ \xi;\ x, x')\,| \leqslant N(|\,x-\xi\,| + |\,x'-\xi\,|) \tag{46}$$

for all ξ, x, x'. This will be the case, for example, if $g''(x)$ exists and is bounded. For if $|\,g''(x)\,| \leqslant 2N$, then

$$|\,\Delta(g;\ \xi, x, x')\,| = \left| \frac{g(x)-g(\xi)}{x-\xi} - \frac{g(x')-g(\xi)}{x'-\xi} \right|$$

$$= \left| \frac{g'(\xi)(x-\xi) + \tfrac{1}{2}g''(\vartheta)(x-\xi)^2}{x-\xi} - \frac{g'(\xi)(x'-\xi) + \tfrac{1}{2}g''(\vartheta')(x'-\xi)^2}{x'-\xi} \right|$$

$$\leqslant N(|\,x-\xi\,| + |\,x'-\xi\,|).$$

From (46)

$$|\,\Delta(f_n;\ \xi;\ x, x')\,| \leqslant \sum_{\nu=0}^{n-1} a^\nu b^\nu \left| \frac{g(b^\nu x)-g(b^\nu \xi)}{b^\nu x - b^\nu \xi} - \frac{g(b^\nu x')-g(b^\nu \xi)}{b^\nu x' - b^\nu \xi} \right|$$

$$\leqslant N(|\,x-\xi\,| + |\,x'-\xi\,|) \sum_{\nu=0}^{n-1} a^\nu b^{2\nu} < N \frac{a^n b^{2n}}{ab^2-1}(|\,x-\xi\,| + |\,x'-\xi\,|),\dagger \tag{47}$$

provided that $ab^2 > 1$. Suppose that, on the other hand, a rough lower bound of $\Delta(f;\ \xi;\ x, x')$ can be found in the following form: for every ξ values x, x' with $|\,x-\xi\,| + |\,x'-\xi\,| \leqslant E$ exist such that

$$\Delta(f;\ \xi;\ x, x') \geqslant D > 0, \tag{48}$$

and, moreover,
$$\Delta(f;\ \xi; x, x') \geqslant L(|\,x-\xi\,| + |\,x'-\xi\,|), \tag{49}$$

where E, D, $L > 0$ are independent of ξ. The function $r_n(x) = a^n f(b^n x)$ will then have this property: for every ξ values x, x' exist such that

$$\Delta(r_n;\ \xi;\ x, x') \geqslant D a^n b^n, \tag{50}$$

$$\Delta(r_n;\ \xi;\ x, x') \geqslant L a^n b^{2n}(|\,x-\xi\,| + |\,x'-\xi\,|). \tag{51}$$

† See Bromwich (4), where this result is used in the case of $g(x) = \cos \pi x$.

As
$$|x-\xi|+|x'-\xi| \leqslant \frac{E}{b^n},$$
x, x' will lie in an arbitrarily small neighbourhood of ξ if n is sufficiently large. Also
$$\begin{aligned}\Delta(f;\ \xi;\ x,x') &\geqslant \Delta(r_n;\ \xi;\ x,x') - N\frac{a^n b^{2n}}{ab^2-1}(|x-\xi|+|x'-\xi|)\\ &\geqslant \Delta(r_n;\ \xi;\ x,x')\left(1-\frac{N}{L}\frac{1}{ab^2-1}\right)\\ &\geqslant Da^n b^n\left(1-\frac{N}{L}\frac{1}{ab^2-1}\right) \geqslant C > 0\end{aligned}$$
for all n, i.e. *f nowhere possesses a finite derivative if*
$$ab^2 > 1+\frac{N}{L} \tag{52}$$
and
$$ab \geqslant 1. \tag{53}$$

The constants L, D can usually be determined in the following way: consider any ξ; choose suitable points x_1, x_2, x_1', x_2' for which the values of f are known, such that
$$x_1 \leqslant \xi \leqslant x_2, \quad x_1' \leqslant \xi \leqslant x_2', \quad x_1 < x_2, \quad x_1' < x_2', \tag{54}$$
and
$$\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_2')-f(x_1')}{x_2'-x_1'} = D^* > 0; \tag{55}$$
then
$$\Delta(f;\ \xi;\ x,x') \geqslant D^*, \tag{56}$$
where x is one of x_1, x_2, x' one of x_1', x_2'. This follows as under (15) and (16). The value of $|x-\xi|+|x'-\xi|$ depends on the values of x, x' which again depend on the position of the point $P = (\xi, f(\xi))$. If, as will generally be the case, the chords used in (55) intersect at a point $S = (\sigma, \tau)$, four cases can be distinguished according to the position of P in one of the four regions shown in Fig. 1.

Choose x and x' as shown in the following table.

P in	x	x'	$\lvert x-\xi\rvert+\lvert x'-\xi\rvert$
I	x_2	x_1'	x_2-x_1'
II	x_2	x_2'	$x_2+x_2'-2\xi \leqslant x_2+x_2'-2\sigma$
III	x_1	x_2'	$x_2'-x_1$
IV	x_1	x_1'	$2\xi-(x_1+x_1') \leqslant 2\sigma-(x_1+x_1')$

If the maximum value in the last column is E^*, we have, in all cases,
$$|x-\xi|+|x'-\xi| \leqslant E^* \tag{57}$$
and
$$\frac{\Delta(f;\ \xi;\ x,x')}{|x-\xi|+|x'-\xi|} \geqslant \frac{D^*}{E^*} = L^*. \tag{58}$$

Generally, the same chords cannot be used for every ξ, and one has to choose L as the minimum of all L^*, D as the minimum of all D^*.

As an illustration, we give a proof, by our elementary method, of Hardy's result: *if* $0<a<1$ *and* b *is an integer* >1, *then Weierstrass's function* (1) *nowhere possesses a finite derivative if* $ab \geqslant 1$.†

We have
$$\left|\frac{d^2 \cos \pi x}{dx^2}\right| \leqslant \pi^2 = 2N, \quad \text{i.e.} \quad N = \tfrac{1}{2}\pi^2. \tag{59}$$

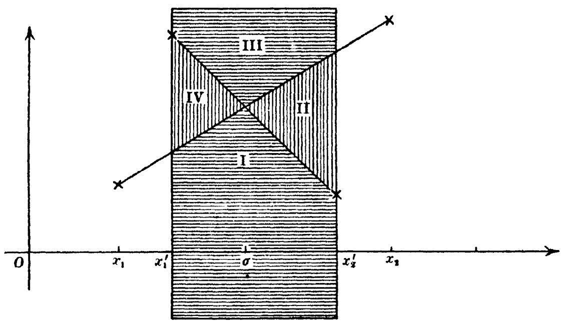

Fig. 1.

As before, it is sufficient to consider the interval $0 \leqslant \xi \leqslant 1$. Choose $x_1 = -1$, $x_2 = 2$, $x_1' = 0$, $x_2' = 1$. Then,

$$f(x_1) = \begin{cases} -a_1, \\ a_1 - 2, \end{cases} \quad f(x_2) = a_1, \quad f(x_1') = a_1, \quad f(x_2') = \begin{cases} -a_1, \\ a_1-2, \end{cases} \quad \text{if } b \text{ is } \begin{cases} \text{odd}, \\ \text{even}, \end{cases}$$

whence
$$D = D^* = \begin{cases} \frac{2}{3}a_1 + 2a_1 = \frac{8}{3}a_1, \\ \frac{2}{3} \quad + 2 \quad = \frac{8}{3}, \end{cases} \quad \text{if } b \text{ is } \begin{cases} \text{odd}, \\ \text{even}. \end{cases}$$

The two chords intersect for $\sigma = \frac{1}{2}$; thus $E^* = 2$ and

$$L = L^* = \frac{D^*}{E^*} = \begin{cases} \frac{4}{3}a_1, \\ \frac{4}{3}, \end{cases} \quad \text{if } b \text{ is } \begin{cases} \text{odd}, \\ \text{even}. \end{cases}$$

Hence (52) and (53) become

$$ab^2 > 1 + \tfrac{3}{8}\pi^2(1-a) \quad (ab \geqslant 1), \quad \text{if } b \text{ is odd,‡} \tag{60}$$

$$ab^2 > 1 + \tfrac{3}{8}\pi^2, \quad (ab \geqslant 1), \quad \text{if } b \text{ is even.‡} \tag{61}$$

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† *Loc. cit.* ((8), 304).

‡ Bromwich (4) only obtains $\frac{3}{4}\pi^2$ instead of $\frac{3}{8}\pi^2$ because he uses $|x_2 - x_1| + |x_2' - x_1'|$ instead of $|x - \xi| + |x' - \xi|$.

But for $b \geqslant 5$, $ab \geqslant 1$ implies the other condition, since

$$ab^2 = ab \,.\, b \geqslant 5 > 1 + \tfrac{3}{8} \,.\, 10 > 1 + \tfrac{3}{8}\pi^2 > 1 + \tfrac{3}{8}\pi^2(1-a).$$

Thus only the cases $b = 2, 3, 4$ remain to be considered.

$b = 3$. We have $a \geqslant \frac{1}{3}$, $\lambda = a_1 \geqslant \frac{3}{2}$. Since $f(\frac{1}{2}+x) = -f(\frac{1}{2}-x)$ we have only to study the interval $0 \leqslant \xi \leqslant \frac{1}{2}$; in order to obtain a sufficiently sharp estimate we have to divide the interval as shown in the following table.

	x_1	x_2	$f(x_1)$	$f(x_2)$	x_1'	x_2'	$f(x_1')$	$f(x_2')$	σ
$0 \leqslant \xi \leqslant \frac{1}{3}$	-1	$\frac{1}{3}$	$-\lambda$	$\frac{3}{2}-\lambda$	0	$\frac{1}{3}$	λ	$\frac{3}{2}-\lambda$	$\frac{1}{3}$
$\frac{1}{3} \leqslant \xi \leqslant \frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{2}-\lambda$	0	0	$\frac{1}{2}$	λ	0	$\frac{1}{2}$

	D^*	E^*	L^*
$0 \leqslant \xi \leqslant \frac{1}{3}$	$\frac{3}{2} \cdot \frac{3}{4} + (2\lambda - \frac{3}{2})\, 3 \geqslant \frac{9}{8} + \frac{9}{2} = \frac{45}{8}$	$\frac{5}{3}$	$\frac{27}{8}$
$\frac{1}{3} \leqslant \xi \leqslant \frac{1}{2}$	$(\lambda - \frac{3}{2})\, 6 + \lambda \,.\, 2 \geqslant 3$	$\frac{2}{3}$	$\frac{9}{2}$

Hence $D = 3 > 0$, $L = \frac{27}{8}$, and condition (52) reads

$$ab^2 > 1 + \tfrac{4}{27}\pi^2,$$

and is satisfied because $ab^2 = ab \,.\, b \geqslant 3$.

$b = 4$. Here $a \geqslant \frac{1}{4}$, $\lambda = a_1 \geqslant \frac{4}{3}$, and the interval $0 \leqslant \xi \leqslant 1$ has to be considered; we get

	x_1	x_2	$f(x_1)$	$f(x_2)$	x_1'	x_2'	$f(x_1')$	$f(x_2')$	σ
$0 \leqslant \xi \leqslant \frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$1-\frac{1}{2}\lambda$	$1-\frac{1}{2}\lambda$	0	$\frac{1}{3}$	λ	$1-\frac{1}{2}\lambda$	$\frac{1}{3}$
$\frac{1}{3} \leqslant \xi \leqslant \frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$1-\frac{1}{2}\lambda$	$\lambda-1$	$\frac{1}{3}$	$\frac{2}{3}$	$1-\frac{1}{2}\lambda$	$-\frac{1}{2}\lambda$	$\frac{1}{3}$
$\frac{1}{2} \leqslant \xi \leqslant \frac{2}{3}$	$\frac{1}{2}$	$\frac{3}{2}$	$\lambda-1$	$\lambda-1$	$\frac{1}{2}$	$\frac{2}{3}$	$\lambda-1$	$-\frac{1}{2}\lambda$	$\frac{1}{2}$
$\frac{2}{3} \leqslant \xi \leqslant 1$	$\frac{2}{3}$	1	$-\frac{1}{2}\lambda$	$\lambda-2$	$\frac{1}{2}$	1	$\lambda-1$	$\lambda-2$	1

	D^*	E^*	L^*
$0 \leqslant \xi \leqslant \frac{1}{3}$	$0 + (\frac{3}{2}\lambda - 1)\, 3 \geqslant 3$	1	3
$\frac{1}{3} \leqslant \xi \leqslant \frac{1}{2}$	$(\frac{3}{2}\lambda - 2)\, 6 + 1.3 \geqslant 3$	$\frac{1}{2}$	6
$\frac{1}{2} \leqslant \xi \leqslant \frac{2}{3}$	$0 + (\frac{3}{2}\lambda - 1)\, 6 \geqslant 6$	$\frac{7}{6}$	$\frac{36}{7}$
$\frac{2}{3} \leqslant \xi \leqslant 1$	$(\frac{3}{2}\lambda - 2)\, 3 + 1.2 \geqslant 2$	$\frac{5}{6}$	$\frac{12}{5}$

Hence $D = 2 > 0$, $L = \frac{12}{5}$, and the condition

$$ab^2 > 1 + \tfrac{5}{24}\pi^2$$

is satisfied because $ab^2 = ab \,.\, b \geqslant 4$.

$b = 2$. Here $a \geqslant \frac{1}{2}$, $\lambda = a_1 \geqslant 2$; we shall also use the inequalities

$$\tfrac{3}{2}\lambda \geqslant 2(1+a), \quad \lambda - 1 - 2a \geqslant 0.$$

We get

	x_1	x_2	$f(x_1)$	$f(x_2)$	x_1'	x_2'	$f(x_1')$	$f(x_2')$	σ
$0\leqslant\xi\leqslant\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$1-\frac{1}{2}\lambda$	$1-\frac{1}{2}\lambda$	0	$\frac{1}{3}$	λ	$1-\frac{1}{2}\lambda$	$\frac{1}{3}$
$\frac{1}{3}\leqslant\xi\leqslant\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$1-\frac{1}{2}\lambda$	$\lambda-1-2a$	$\frac{1}{3}$	$\frac{2}{3}$	$1-\frac{1}{2}\lambda$	$-\frac{1}{2}\lambda$	$\frac{1}{3}$
$\frac{1}{2}\leqslant\xi\leqslant\frac{2}{3}$	$\frac{1}{2}$	1	$\lambda-1-2a$	$\lambda-2$	$\frac{1}{2}$	$\frac{2}{3}$	$\lambda-1-2a$	$-\frac{1}{2}\lambda$	$\frac{1}{2}$
$\frac{2}{3}\leqslant\xi\leqslant 1, f(\xi)\geqslant-\frac{1}{3}$	$\frac{2}{3}$	ξ	$-\frac{1}{2}\lambda$	$f(\xi)$	ξ	$\frac{4}{3}$	$f(\xi)$	$-\frac{1}{2}\lambda$	ξ
$\frac{2}{3}\leqslant\xi\leqslant 1, f(\xi)\leqslant-\frac{1}{3}$	ξ	1	$f(\xi)$	$\lambda-2$	$\frac{1}{2}$	ξ	$\lambda-1-2a$	$f(\xi)$	ξ

	D^*	E^*	L^*
$0\leqslant\xi\leqslant\frac{1}{3}$	$0+(\frac{3}{2}\lambda-1)\,3\geqslant 6$	1	6
$\frac{1}{3}\leqslant\xi\leqslant\frac{1}{2}$	$(\frac{3}{2}\lambda-2(1+a))\,6+1.3\geqslant 3$	$\frac{1}{2}$	6
$\frac{1}{2}\leqslant\xi\leqslant\frac{2}{3}$	$(2a-1)\,2+(\frac{3}{2}\lambda-2(1+a)+1)\,6\geqslant 6$	$\frac{2}{3}$	9
$\frac{2}{3}\leqslant\xi\leqslant 1, f(\xi)\geqslant-\frac{1}{3}$	$\dfrac{f(\xi)+\frac{1}{2}\lambda}{\xi-\frac{2}{3}}+\dfrac{f(\xi)+\frac{1}{2}\lambda}{\frac{4}{3}-\xi}\geqslant\frac{2}{3}.6=4$	$\frac{2}{3}$	6
$\frac{2}{3}\leqslant\xi\leqslant 1, f(\xi)\leqslant-\frac{1}{3}$	$\dfrac{\lambda-2-f(\xi)}{1-\xi}+\dfrac{\lambda-1-2a-f(\xi)}{\xi-\frac{1}{2}}\geqslant\dfrac{1}{3}\left(\dfrac{1}{1-\xi}+\dfrac{1}{\xi-\frac{1}{2}}\right)\geqslant\frac{8}{3}$	$\frac{1}{2}$	$\frac{16}{3}$

Hence $D=\frac{8}{3}>0$, $L=\frac{16}{3}$, and the condition

$$ab^2>1+\tfrac{3}{32}\pi^2$$

is satisfied because $ab^2=ab.b\geqslant 2$. This proves the statement.

Now let b be arbitrary, $0<\epsilon<\frac{1}{2}$, $b\geqslant 1+\epsilon^{-1}$. We use the special points x_k constructed above; let $P_k=(x_k, f(x_k))$, $P=(\xi, f(\xi))$; then

$$|k-x_k|\leqslant\epsilon,\quad f(x_{2m})\geqslant\delta a_1,\quad f(x_{2m+1})\leqslant-\delta a_1, \tag{62}$$

$$x_{2m}<x_{2m+1}<x_{2m+2}, \tag{63}$$

for all integral k, m. Suppose that ξ is in the interval $x_{2m}\leqslant\xi\leqslant x_{2m+1}$ (the case $x_{2m+1}\leqslant\xi\leqslant x_{2m+2}$ is treated similarly). Let

$$Q_{2m-1}=(2m-1-\epsilon,\,-\delta a_1),\quad Q_{2m}=(2m-\epsilon,\,\delta a_1),$$
$$Q_{2m+1}=(2m+1+\epsilon,\,-\delta a_1),\quad Q_{2m+2}=(2m+2+\epsilon,\,\delta a_1).$$

$Q_{2m}Q_{2m+1}$, $Q_{2m-1}Q_{2m+1}$ meet at $R=(2m+\frac{1}{2},0)$; P lies in one of the regions I, II, III and IV shown in Fig. 2. If P lies in I, we have

$$\Delta(f;\,\xi;\,x_{2m+2},x_{2m})\geqslant\text{gradient } Q_{2m+2}R-\text{gradient } Q_{2m}R$$

$$=\frac{\delta}{(1-a)(\frac{3}{2}+\epsilon)}+\frac{\delta}{(1-a)(\frac{1}{2}+\epsilon)}=\frac{8\delta(1+\epsilon)}{(1-a)(3+2\epsilon)(1+2\epsilon)}=D \tag{64}$$

and $$|x_{2m+2}-\xi|+|x_{2m}-\xi|=x_{2m+2}-x_{2m}\leqslant 2(1+\epsilon)=E; \tag{65}$$

thus $$L=\frac{4\delta}{(3+2\epsilon)(1+2\epsilon)(1-a)}. \tag{66}$$

If P lies in II, we have

$$\Delta(f;\, \xi;\, x_{2m+2}, x_{2m+1}) \geqslant \text{gradient } Q_{2m+2}R - \text{gradient } Q_{2m+1}R = D \tag{67}$$

and
$$|x_{2m+2}-\xi| + |x_{2m+1}-\xi| \leqslant (2m+2+\epsilon)-(2m+\tfrac{1}{2}) + (2m+1+\epsilon)-(2m+\tfrac{1}{2}) = E, \tag{68}$$

and the same values are obtained for III and IV. Thus the value (66) of L holds for all ξ, and the conditions for the non-existence of a finite differential coefficient become

$$ab^2 > 1 + \frac{(3+2\epsilon)(1+2\epsilon)}{8\cos\pi\epsilon}\pi^2(1-a), \quad ab \geqslant 1, 0<\epsilon<\tfrac{1}{2}, b \geqslant 1+\epsilon^{-1}. \tag{69}$$

Satisfying the last condition by putting $b = 1+\epsilon^{-1}$, i.e. $\epsilon = b_1$, we find that *Weierstrass's function* (1) *nowhere possesses a finite differential coefficient if*

$$ab^2 > 1 + \frac{(3+2b_1)(1+2b_1)}{8\cos\pi b_1}\pi^2(1-a), \quad ab \geqslant 1, b \geqslant 3. \tag{70}$$

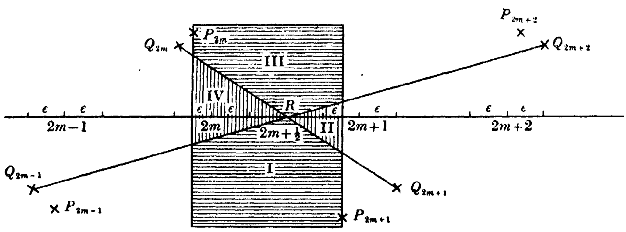

Fig. 2.

This is stronger and less restricted than Dini's (*loc. cit.* (**3**))

$$ab^2 > 1 + 15\pi^2\frac{1-a}{5-21a}, \quad ab \geqslant 1, a < \tfrac{5}{21},$$

but considerably weaker than Hardy's $ab \geqslant 1$ (*loc. cit.* (**5**)). Hardy's result can only be established in this way for $b \geqslant b_0$ (e.g. $b \geqslant \frac{20}{3}$). The conditions

$$ab^2 > 1 + \kappa(1-a), \quad ab \geqslant 1 \tag{71}$$

are certainly satisfied if
$$b > \kappa, \quad ab \geqslant 1; \tag{72}$$

for the first inequality in (71) is equivalent to $a > (1+\kappa)/(b^2+\kappa)$, and, since $a \geqslant 1/b$, this is satisfied if

$$\frac{1}{b} > \frac{1+\kappa}{b^2+\kappa} \quad \text{i.e.} \quad \frac{(b-1)(b-\kappa)}{b(b^2+\kappa)} > 0 \quad \text{i.e.} \quad b > \kappa.$$

Hence (70) will be satisfied if

$$b > \frac{(3+2b_1)(1+2b_1)}{8\cos \pi b_1}\pi^2, \quad ab \geqslant 1, \tag{73}$$

which imply $b \geqslant 3$; and it is easily verified that (73) holds if

$$b \geqslant \tfrac{20}{3}, \quad ab \geqslant 1. \tag{74}$$

The same procedure obviously applies to more general functions $g(x)$ and the result will be:

THEOREM II. *If $g(x)$ is defined for all x, continuous, bounded and "pseudo-periodic", and satisfies a "Lipschitz condition of the second order"† then*

$$f(x) = \sum_{\nu=0}^{\infty} a^\nu g(b^\nu x) \quad (0 < a < 1),$$

nowhere possesses a finite derivative if $ab \geqslant 1$ and b is sufficiently large.‡

It should be noted that this theorem does not apply to functions $g(x)$ which are piecewise linear (polygonal), since such functions do not satisfy a Lipschitz condition of the second order. However, we can prove

THEOREM III. *If $g(x)$ is piecewise linear (polygonal) but not constant, and periodic with period 1, with a finite number of vertices in the interval $0 < x \leqslant 1$ all of which have rational abscissae, then*

$$f(x) = \sum_{\nu=0}^{\infty} a^\nu g(b^\nu x)$$

is a continuous function without finite derivative at any point for all a, b satisfying the conditions

$$0 < a < 1, \quad b \text{ an integer}, \quad ab \geqslant 1,$$

with the possible exception of a finite number of cases in which $f(x)$ is itself a polygon.

Proof. (1) LEMMA. *If $f(\xi)$ is continuous for $\alpha \leqslant \xi \leqslant \beta$, and not linear in this interval, then there is a constant $D > 0$ such that*

$$\Delta(f;\, \xi;\, x, x') \geqslant D \tag{75}$$

for all ξ and suitably chosen x, x' in the interval.

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† Which implies a Lipschitz condition of the first order needed for the construction of special points.

‡ It may be added that the non-existence of a finite right- and left-hand derivative for large b and $ab \geqslant 1$ can be established in the same manner; one has only to choose x, x' on the same side of ξ which will result in a smaller value of L, i.e. in a less favourable estimate for b.

For choose γ so that the points $(\alpha, f(\alpha))$, $(\beta, f(\beta))$, $(\gamma, f(\gamma))$ are not collinear; then

$$\Delta(f;\, \xi;\, x, x') \geqslant \left| \frac{f(\alpha)-f(\gamma)}{\alpha-\gamma} - \frac{f(\alpha)-f(\beta)}{\alpha-\beta} \right| = D_1 > 0 \quad (\alpha \leqq \xi \leqq \gamma),$$

$$\Delta(f;\, \xi;\, x, x') \geqslant \left| \frac{f(\beta)-f(\gamma)}{\beta-\gamma} - \frac{f(\alpha)-f(\beta)}{\alpha-\beta} \right| = D_2 > 0 \quad (\gamma \leqq \xi \leqq \beta),$$

where x, x' are suitably chosen from α, β, γ; then $D = \min(D_1, D_2)$ will satisfy the requirement.

(2) $f(x)$ is continuous, with period 1; write, as above,

$$f(x) = f_n(x) + r_n(x), \tag{76}$$

where

$$f_n(x) = \sum_{\nu=0}^{n-1} a^\nu g(b^\nu x) \tag{77}$$

is a polygon, and

$$r_n(x) = a^n f(b^n x). \tag{78}$$

(3) Suppose that $f(x)$ is linear in some interval $\alpha \leqslant x \leqslant \beta$, $f(x) = l(x)$, say. Choose n so that $b^n(\beta-\alpha) \geqslant 1$; then

$$a^n f(b^n x) = r_n(x) = f(x) - f_n(x) = l(x) - f_n(x)$$

will be a polygon for $\alpha \leqslant x \leqslant \beta$, or, with $b^n x = X$, $f(X)$ will be a polygon for $b^n\alpha \leqslant X \leqslant b^n\beta$, i.e. in an interval of length $b^n(\beta-\alpha) \geqslant 1$; but since f has the period 1 it will be polygonal throughout. Assuming, then, that f is not a polygon, f will not be linear in any interval.

(4) Now let x_ρ $(\rho = 1, 2, \ldots, r)$ be the abscissae of the vertices of $g(x)$ in $0 < x \leqslant 1$, and

$$0 < x_1 < x_2 < \ldots < x_r \leqslant 1. \tag{79}$$

They may be written in the form

$$x_\rho = p_\rho / q \quad (\rho = 1, 2, \ldots, r), \tag{80}$$

where p_ρ, q are integers, q a common denominator. Then

$$f_n(x) = \sum_{\nu=0}^{n-1} a^\nu g(b^\nu x)$$

will be a polygon, and the abscissae of its vertices will be of the form m/qb^ν and can all be written with the common denominator qb^{n-1}; hence $f_n(\xi)$ will be linear in any interval of the form

$$k/(qb^{n-1}) \leqq \xi \leqq (k+1)/(qb^{n-1}), \tag{81}$$

and

$$\Delta(f_n;\, \xi;\, x, x') = 0 \tag{82}$$

for any ξ, x, x' in this interval.

(5) If $f(x)$ is not a polygon it will, by (1) and (3), possess, for every one of the q intervals

$$m/q \leqq \xi \leqq (m+b)/q \quad (m = 0, 1, \ldots, q-1) \tag{83}$$

a constant $D_m > 0$, satisfying the lemma; $D = \min(D_0, D_1, \ldots, D_{q-1})$ can be used simultaneously in all q intervals, and, because of the periodicity of f, for all intervals (83) with arbitrary integral m. Thus

$$\Delta(f;\, \xi;\, x, x') \geqslant D > 0 \tag{84}$$

for any ξ and suitable x, x' in (83).

Passing from $f(x)$ to $r_n(x) = a^n f(b^n x)$, we see that, if ξ is in an interval of the form

$$m/(qb^n) \leqq \xi \leqq (m+b)/(qb^n), \tag{85}$$

then

$$\Delta(r_n;\, \xi;\, x, x') \geqslant a^n b^n D \geqslant D \tag{86}$$

for suitable x, x' in (85). Now, for every n, ξ will be in an interval (81), and (81) is of the form (85); hence (86) can be satisfied within (81), and at the same time (82) will hold, so that

$$\Delta(f;\, \xi;\, x, x') = \Delta(r_n;\, \xi;\, x, x') \geqslant D \tag{87}$$

for suitable x, x' in an arbitrarily small neighbourhood of ξ, which proves that f nowhere possesses a finite derivative.

(6) Assume now that, for some value of a, b, $f(x)$ is a polygon; let t_σ ($\sigma = 1, 2, \ldots, s$) be the abscissae of its vertices in $0 < x \leqslant 1$. We may assume that

$$g(0) = 0.\dagger \tag{88}$$

Then

$$f(0) = 0 \tag{89}$$

and

$$g(x) = hx \quad (0 \leqslant x \leqslant x_1), \tag{90}$$

$$f(x) = Hx \quad (0 \leqslant x \leqslant t_1). \tag{91}$$

Now

$$g(x) = f(x) - af(bx). \tag{92}$$

The first vertex x_1 of $g(x)$ must coincide with the first vertex t_1/b of the right-hand side; hence

$$b = t_1/x_1 < 1/x_1. \tag{93}$$

This leaves only a finite number of values for b; for any one of these values we have, by (90), (91) and (92)

$$hx = Hx - aHbx \tag{94}$$

for $0 \leqslant x \leqslant x_1$, whence

$$h = H(1-ab). \tag{95}$$

† If $g(0) \neq 0$, consider $G(x) = g(x) - g(0)$; then $F(x) = \sum_{\nu=0}^{\infty} a^\nu G(b^\nu x) = f(x) - f(0)$.

Assuming that $ab > 1$ (which excludes at most the finite number of cases $a = b^{-1}$, which can only occur when $h = 0$), we have

$$H = \frac{h}{1-ab}. \tag{96}$$

As $g(x) \not\equiv 0$,
$$g(k/b^n) \neq 0 \tag{97}$$

for some integral k, n. Choose the smallest n for which there is a k satisfying (97): then $n \geqslant 1$ since, by (88), $g(k/b^n) = 0$ for $n \leqslant 0$. Further, choose the smallest positive k corresponding to this n; and let m be the smallest integer $\geqslant n$ for which

$$k/b^m \leqslant t_1. \tag{98}$$

Then
$$f\left(\frac{k}{b^m}\right) = H\frac{k}{b^m} = \frac{hk}{b^m(1-ab)} = \sum_{\nu=0}^{\infty} a^\nu g\left(\frac{k}{b^m}b^\nu\right) = \sum_{\nu=0}^{m-n} a^\nu g\left(\frac{k}{b^{m-\nu}}\right), \tag{99}$$

or
$$(1-ab)\sum_{\nu=0}^{m-n} a^\nu g\left(\frac{k}{b^{m-\nu}}\right) - \frac{hk}{b^m} = 0. \tag{100}$$

(100) is not satisfied identically because the coefficient of a^{m-n+1}, namely $-bg(k/b^n)$, is not 0; hence there are at most $m-n+1$ values of a satisfying (100). Adding the possible value $a = 1/b$, we see that for each b there are at most $m-n+2$ values of a for which $f(x)$ can become a polygon.

This estimate is rather crude and could certainly be improved by finer considerations. But it should be noted that the occurrence of exceptional values, for which $f(x)$ is polygonal, cannot be ruled out altogether; in fact, every $f(x)$ can be represented as $\Sigma a^\nu g(b^\nu x)$ for any given a, b; one has simply to define $g(x)$ by (92), and if $f(x)$ is polygonal, $g(x)$ automatically becomes polygonal. But in certain cases the existence of exceptional values can easily be disproved. If, for example, $g(x) \geqslant 0$, $g(0) = 0$, and $h > 0$, the case $a = 1/b$ cannot occur,† and (99) cannot be satisfied by any a between 0 and 1 because its two sides are of different sign.

As examples consider:

(1) $g(x) = \psi(x)$;‡ we have $\psi(0) = 0$, $h = 1 > 0$, and $\psi(x) \geqslant 0$. Hence

$$f(x) = \sum_{\nu=0}^{\infty} a^\nu \psi(b^\nu x)$$

nowhere possesses a finite derivative for integral $b > 1$ and $ab \geqslant 1$. This result includes the examples given by van der Waerden (**9**), Landau (**12**) and Hildebrandt (**11**). The same result holds for $\phi(x)$.‡

† See the remark after (95).

‡ As defined in the first footnote of the paper.

(2) $g(x) = \chi(x)$;† consider the corresponding function with period 1:

$$g(x) = \tfrac{1}{2}\chi(2x) = \begin{cases} x & \text{for } 0 \leqslant x \leqslant \frac{1}{4}, \\ \frac{1}{2}-x & \text{for } \frac{1}{4} \leqslant x \leqslant \frac{3}{4}, \\ -1+x & \text{for } \frac{3}{4} \leqslant x \leqslant 1, \end{cases} \quad g(x+1) = g(x).$$

Here $x_1 = \frac{1}{4}$; hence, for exceptional b, $b < 4$, $g(0) = 0$, $h = 1$, and hence $ab > 1$.

Now for $b = 2$, $g(\frac{1}{2}k) = 0$, $g(\frac{1}{4}) = \frac{1}{4} \neq 0$, and $\frac{1}{4} < \frac{1}{2} = 2x_1 = t_1$; and hence, with $k = 1$, $m = n = 2$, (100) becomes

$$(1-ab)\,g(\tfrac{1}{4}) - \tfrac{1}{4} = (1-ab)\,\tfrac{1}{4} - \tfrac{1}{4} = 0$$

with the only solution $a = 0$. And for $b = 3$, $g(\frac{1}{3}) = \frac{1}{6} \neq 0$, $\frac{1}{3} < \frac{3}{4} = 3x_1 = t_1$. Now if $k = 1$, $m = n = 1$, (100) becomes

$$(1-ab)\,g(\tfrac{1}{3}) - \tfrac{1}{3} = (1-3a)\,\tfrac{1}{6} - \tfrac{1}{3} = 0,$$

with the only solution $a = -\frac{1}{3}$. Hence

$$f(x) = \sum_{\nu=0}^{\infty} a^{\nu}\chi(b^{\nu}x)$$

nowhere possesses a finite derivative for integral $b > 1$, and $ab \geqslant 1$.

References

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The following is a short list (not claiming completeness) of references relevant to the present paper:

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Department of Mathematics
The University of Melbourne

† As defined in the first footnote of the paper.

NOTE ON CONVERGENCE AND SUMMABILITY FACTORS (III)

By L. S. BOSANQUET

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1. *Introduction*

In recent notes* I have discussed the problem of finding necessary and sufficient conditions to be satisfied by a sequence $\{\epsilon_n\}$ in order that the series† $\Sigma a_n \epsilon_n$ may be summable (C, β) whenever Σa_n is summable (C, α). In the case where α and β are integers, necessary and sufficient conditions have been formulated without proof by Schur,‡ and in (**7**) and (**8**) I have given proofs of Schur's theorem, and an extension to series whose partial Cesàro means are $o(n^p)$, where p is any real number. In the case where $\alpha = \beta = \kappa$, and κ is an integer, the sufficiency part of the theorem is the Bohr-Hardy theorem, found independently by Bohr§ and Hardy,|| and the necessity part is a result of Fekete.¶ In the case where $\beta = 0$ and α is an integer the sufficiency part is due to Bromwich,** and the necessity to Kojima.†† Bromwich's theorem was extended by Chapman‡‡ to the case where α is fractional, and Andersen§§ extended the Bohr-Hardy theorem similarly, stating the corresponding result with a parameter $p \geqslant 0$. I have completed Andersen's result, in necessary and sufficient form, in (**6**). Here I give a solution of the general problem, where α and β may be fractional. The form of the conditions is the same as in the integral case, but the presence of a fractional difference gives the result a different character. It is convenient at the same time to consider the case with a parameter $p \geqslant 0$, but the case where $p < 0$ is not discussed, nor is the case where α or β is negative.

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* Bosanquet (**6, 7, 8**).

† Here and elsewhere Σ denotes $\sum_0^\infty$.

‡ Schur (**14**).

§ Bohr (**3, 4**).

|| Hardy (**12**).

¶ Fekete (**11**).

** Bromwich (**9**).

†† Kojima (**13**).

‡‡ Chapman (**10**).

§§ Andersen (**1, 2**).

2. *Notation*

We write $$S_n^\alpha(s_\nu) = S_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu \quad (\alpha \text{ real}), \tag{1}$$

where A_n^α is given by the identity

$$(1-x)^{-\alpha-1} = \Sigma A_n^\alpha x^n \quad (|x|<1). \tag{2}$$

Then it is familiar that*

$$S_n^{\alpha+\beta} = S_n^\alpha(S_\nu^\beta) \quad (\alpha, \beta \text{ real}), \tag{3}$$

$$A_\alpha^{\alpha+\beta} = S_n^\alpha(A_\nu^\beta), \tag{4}$$

$$A_0^\alpha = 1, \quad A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{1.2.\dots.n} \quad (n \geqslant 1), \tag{5}$$

$$A_n^\alpha \sim \frac{n^\alpha}{\Gamma(\alpha+1)} \quad (\alpha \neq -1, -2, \dots). \tag{6}$$

We also write $$\Delta^\alpha \epsilon_n = \sum_{\nu=n}^{\infty} A_{\nu-n}^{-\alpha-1} \epsilon_\nu \tag{7}$$

whenever this series converges.† In particular

$$\Delta^1 \epsilon_n = \Delta \epsilon_n = \epsilon_n - \epsilon_{n+1}, \tag{8}$$

and, if $\epsilon_n = O(1)$, $\Delta^\alpha \epsilon_n$ exists for every $\alpha \geqslant 0$.

When discussing a series Σa_n we shall always take

$$s_n = S_n^1(a_\nu) = \sum_{\nu=0}^{n} a_\nu, \tag{9}$$

so that $$a_n = S_n^{-1}(s_\nu) = \begin{cases} s_n - s_{n-1} & (n \geqslant 1), \\ s_0 & (n = 0). \end{cases} \tag{10}$$

A series Σa_n can always be chosen so that S_n^α is any given sequence. For, by (3), $S_n^\alpha = u_n$ if $a_n = S_n^{-\alpha-1}(u_\nu)$.

3. *Preliminary lemmas*

We require the following lemmas.

LEMMA 1. *If $\alpha+p > -1$, $\delta > 0$ and $S_n^\alpha = o(n^{\alpha+p})$, then $S_n^{\alpha+\delta} = o(n^{\alpha+p+\delta})$. The same holds with O in place of o.*

---

* It is to be understood here and elsewhere that $S_n^\alpha(u_\nu) = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} u_\nu$, where $\{u_\nu\}$ may be any sequence, while S_n^α stands only for $S_n^\alpha(s_\nu)$.

† This implies $\epsilon_n = o(n^{\alpha+1})$, by (6), unless $\alpha = 0, 1, \dots$.

By (3) and (6)

$$S_n^{\alpha+\delta} = \sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1} S_\nu^\alpha = \sum_{\nu=0}^{n} O\{(n-\nu+1)^{\delta-1}\} o(\nu^{\alpha+p}) = o(n^{\alpha+p+\delta}).$$

LEMMA 2. *If $\alpha+p > -1$, $\delta > 0$ and $S_n^\alpha = o(n^{\alpha+p})$, then $S_n^{\alpha-\delta} = o(n^{\alpha+p})$.*

By (3) and (6)

$$S_n^{\alpha-\delta} = \sum_{\nu=0}^{n} A_{n-\nu}^{-\delta-1} S_\nu^\alpha = \sum_{\nu=0}^{n} O\{(n-\nu+1)^{-\delta-1}\} o(\nu^{\alpha+p}) = o(n^{\alpha+p}).$$

LEMMA 3. *If $\epsilon_n = O(1)$, then*

$$\Delta^{\alpha+\beta}\epsilon_n = \Delta^\beta(\Delta^\alpha \epsilon_n) \tag{11}$$

for $\alpha \geqslant 0$, $\beta > -1$, $\alpha+\beta > 0$.

If $\epsilon_n = o(1)$ the last two conditions may be replaced by $\beta \geqslant -1$, $\alpha+\beta \geqslant 0$.

This is due to Andersen.*

LEMMA 4. *If $\alpha \geqslant 0$, $p \geqslant 0$ and*

$$\text{(i) } \epsilon_n = O(n^{-p}), \quad \text{(ii) } \Sigma n^{\alpha+p} \,|\, \Delta^{\alpha+1}\epsilon_n \,| < \infty, \tag{12}$$

then

$$(a) \qquad \Sigma A_n^{\beta+p} \,|\, \Delta^{\beta+1}\epsilon_n \,| < \infty \tag{13}$$

for $-1 \leqslant \beta < \alpha$,

$$(b) \qquad \Delta^\beta \epsilon_n = o(n^{-\beta-p}) \tag{14}$$

for $0 < \beta \leqslant \alpha$.

This is due to Andersen.†

LEMMA 5.‡ *If $0 \leqslant \delta \leqslant 1$ and $0 \leqslant m < n$, then*

$$\left| \sum_{\nu=0}^{m} A_{n-\nu}^{\delta-1} s_\nu \right| \leqslant \max_{0 \leqslant \mu \leqslant m} |\, S_\mu^\delta \,|. \tag{15}$$

The result being trivial when $\delta = 0$ or 1, we suppose that $0 < \delta < 1$. By repeated applications of Abel's Lemma,§ since

$$\frac{A_{n-\nu}^{\delta-1}}{A_{m-\nu}^{\delta-1}} = \frac{\delta-1+n-\nu}{n-\nu} \, \frac{m-\nu}{\delta-1+m-\nu} \, \frac{A_{n-\nu-1}^{\delta-1}}{A_{m-\nu-1}^{\delta-1}} > \frac{A_{n-\nu-1}^{\delta-1}}{A_{m-\nu-1}^{\delta-1}}$$

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* Andersen (**1**, **2**). A proof is given in Bosanquet (**6**).

† Andersen (**1**, **2**). A proof is given in Bosanquet (**6**). In (13) $A_n^{\beta+p}$ may be replaced by $(n+1)^{\beta+p}$ by (6), except when $\beta = -1$, $p = 0$, as is shown by the example $\epsilon_n = \{\log(n+2)\}^{-1}$.

‡ Bosanquet (**5**). Here we give an alternative proof.

§ If $\max_{0 \leqslant \nu \leqslant m} |\, a_0 + a_1 + \ldots + a_\nu \,| = M_m$ and $d_0 \geqslant d_1 \geqslant \ldots \geqslant d_m \geqslant 0$, then

$$|\, d_0 a_0 + d_1 a_1 + \ldots + d_m a_m \,| \leqslant d_0 M_m.$$

for $0 \leqslant \nu < m < n$, there exist integers m_ν such that $m \geqslant m_1 \geqslant m_2 \geqslant \ldots \geqslant 0$, and

$$\begin{aligned}\left|\sum_{\nu=0}^{m} A_{n-\nu}^{\delta-1} s_\nu\right| &= \left|\sum_{\nu=0}^{m} \frac{A_{n-\nu}^{\delta-1}}{A_{m-\nu}^{\delta-1}} A_{m-\nu}^{\delta-1} s_\nu\right| \\ &\leqslant \frac{A_n^{\delta-1}}{A_m^{\delta-1}}\left|\sum_{\nu=0}^{m_1} A_{m-\nu}^{\delta-1} s_\nu\right| \\ &\leqslant \frac{A_n^{\delta-1}}{A_m^{\delta-1}} \frac{A_m^{\delta-1}}{A_{m_1}^{\delta-1}}\left|\sum_{\nu=0}^{m_2} A_{m_1-\nu}^{\delta-1} s_\nu\right| \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\leqslant \frac{A_n^{\delta-1}}{A_{m_k}^{\delta-1}}\left|\sum_{\nu=0}^{m_{k+1}} A_{m_k-\nu}^{\delta-1} s_\nu\right|.\end{aligned}$$

Now since $m_1, m_2, \ldots$ is a non-increasing sequence of non-negative integers, there is an integer ρ such that $m_\rho = m_{\rho+1}$. Therefore, since

$$0 \leqslant m_\rho \leqslant m < n,$$

$$\begin{aligned}\left|\sum_{\nu=0}^{m} A_{n-\nu}^{\delta-1} s_\nu\right| &\leqslant \frac{A_n^{\delta-1}}{A_{m_\rho}^{\delta-1}}\left|\sum_{\nu=0}^{m_\rho} A_{m_\rho-\nu}^{\delta-1} s_\nu\right| \\ &\leqslant |S_{m_\rho}^\delta| \\ &\leqslant \max_{0\leqslant\mu\leqslant m} |S_\mu^\delta|.*\end{aligned}$$

LEMMA 6. *If $\{d_{m,n}\}$ is a given double sequence, then necessary and sufficient conditions that the series $\sum_{m=0}^{\infty} s_m d_{m,n}$ should converge and its sum tend to a limit as $n \to \infty$, whenever $s_n = o(1)$, are that*

(i) $\sum_{m=0}^{\infty} |d_{m,n}| \leqslant M$, *where M is independent of n, and*

(ii) $\lim_{n\to\infty} d_{m,n}$ *should exist for* $m = 0, 1, \ldots$.

If o is replaced by O, then condition (i) *must be replaced by the condition that*

(iii) $\sum_{m=0}^{\infty} |d_{m,n}|$ *should converge uniformly with respect to n.*

This is due essentially to Schur.†

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* Incidentally we note that, if $0 < \delta < 1$, then $A_n^{\delta-1}/A_{m_\rho}^{\delta-1} < 1$, and hence there is strict inequality in (15) unless $S_\mu^\delta = 0$ for $0 \leqslant \mu \leqslant m$, i.e. unless $s_0 = s_1 = \ldots = s_m = 0$.

† Schur (14). Instead of the class of *nul sequences* $\{s_n\}$ Schur considered the class of *convergent sequences* $\{s_n\}$. In the latter case a further condition must be added to (i) and (ii), namely, (iv) *that* $\lim_{n\to\infty} \sum_{m=0}^{\infty} d_{m,n}$ *should exist.* I am indebted to Mr C. A. Rogers for pointing out that this condition should not be included in the lemma as given here. In his proofs that (i) and (ii) are necessary Schur used nul sequences.

4. *The main lemma*

Before giving the main result we prove one more lemma.

LEMMA 7. *If* $0 \leqslant \beta \leqslant \alpha$, $p \geqslant 0$, $q \geqslant 0$, $q < p+1$ *and*

(i) $$S_n^\alpha = o(n^{\alpha+p}), \tag{16}$$

(ii) $$\epsilon_n = O(n^{\beta-\alpha-q}), \tag{17}$$

(iii) $$\sum n^{\alpha+q} \,|\, \Delta^{\alpha+1}\epsilon_n \,| < \infty, \tag{18}$$

then $$s_n \epsilon_n = o(n^{p-q}) \quad (C, \beta). \tag{19}$$

*The conclusion also holds if o and O are interchanged in the hypotheses.**

Proof. We consider only the first of the alternative sets of hypotheses. The proof in the other case is similar.

Case 1. *Suppose that* $\beta = 0$. Then by (16), (17) and lemma 2,

$$s_n = S_n^0 = o(n^{\alpha+p}) \quad \text{and} \quad \epsilon_n = O(n^{-\alpha-q}).$$

Hence $$s_n \epsilon_n = o(n^{\alpha+p})\, O(n^{-\alpha-q}) = o(n^{p-q}).$$

Case 2. *Suppose that* $0 < \beta \leqslant \alpha < 1$. Then by (1) and (3), since

$$s_n = S_n^0 = S_n^{-\alpha}(S_\nu^\alpha),$$

$$\begin{aligned} S_n^\beta(s_\nu \epsilon_\nu) &= \sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1} s_\nu \epsilon_\nu \\ &= \sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1} \epsilon_\nu \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{-\alpha-1} S_\mu^\alpha \\ &= \sum_{\mu=0}^{n} S_\mu^\alpha \sum_{\nu=\mu}^{n} A_{n-\nu}^{\beta-1} A_{\nu-\mu}^{-\alpha-1} \epsilon_\nu \\ &= \sum_{\mu=0}^{n} S_\mu^\alpha I, \end{aligned}$$

where, by partial summation,

$$\begin{aligned} I &= \sum_{\nu=\mu}^{n} A_{n-\nu}^{\beta-1} A_{\nu-\mu}^{-\alpha-1} \epsilon_\nu \\ &= \epsilon_{n+1} \sum_{\nu=\mu}^{n} A_{n-\nu}^{\beta-1} A_{\nu-\mu}^{-\alpha-1} + \sum_{\nu=\mu}^{n} \Delta\epsilon_\nu \sum_{\rho=\mu}^{\nu} A_{n-\rho}^{\beta-1} A_{\rho-\mu}^{-\alpha-1} \\ &= I_1 + I_2. \end{aligned}$$

---

* In the case where α and β are both integers this result has been proved in (8), with condition (iii) replaced by the weaker condition

$$\text{(iii)}' \quad \Delta^\alpha \epsilon_n = O(n^{-\alpha-q}),$$

and with the conditions $p \geqslant 0$, $q \geqslant 0$ removed.

Now, by (4),

$$I_1 = \epsilon_{n+1} \sum_{\rho=0}^{n-\mu} A_{n-\mu-\rho}^{\beta-1} A_{\rho}^{-\alpha-1} = \epsilon_{n+1} A_{n-\mu}^{\beta-\alpha-1}. \tag{20}$$

Hence, if $\beta = \alpha$,

$$I_1 = \begin{cases} \epsilon_{n+1} & (\mu = n), \\ 0 & (0 \leqslant \mu \leqslant n-1), \end{cases}$$

and

$$\begin{aligned} \sum_{\mu=0}^{n} S_\mu^\alpha I_1 &= S_n^\alpha \epsilon_{n+1} = o(n^{\alpha+p})\, O(n^{-q}) \\ &= o(n^{\alpha+p-q}) = o(n^{\beta+p-q}). \end{aligned}$$

If $0 < \beta < \alpha$, by (17), (6) and (20),

$$I_1 = O(n^{\beta-\alpha-q})\, O\{(n-\mu+1)^{\beta-\alpha-1}\},$$

and, since $\alpha + p \geqslant 0$,

$$\begin{aligned} \sum_{\mu=0}^{n} S_\mu^\alpha I_1 &= O(n^{\beta-\alpha-q}) \sum_{\mu=0}^{n} o(\mu^{\alpha+p})\, O\{(n-\mu+1)^{\beta-\alpha-1}\} \\ &= O(n^{\beta-\alpha-q})\, o(n^{\alpha+p}) \sum_{\rho=1}^{\infty} \rho^{\beta-\alpha-1} \\ &= o(n^{\beta+p-q}). \end{aligned}$$

Next

$$I_2 = \sum_{\nu=\mu}^{n} \Delta\epsilon_\nu J,$$

where

$$J = \sum_{\rho=\mu}^{\nu} A_{n-\rho}^{\beta-1} A_{\rho-\mu}^{-\alpha-1}. \tag{21}$$

We show now that, for $0 \leqslant \mu \leqslant \nu \leqslant n$,

$$|J| \leqslant K A_{n-\mu}^{\beta-1} A_{\nu-\mu}^{-\alpha}, \tag{22}$$

where K is independent of μ, ν, n.

We first observe that $A_0^{-\alpha-1} = 1 > 0$ and, since $0 < \alpha < 1$,

$$A_\mu^{-\alpha-1} = \frac{-\alpha(-\alpha+1)\ldots(-\alpha+\mu-1)}{1.2.\ldots.\mu} < 0 \quad (\mu \geqslant 1). \tag{23}$$

Also $A_0^{\beta-1} = 1 > 0$, and, since $0 < \beta < 1$,

$$0 < A_\mu^{\beta-1} = \frac{\beta+\mu-1}{\mu} A_{\mu-1}^{\beta-1} < A_{\mu-1}^{\beta-1} \quad (\mu \geqslant 1). \tag{24}$$

It follows,* by (23), (24) and (4), that

$$\begin{aligned} J &= \sum_{\rho=\mu}^{\nu} A_{n-\rho}^{\beta-1} A_{\rho-\mu}^{-\alpha-1} \\ &= A_{n-\mu}^{\beta-1} |A_0^{-\alpha-1}| - A_{n-\mu-1}^{\beta-1} |A_1^{-\alpha-1}| - \ldots - A_{n-\nu}^{\beta-1} |A_{\nu-\mu}^{-\alpha-1}| \\ &\leqslant A_{n-\mu}^{\beta-1}(|A_0^{-\alpha-1}| - |A_1^{-\alpha-1}| - \ldots - |A_{\nu-\mu}^{-\alpha-1}|) \\ &= A_{n-\mu}^{\beta-1} \sum_{\sigma=0}^{\nu-\mu} A_\sigma^{-\alpha-1} \\ &= A_{n-\mu}^{\beta-1} A_{\nu-\mu}^{-\alpha}. \end{aligned}$$

* Cf. Andersen (2), 70, for a similar argument.

On the other hand, by (23), (24), (4) and (6), if $0 \leqslant \mu \leqslant \nu \leqslant n$,

$$J = \sum_{\rho=\mu}^{\nu} A_{n-\rho}^{\beta-1} A_{\rho-\mu}^{-\alpha-1} \geqslant \sum_{\rho=\mu}^{n} A_{n-\rho}^{\beta-1} A_{\rho-\mu}^{-\alpha-1}$$
$$= A_{n-\mu}^{\beta-\alpha-1}$$
$$\geqslant -K_1(n-\mu+1)^{\beta-\alpha-1}$$
$$\geqslant -K_1(n-\mu+1)^{\beta-1}(\nu-\mu+1)^{-\alpha}$$
$$\geqslant -K_2 A_{n-\mu}^{\beta-1} A_{\nu-\mu}^{-\alpha}.$$

Thus (22) holds, with $K = \max(1, K_2)$.

We now have,* by (22), (7), (4), (5) and (6), since $\Delta\epsilon_n = \Delta^{-\alpha}(\Delta^{\alpha+1}\epsilon_n)$, by lemma 3,

$$| I_2 | \leqslant \sum_{\nu=\mu}^{n} | \Delta\epsilon_\nu | \, | J |$$
$$\leqslant K A_{n-\mu}^{\beta-1} \sum_{\nu=\mu}^{n} A_{\nu-\mu}^{-\alpha} | \Delta\epsilon_\nu |$$
$$= K A_{n-\mu}^{\beta-1} \sum_{\nu=\mu}^{n} A_{\nu-\mu}^{-\alpha} \left| \sum_{\rho=\nu}^{\infty} A_{\rho-\nu}^{\alpha-1} \Delta^{\alpha+1}\epsilon_\rho \right|$$
$$\leqslant K A_{n-\mu}^{\beta-1} \sum_{\nu=\mu}^{\infty} A_{\nu-\mu}^{-\alpha} \sum_{\rho=\nu}^{\infty} A_{\rho-\nu}^{\alpha-1} | \Delta^{\alpha+1}\epsilon_\rho |$$
$$= K A_{n-\mu}^{\beta-1} \sum_{\rho=\mu}^{\infty} | \Delta^{\alpha+1}\epsilon_\rho | \sum_{\nu=\mu}^{\rho} A_{\rho-\nu}^{\alpha-1} A_{\nu-\mu}^{-\alpha}$$
$$= K A_{n-\mu}^{\beta-1} \sum_{\rho=\mu}^{\infty} | \Delta^{\alpha+1}\epsilon_\rho | A_{\rho-\mu}^{0}$$
$$= K A_{n-\mu}^{\beta-1} \sum_{\rho=\mu}^{\infty} | \Delta^{\alpha+1}\epsilon_\rho |$$
$$\leqslant K A_{n-\mu}^{\beta-1} \mu^{-\alpha-q} \sum_{\rho=\mu}^{\infty} \rho^{\alpha+q} | \Delta^{\alpha+1}\epsilon_\rho |$$
$$= O\{(n-\mu+1)^{\beta-1}\} o(\mu^{-\alpha-q}),$$

since $\alpha + q \geqslant 0$. It follows that

$$\sum_{\mu=0}^{n} S_\mu^\alpha I_2 = \sum_{\mu=0}^{n} o(\mu^{\alpha+p}) O\{(n-\mu+1)^{\beta-1}\} o(\mu^{-\alpha-q})$$
$$= \sum_{\mu=0}^{n} O\{(n-\mu+1)^{\beta-1}\} o(\mu^{p-q})$$
$$= o(n^{\beta+p-q}),$$

since $\beta > 0$ and $p - q > -1$.

* Cf. Andersen (2), lemma 4.

Thus, collecting our results, we have, in case 2,

$$S_n^\beta(s_\nu \epsilon_\nu) = o(n^{\beta+p-q}),$$

i.e.

$$s_n \epsilon_n = o(n^{p-q}) \quad (C, \beta).$$

Case 3. *Suppose that* $0 < \beta < 1 \leqslant \alpha$, *and assume the lemma with* α, β, p, q *replaced by* $\alpha - 1$, τ, $p+1$, $q+1$, *where* $\tau = \min(\beta, \alpha - 1)$.

We have, by partial summation,

$$\sum_{\nu=0}^{n} s_\nu \epsilon_\nu = S_n^1 \epsilon_{n+1} + \sum_{\nu=0}^{n} S_\nu^1 \Delta\epsilon_\nu, \tag{25}$$

i.e.

$$S_n^1(s_\nu \epsilon_\nu) = S_n^1 \epsilon_{n+1} + S_n^1(S_\nu^1 \Delta\epsilon_\nu), \tag{26}$$

and hence, by (3),

$$\begin{aligned} S_n^\beta(s_\nu \epsilon_\nu) &= S_n^{\beta-1}\{S_\nu^1(s_\mu \epsilon_\mu)\} \\ &= S_n^{\beta-1}(S_\nu^1 \epsilon_{\nu+1}) + S_n^{\beta-1}\{S_\nu^1(S_\mu^1 \Delta\epsilon_\mu)\}. \end{aligned}$$

Thus, by (3),

$$S_n^\beta(s_\nu \epsilon_\nu) = S_n^{\beta-1}(S_\nu^1 \epsilon_{\nu+1}) + S_n^\beta(S_\nu^1 \Delta\epsilon_\nu). \tag{27}$$

Now, by (16) and lemma 2, since $\alpha \geqslant 1$, $S_n^1 = o(n^{\alpha+p})$. Also, by (17), $\epsilon_{n+1} = O(n^{\beta-\alpha-q})$. Hence

$$\begin{aligned} S_n^1 \epsilon_{n+1} &= o(n^{\alpha+p})\, O(n^{\beta-\alpha-q}) \\ &= o(n^{\beta+p-q}). \end{aligned}$$

It then follows, by lemma 2, since $\beta - 1 < 0$ and $\beta + p - q > -1$, that

$$S_n^{\beta-1}(S_\nu^1 \epsilon_{\nu+1}) = o(n^{\beta+p-q}). \tag{28}$$

We next show that S_n^1, $\Delta\epsilon_n$ satisfy the hypotheses of s_n, ϵ_n in the case assumed. In the first place, by (16) and (3),

$$S_n^{\alpha-1}(S_\nu^1) = S_n^\alpha = o(n^{\alpha+p}) = o\{n^{(\alpha-1)+(p+1)}\}. \tag{29}$$

Next, by (17),

$$\begin{aligned} \Delta\epsilon_n &= \epsilon_n - \epsilon_{n-1} \\ &= O(n^{\beta-\alpha-q}) + O(n^{\beta-\alpha-q}) \\ &= O\{n^{\beta-(\alpha-1)-(q+1)}\}. \end{aligned}$$

On the other hand, by (17), (18) and lemma 4 (b), since

$$\epsilon_n = O(n^{\beta-\alpha-q}) = O(n^{-q}),$$

we have

$$\Delta\epsilon_n = O(n^{-q-1}) = O\{n^{(\alpha-1)-(\alpha-1)-(q+1)}\}.$$

Thus

$$\Delta\epsilon_n = O\{n^{\tau-(\alpha-1)-(q+1)}\}, \tag{30}$$

where $\tau = \min(\beta, \alpha - 1)$.

Also, by (18) and lemma 3, since $\epsilon_n = O(1)$,

$$\sum n^{(\alpha-1)+(q+1)} \left| \Delta^{(\alpha-1)+1}(\Delta\epsilon_n) \right| = \sum n^{\alpha+q} \left| \Delta^{\alpha+1}\epsilon_n \right| < \infty. \tag{31}$$

Thus S_n^1, $\Delta\epsilon_n$ satisfy the hypotheses of s_n, ϵ_n in the lemma, with α, β, p, q replaced by $\alpha-1$, τ, $p+1$, $q+1$, and it follows from our assumption that

$$S_n^1\Delta\epsilon_n = o\{n^{(p+1)-(q+1)}\} \quad (C,\tau), \tag{32}$$

and hence, by lemma 1, that

$$S_n^1\Delta\epsilon_n = o(n^{p-q}) \quad (C,\beta), \tag{33}$$

i.e.
$$S_n^\beta(S_\nu^1\Delta\epsilon_\nu) = o(n^{\beta+p-q}). \tag{34}$$

We therefore have, by (28), (34) and (27),

$$\begin{aligned} S_n^\beta(s_\nu\epsilon_\nu) &= o(n^{\beta+p-q}) + o(n^{\beta+p-q}) \\ &= o(n^{\beta+p-q}), \end{aligned}$$

and case 3 follows by induction from cases 1 and 2.

Case 4. *Suppose that* $1 \leqslant \beta \leqslant \alpha$, *and assume the lemma with* α, β, p, q *replaced by* $\alpha-1$, $\beta-1$, $p+1$, q, *or by* $\alpha-1$, τ, $p+1$, $q+1$, *where*

$$\tau = \min(\beta, \alpha-1).$$

We again use (27), and we begin by showing that S_n^1, ϵ_{n+1} satisfy the hypotheses of s_n, ϵ_n in the first case assumed.

We have, by (16), (17) and (3),

$$S_n^{\alpha-1}(S_\nu^1) = S_n^\alpha = o\{n^{(\alpha-1)+(p+1)}\}, \tag{35}$$

$$\epsilon_n = O\{n^{(\beta-1)-(\alpha-1)-q}\}, \tag{36}$$

and, by (17), (18), (6) and lemma 4 (a), since $\epsilon_n = O(n^{\beta-\alpha-q}) = O(^{-q})$,

$$\sum_0^\infty (n+1)^{(\alpha-1)+q} \,|\, \Delta^{(\alpha-1)+1}\epsilon_{n+1} \,| = \sum_1^\infty n^{\alpha-1+q} \,|\, \Delta^\alpha\epsilon_n \,| < \infty. \tag{37}$$

Thus S_n^1, ϵ_{n+1} satisfy the hypotheses of s_n, ϵ_n with α, β, p, q replaced by $\alpha-1$, $\beta-1$, $p+1$, q, and it follows from the first case assumed that

$$S_n^1\epsilon_{n+1} = o(n^{p+1-q}) \quad (C,\beta-1), \tag{38}$$

i.e.
$$S_n^{\beta-1}(S_\nu^1\epsilon_{\nu+1}) = o\{n^{(\beta-1)+(p+1-q)}\} = o(n^{\beta+p-q}). \tag{39}$$

Again, as in case 3, S_n^1, $\Delta\epsilon_n$ satisfy the hypotheses of s_n, ϵ_n with α, β, p, q replaced by $\alpha-1$, τ, $p+1$, $q+1$, where $\tau = \min(\beta, \alpha-1)$, and it follows from the second case assumed that

$$S_n^1\Delta\epsilon_n = o\{n^{(p+1)-(q+1)}\} \quad (C,\tau), \tag{40}$$

and hence, by lemma 1, that

$$S_n^1\Delta\epsilon_n = o(n^{p-q}) \quad (C,\beta), \tag{41}$$

i.e.
$$S_n^\beta(S_\nu^1\Delta\epsilon_\nu) = o(n^{\beta+p-q}). \tag{42}$$

Thus it follows, by (39), (42) and (27), that

$$S_n^\beta(s_\nu \epsilon_\nu) = o(n^{\beta+p-q}) + o(n^{\beta+p-q})$$
$$= o(n^{\beta+p-q}),$$

and case 4 follows by induction from cases 1, 2 and 3.

This completes the proof of the lemma.

5. *The main result*

THEOREM A. *If* $0 \leqslant \beta \leqslant \alpha$, $p \geqslant 0$, *then necessary and sufficient conditions that* $\Sigma a_n \epsilon_n$ *should be summable* (C, β) *whenever* $S_n^\alpha = o(n^{\alpha+p})$ *are*

$$\text{(i) } \epsilon_n = O(n^{\beta-\alpha-p}), \quad \text{(ii) } \Sigma n^{\alpha+p} \,|\, \Delta^{\alpha+1}\epsilon_n \,| < \infty. \tag{43}$$

If $\beta > \alpha \geqslant 0$, $p \geqslant 0$, *the conditions are*

$$\text{(i)}' \ \epsilon_n = O(n^{-p}), \quad \text{(ii) } \Sigma n^{\alpha+p} \,|\, \Delta^{\alpha+1}\epsilon_n \,| < \infty. \tag{44}$$

*The same holds if O and o are interchanged throughout.**

Proof. We shall consider only the case where $S_n^\alpha = o(n^{\alpha+p})$. The alternative theorem is proved by a similar argument.

Sufficiency. Suppose that $0 \leqslant \beta \leqslant \alpha$, *that* (i) *and* (ii) *hold, and that*

$$S_n^\alpha = o(n^{\alpha+p}).$$

Write
$$\sum_{\nu=0}^{n} a_\nu \epsilon_\nu = s_n \epsilon_{n+1} + \sum_{\nu=0}^{n} s_\nu \Delta\epsilon_\nu. \tag{45}$$

Now
$$S_n^\alpha = o(n^{\alpha+p}), \tag{46}$$

$$\epsilon_{n+1} = O(n^{\beta-\alpha-p}) \tag{47}$$

and
$$\Sigma n^{\alpha+p} \,|\, \Delta^{\alpha+1}\epsilon_{n+1} \,| < \infty. \tag{48}$$

Hence, by lemma 7, with $q = p$,

$$s_n \epsilon_{n+1} = o(1) \quad (C, \beta). \tag{49}$$

We shall show that the series $\Sigma s_\nu \Delta\epsilon_\nu$ is convergent if $0 \leqslant \alpha \leqslant 1$, and summable (C, τ), where $\tau = \min(\beta, \alpha - 1)$, if $\alpha > 1$.

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* There is also a version where the condition that $S_n^\alpha = o(n^{\alpha+p})$ is replaced by $S_n^\alpha \sim sA_n^{\alpha+p}$ *for some s*. The necessity part is then included in theorem A. To prove the sufficiency part we may subtract sA_n^{p-1} from a_n and $s\Sigma A_n^{p-1}\epsilon_n$ from $\Sigma a_n \epsilon_n$, and so deduce the result from theorem A. The convergence of $\Sigma A_n^{p-1}\epsilon_n$ follows from conditions (i) [or (i)′] and (ii) by lemma 4(*a*).

Case 1. *Suppose that* $0 \leqslant \beta \leqslant \alpha < 1$.

We write, since $\Delta\epsilon_\nu = \Delta^{-\alpha}(\Delta^{\alpha+1}\epsilon_\nu)$, by lemma 3,

$$\sum_{\nu=0}^{n} s_\nu \Delta\epsilon_\nu = \sum_{\nu=0}^{n} s_\nu \sum_{\mu=\nu}^{\infty} A_{\mu-\nu}^{\alpha-1}\Delta^{\alpha+1}\epsilon_\mu$$
$$= \sum_{\nu=0}^{n}\sum_{\mu=\nu}^{n} + \sum_{\nu=0}^{n}\sum_{\mu=n+1}^{\infty}$$
$$= J_1 + J_2.$$

Then
$$J_1 = \sum_{\nu=0}^{n} s_\nu \sum_{\mu=\nu}^{n} A_{\mu-\nu}^{\alpha-1}\Delta^{\alpha+1}\epsilon_\mu$$
$$= \sum_{\mu=0}^{n} \Delta^{\alpha+1}\epsilon_\mu \sum_{\nu=0}^{\mu} A_{\mu-\nu}^{\alpha-1} s_\nu$$
$$= \sum_{\mu=0}^{n} S_\mu^\alpha \Delta^{\alpha+1}\epsilon_\mu$$
$$\rightarrow \sum_{\mu=0}^{\infty} S_\mu^\alpha \Delta^{\alpha+1}\epsilon_\mu,$$

as $n \to \infty$, since $\Sigma \mid S_\mu^\alpha \Delta^{\alpha+1}\epsilon_\mu \mid \leqslant A\Sigma(\mu+1)^{\alpha+p} \mid \Delta^{\alpha+1}\epsilon_\mu \mid < \infty$, by (ii).

Also
$$J_2 = \sum_{\nu=0}^{n} s_\nu \sum_{\mu=n+1}^{\infty} A_{\mu-\nu}^{\alpha-1}\Delta^{\alpha+1}\epsilon_\mu$$
$$= \sum_{\mu=n+1}^{\infty} \Delta^{\alpha+1}\epsilon_\mu \sum_{\nu=0}^{n} A_{\mu-\nu}^{\alpha-1} s_\nu,$$

and hence, by lemma 5, since $\alpha + p \geqslant 0$,

$$\mid J_2 \mid \leqslant \sum_{\mu=n+1}^{\infty} \mid \Delta^{\alpha+1}\epsilon_\mu \mid \max_{0 \leqslant \rho \leqslant n} \mid S_\rho^\alpha \mid$$
$$= O(n^{\alpha+p}) \sum_{\mu=n+1}^{\infty} \mid \Delta^{\alpha+1}\epsilon_\mu \mid$$
$$= \sum_{\mu=n+1}^{\infty} O(\mu^{\alpha+p}) \mid \Delta^{\alpha+1}\epsilon_\mu \mid$$
$$= o(1).$$

Thus $\Sigma s_\nu \Delta\epsilon_\nu$ converges, and so is summable (C, β), by lemma 1. It follows from (45) that $\Sigma a_n \epsilon_n$ is summable (C, β) in case 1.

Case 2. *Suppose that* $0 \leqslant \beta \leqslant \alpha$, $\alpha \geqslant 1$, *and assume the theorem with* α, β, p *replaced by* $\alpha - 1$, τ, $p+1$, *where* $\tau = \min(\beta, \alpha - 1)$.

We have, as in the proof of lemma 7 (cases 3 and 4),

$$S_n^{\alpha-1}(S_\nu^1) = S_n^\alpha = o\{n^{(\alpha-1)+(p+1)}\}, \tag{50}$$
$$\Delta\epsilon_n = O\{n^{\tau-(\alpha-1)-(p+1)}\}, \tag{51}$$
$$\Sigma n^{(\alpha-1)+(p+1)} \mid \Delta^\alpha(\Delta\epsilon_n) \mid < \infty, \tag{52}$$

where $\tau = \min(\beta, \alpha - 1)$, i.e. S_n^1, $\Delta\epsilon_n$ satisfy the hypotheses of s_n, ϵ_n in the theorem, with α, β, p replaced by $\alpha - 1, \tau, p+1$. It follows from our assumption that $\Sigma s_n \Delta\epsilon_n$ is summable (C, τ), and hence, by lemma 1, is summable (C, β). The summability (C, β) of $\Sigma s_n \Delta\epsilon_n$ is thus proved by induction in case 2, and it again follows from (45) that $\Sigma a_n \epsilon_n$ is summable (C, β).

This completes the proof of the sufficiency when $0 \leqslant \beta \leqslant \alpha$.

Suppose that $\beta > \alpha \geqslant 0$. Then conditions (i)′ and (ii) are the same as conditions (i) and (ii), with $\beta = \alpha$, and hence $\Sigma a_n \epsilon_n$ is summable (C, α) whenever $S_n^\alpha = o(n^{\alpha+p})$. It follows from lemma 1 that $\Sigma a_n \epsilon_n$ is summable (C, β).

This completes the proof of the sufficiency part of the theorem.

Necessity. We first show that condition (i)′ is necessary.* We observe that, if $\Sigma a_n \epsilon_n$ is summable (C) whenever $S_n^\alpha = o(n^{\alpha+p})$, then in particular, by lemma 1, it is summable (C) whenever $s_n = o(n^p)$.

Now suppose, if possible, that $\overline{\lim}\, n^p |\epsilon_n| = +\infty$. Then there are positive integers n_ν such that $n_1 < n_2 < \ldots$, and $n_\nu^p |\epsilon_{n_\nu}| > \nu^3$ $(\nu \geqslant 1)$. Choose a_n so that

$$a_{n_\nu} = \nu^{-3} n_\nu^p \operatorname{sgn} \bar{\epsilon}_{n_\nu}, \quad a_n = 0 \quad (n \neq n_\nu).$$

Then, if $p > 0$ and $n_\mu \leqslant n < n_{\mu+1}$, we have

$$|s_n| = |s_{n_\mu}| \leqslant \sum_{\nu=1}^{\mu} \nu^{-3} n_\nu^p = \sum_{\nu=1}^{\mu} \nu^{-2} o(n_\nu^p) = o(n_\mu^p) = o(n^p),$$

while, if $p = 0$, we can make $s_n = o(1)$ by altering a_0 suitably. But, for $\nu \geqslant 1$,

$$a_{n_\nu}\epsilon_{n_\nu} = \nu^{-3} n_\nu^p \operatorname{sgn} \bar{\epsilon}_{n_\nu} . \epsilon_{n_\nu} = \nu^{-3} n_\nu^p |\epsilon_{n_\nu}| > 1,$$

and $a_n \epsilon_n = 0$ $(n \neq n_\nu)$.† Hence $\Sigma a_n \epsilon_n = +\infty$, which contradicts the summability (C) of $\Sigma a_n \epsilon_n$. Thus (i)′ is necessary.

Next we have, by (1), (3) and (10),

$$\begin{aligned}
\frac{S_n^\beta\left(\sum_{\mu=0}^{\nu} a_\mu \epsilon_\mu\right)}{A_n^\beta} &= \frac{S_n^{\beta+1}(a_\nu \epsilon_\nu)}{A_n^\beta} \\
&= \frac{1}{A_n^\beta} \sum_{\nu=0}^{n} A_{n-\nu}^\beta a_\nu \epsilon_\nu \\
&= \frac{1}{A_n^\beta} \sum_{\nu=0}^{n} A_{n-\nu}^\beta \epsilon_\nu \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{-\alpha-2} S_\mu^\alpha \\
&= \frac{1}{A_n^\beta} \sum_{\mu=0}^{n} S_\mu^\alpha \sum_{\nu=\mu}^{n} A_{n-\nu}^\beta A_{\nu-\mu}^{-\alpha-2} \epsilon_\nu \\
&= \sum_{\mu=0}^{n} \frac{S_\mu^\alpha}{A_\mu^{\alpha+p}} d_{\mu,n},
\end{aligned}$$

* When $0 \leqslant \beta \leqslant \alpha$ condition (i)′ is included in condition (i).

† Except possibly for $n = 0$, if $p = 0$.

where
$$d_{\mu,n} = \frac{A_\mu^{\alpha+p}}{A_n^\beta} \sum_{\nu=\mu}^{n} A_{n-\nu}^\beta A_{\nu-\mu}^{-\alpha-2} \epsilon_\nu. \tag{53}$$

It follows that, if $\Sigma a_n \epsilon_n$ is summable (C, β) whenever $S_n^\alpha / A_n^{\alpha+p} = o(1)$, there is, by lemma 6, a number M, independent of n, such that

$$\sum_{\mu=0}^{n} | d_{\mu,n} | \leqslant M, \tag{54}$$

where $d_{\mu,n}$ is given by (53).

It follows from (54), in particular, that

$$| d_{n,n} | \leqslant M, \tag{55}$$

i.e.
$$\frac{A_n^{\alpha+p}}{A_n^\beta} | \epsilon_n | \leqslant M, \tag{56}$$

and hence, by (6),
$$\epsilon_n = O(n^{\beta-\alpha-p}). \tag{57}$$

Thus condition (i) is necessary.*

Next since $\epsilon_n = O(1)$, by (i) or (i)$'$, the series $\sum_{\nu=\mu}^{\infty} A_{\nu-\mu}^{-\alpha-2} \epsilon_\nu$ converges, and so, by lemma 1, is summable (C, β).† Hence

$$\frac{1}{A_n^\beta} \sum_{\nu=\mu}^{n} A_{n-\nu}^\beta A_{\nu-\mu}^{-\alpha-2} \epsilon_\nu \to \sum_{\nu=\mu}^{\infty} A_{\nu-\mu}^{-\alpha-2} \epsilon_\nu = \Delta^{\alpha+1} \epsilon_\mu \tag{58}$$

as $n \to \infty$, and therefore, for each $\mu \geqslant 0$,

$$\lim_{n\to\infty} d_{\mu,n} = A_\mu^{\alpha+p} \Delta^{\alpha+1} \epsilon_\mu.\ddagger \tag{59}$$

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* When $\beta > \alpha \geqslant 0$ condition (i) is included in condition (i)$'$.

We may also prove the necessity of condition (i) without using lemma 6. For suppose if possible that $\overline{\lim} \mid n^{-\beta+\alpha+p} \epsilon_n \mid = +\infty$. Then there is an increasing sequence of positive integers n_ν such that

$$n_\nu / n_{\nu-1} \to +\infty \quad \text{and} \quad n_\nu^{-\beta+\alpha+p} \mid \epsilon_{n_\nu} \mid > \nu \quad (\nu \geqslant 1).$$

Now choose a_n so that $S_n^\alpha = \nu^{-1} n_\nu^{\alpha+p}$, $S_n^\alpha = 0$ $(n \neq n_\nu)$. Then $S_n^\alpha = o(n^{\alpha+p})$, and

$$a_{n_\nu} = S_{n_\nu}^{-1} = S_{n_\nu}^{-\alpha-1}(S_\mu^\alpha) = \sum_{\mu=0}^{n_\nu} A_{n_\nu-\mu}^{-\alpha-2} S_\mu^\alpha = \sum_{\rho=1}^{\nu} A_{n_\nu - n_\rho}^{-\alpha-2} \rho^{-1} n_\rho^{\alpha+p} \sim \nu^{-1} n_\nu^{\alpha+p}.$$

Hence
$$\mid n_\nu^{-\beta} a_{n_\nu} \epsilon_{n_\nu} \mid \sim \nu^{-1} n_\nu^{-\beta+\alpha+p} \mid \epsilon_{n_\nu} \mid > 1,$$

and so $a_n \epsilon_n$ is not $o(n^\beta)$. This, by lemma 2, contradicts the summability (C, β) of $\Sigma a_n \epsilon_n$.

† We may consider the series as starting at the term for which $\nu = 0$, the first μ terms being zero.

‡ This is condition (ii) of lemma 6.

It follows from (54) that, for each $m \geqslant 0$,

$$\sum_{\mu=0}^{m} \lim_{n\to\infty} |d_{\mu,n}| \leqslant M, \tag{60}$$

and hence

$$\sum_{\mu=0}^{\infty} \lim_{n\to\infty} |d_{\mu,n}| \leqslant M, \tag{61}$$

i.e.

$$\sum_{\mu=0}^{\infty} A_{\mu}^{\alpha+p} |\Delta^{\alpha+1}\epsilon_{\mu}| \leqslant M < \infty, \tag{62}$$

which, by (6), shows that (ii) is necessary.

This completes the proof of the necessity part of the theorem.

6. *Additional remarks*

I mentioned in the introduction that theorem A (with $p = 0$) was formulated by Schur in the case where α and β are integers. It is natural to inquire how he proved the theorem.

He observed in effect* that

$$\sum_{\nu=0}^{n} A_{n-\nu}^{\beta} a_{\nu} \epsilon_{\nu} = \sum_{\nu=0}^{n} S_{\nu}^{\alpha} \Delta_{n}^{\alpha+1}(A_{n-\nu}^{\beta}\epsilon_{\nu}), \tag{63}$$

where the broken difference $\Delta_n^{\lambda}\epsilon_{\nu}$ is obtained from the difference $\Delta^{\lambda}\epsilon_{\nu}$ by putting $\epsilon_{\mu} = 0$ for $\mu = n+1, n+2, \ldots$, and he stated that the result could then be obtained by using lemma 6 (above). So far as the necessity of conditions (i) and (ii) is concerned his argument was probably much the same as the one I have given above, while the necessity of (i)′ may be established in various ways.† Concerning the sufficiency Prof. Hardy has recently shown me how the Bohr-Hardy theorem may be deduced from (63) by means of lemma 6, and I have verified that his method may be adapted to the case where α and β are unequal integers, and there is a parameter p. His method shows that, if α and β are integers and p is real, conditions (i) and (ii) of theorem A imply that

$$\frac{1}{A_n^{\beta}} \sum_{\nu=0}^{n} A_{\nu}^{\alpha+p} |\Delta_n^{\alpha+1}(A_{n-\nu}^{\beta}\epsilon_{\nu})| \leqslant M, \tag{64}$$

where M is independent of n. This is condition (i) of lemma 6, with $d_{\mu,n}$ given by (53), while condition (ii) is trivial.

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* I use the notation of Andersen (**1**, **2**), and of the present paper.

† Cf. Bosanquet (**7**, **8**). When α is an integer, the series defining $\Delta^{\alpha+1}\epsilon_n$ terminates and, if a parameter p is inserted in the theorem, p may be any real number.

Still another way of establishing the sufficiency, when α and β are integers and p is real, is to adapt Hardy's original method of proof of the Bohr-Hardy theorem, starting from (63). This is easily carried out if we follow Andersen's* compact proof of the Bohr-Hardy theorem, making suitable modifications.

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University College
London

* Andersen (2). Hardy's proof is spread over two papers, the second correcting in a footnote an oversight in the first.

ARITHMETICAL PATTERN PROBLEMS RELATING TO DIVISIBILITY BY rTH POWERS*

By L. MIRSKY

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Introduction

Let r be an integer at least equal to 2. A natural number will be called an *r-number* if it is divisible by the rth power of some prime; otherwise it will be called an *r-free number*.† It is well known‡ that the number of r-free numbers not exceeding x is

$$x/\zeta(r)+O(x^{1/r}) \tag{1}$$

as $x\to\infty$, and it is natural to investigate the frequency of occurrence not only of r-free numbers but also of systems of r-free numbers. This problem was studied (for $r=2$) by S. S. Pillai§ who established an asymptotic formula, with an error term $O(x/\log x)$, for the number $N(x)=N(x;d_1,\ldots,d_{r-1})$ of systems of 2-free numbers $q_1,q_1+d_1,\ldots,q_1+d_{r-1}$ not exceeding x.

We shall here be concerned with a still more general question, namely the frequency of occurrence of certain patterns of numbers whose character with regard to divisibility by rth powers is prescribed. Let $a_1,\ldots,a_l;b_1,\ldots,b_m$ be any distinct positive integers.|| We denote by $H(x)=H_r(x;a_1,\ldots,a_l;b_1,\ldots,b_m)$ the number of systems of positive integers $n+a_1,\ldots,n+a_l;n+b_1,\ldots,n+b_m$, not exceeding x, and such that the first l of these numbers are r-free whilst the remaining m are not.

* This paper includes some material from the author's M.Sc. thesis (London, 1941).

† The customary term for 2-free is *square-free* or *quadratfrei*.

‡ For the case $r=2$, see, for example, E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen* (Leipzig and Berlin, 1909), 2, 580–582. The general case can be dealt with in precisely the same way.

§ "On sets of square-free integers", *Journal Indian Math. Soc.*, N.S., 2 (1936), 116–118.

|| We assume that $l+m>0$, but the cases $l>0$, $m=0$ and $l=0$, $m>0$ are admitted. In the former case, for instance, the system of numbers $a_1,\ldots,a_l;b_1,\ldots,b_m$ is interpreted as $a_1,\ldots,a_l$.

Our principal result (theorem 2) is the asymptotic formula

$$H(x) = hx + O(x^{\alpha}) \quad (x \to \infty),$$

where $\alpha = (l+m)/(r+l+m-1)+\epsilon$, and, except in certain trivial cases when $H(x)$ vanishes identically, $h = h_r(a_1, \ldots, a_l; b_1, \ldots, b_m)$ is a positive constant whose value will be determined. The proof of this formula will depend upon a generalization of Pillai's result in a different direction (theorem 1).

In the final section we consider an application of theorem 2 to the study of frequency of systems of *consecutive* r-free numbers and r-numbers.

I wish to thank Dr T. Estermann for pointing out to me that a method used by him in estimating certain sums* was also applicable in the present problem. Dr Estermann's remark enabled me to improve considerably on my previous results. I am also under a great obligation to Dr R. Rado who read the manuscript and made many valuable suggestions.

Notation

If n is any variable and $P_1(n)$, $P_2(n)$ are two propositions concerning this variable, then

$$P_1(n) \quad (P_2(n))$$

means that for *every* n for which $P_2(n)$ holds, $P_1(n)$ holds also;

$$P_1(n) \quad [P_2(n)]$$

means that for *some* n for which $P_2(n)$ holds, $P_1(n)$ holds also.

For instance,

$$\frac{1}{n} \leqslant 1 \quad (n \geqslant 1),$$

$$\sqrt{n} \sin \frac{\pi n}{3} > 100 \quad [n \geqslant 1].$$

Throughout x denotes a positive real number; ϵ denotes an arbitrarily small positive number; p, p_1, etc., denote prime numbers.

The O-notation refers to the passage $x \to \infty$, and the constants implied by it may depend upon parameters other than x.

$(n_1, \ldots, n_s)$ and $\{n_1, \ldots, n_s\}$ denote respectively the highest common factor and the least common multiple of $n_1, \ldots, n_s$.

$n \leqslant n_1, \ldots, n_s$ means $n \leqslant n_i$ $(1 \leqslant i \leqslant s)$.

$d(n)$ is the number of positive divisors of n.

$\mu(n)$ is Möbius's function; $\mu_r(n) = 1$ or 0 according as n is or is not r-free.

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* See T. Estermann, "On the representation of a number as the sum of two numbers not divisible by kth powers", *J. London Math. Soc.* 6 (1931), 37–40.

Let Φ be a function having for its argument any finite system of integers. For simplicity we shall denote such a function by, say, $\Phi(n_1, \ldots, n_s)$ where it is understood that s may vary. Then, for $k = 0$, the symbol

$$\sum_{1 \leqslant \nu_1 < \ldots < \nu_k \leqslant m} \Phi(\lambda_1, \ldots, \lambda_t, \nu_1, \ldots, \nu_k)$$

is interpreted as $\Phi(\lambda_1, \ldots, \lambda_t)$.

We write

$$M_r(x; q; c_1, \ldots, c_s) = \sum_{1 \leqslant nq \leqslant x} \mu_r(nq + c_1) \ldots \mu_r(nq + c_s),$$

$$M_r(x; c_1, \ldots, c_s) = M_r(x; 1; c_1, \ldots, c_s).$$

For $\sigma > 0$, $D(\sigma \mid n_1, \ldots, n_s)$ denotes the number of different residue classes $(\operatorname{mod} \sigma)$ represented by $n_1, \ldots, n_s$, i.e. the number of integers ν satisfying the conditions

$$1 \leqslant \nu \leqslant \sigma,$$

$$\nu \equiv n_i \,(\operatorname{mod} \sigma) \quad [1 \leqslant i \leqslant s].$$

More generally we define, for σ, $\tau > 0$, $D(\sigma, \tau \mid n_1, \ldots, n_s)$ as the number of integers ν satisfying

$$1 \leqslant \nu \leqslant \sigma,$$

$$(\sigma, \tau) \mid \nu,$$

$$\nu \equiv n_i \,(\operatorname{mod} \sigma) \quad [1 \leqslant i \leqslant s].$$

Clearly $D(\sigma, \tau \mid n_1, \ldots, n_s)$ is also equal to the number of integers ν' satisfying

$$1 \leqslant \nu' \leqslant (\sigma, \tau)^{-1} \sigma,$$

$$(\sigma, \tau)\, \nu' \equiv n_i \,(\operatorname{mod} \sigma) \quad [1 \leqslant i \leqslant s].$$

Furthermore, $$0 \leqslant D(\sigma, \tau \mid n_1, \ldots, n_s) \leqslant (\sigma, \tau)^{-1} \sigma,$$

and $$D(\sigma, 1 \mid n_1, \ldots, n_s) = D(\sigma \mid n_1, \ldots, n_s).$$

If $\lambda_1, \ldots, \lambda_s$ are integers and $n_1, \ldots, n_s$ are positive integers, then

$$E\begin{pmatrix} n_1, \ldots, n_s \\ \lambda_1, \ldots, \lambda_s \end{pmatrix} = 1 \text{ or } 0$$

according as the system of congruences in ξ,

$$\xi + \lambda_\nu \equiv 0 \,(\operatorname{mod} n_\nu) \quad (1 \leqslant \nu \leqslant s) \tag{2}$$

is or is not soluble.

Lemmas

LEMMA 1. *The system of congruences* (2) *is soluble if and only if*

$$(n_i, n_j) \mid (\lambda_i - \lambda_j) \quad (1 \leqslant i < j \leqslant s).$$

In the case of solubility the solutions form exactly one residue class

$$(\mathrm{mod}\,\{n_1, \ldots, n_s\}).$$

This result is well known.*

LEMMA 2. *Let T be the number of positive integers $\xi \leqslant x$ satisfying* (2). *Then*

$$\left| T - x\,\frac{E\begin{pmatrix} n_1, \ldots, n_s \\ \lambda_1, \ldots, \lambda_s \end{pmatrix}}{\{n_1, \ldots, n_s\}} \right| \leqslant 1.$$

This follows easily by lemma 1.

LEMMA 3. *Let $n_1, \ldots, n_s, c_1, \ldots, c_s, q$ be given positive integers, and T the number of systems of positive integers $\xi, \xi_1, \ldots, \xi_s$ such that*

$$n_\nu \xi_\nu - q\xi = c_\nu \quad (1 \leqslant \nu \leqslant s),$$

$$q\xi \leqslant x.$$

Then

$$\left| T - x\,\frac{E\begin{pmatrix} q, n_1, \ldots, n_s \\ 0, c_1, \ldots, c_s \end{pmatrix}}{\{q, n_1, \ldots, n_s\}} \right| \leqslant 1.$$

Proof. T is clearly equal to the number of positive integers ξ such that

$$q\xi + c_\nu \equiv 0\,(\mathrm{mod}\,n_\nu) \quad (1 \leqslant \nu \leqslant s),$$

$$q\xi \leqslant x.$$

Hence it is equal to the number of positive integers η such that

$$\eta \equiv 0\,(\mathrm{mod}\,q),$$

$$\eta + c_\nu \equiv 0\,(\mathrm{mod}\,n_\nu) \quad (1 \leqslant \nu \leqslant s),$$

$$\eta \leqslant x,$$

and the required result follows by lemma 2.

LEMMA 4. *Let $f(n_1, \ldots, n_s)$ be a function of s integral variables.*

If (i) $\sum_{n_1, \ldots, n_s \geqslant 1} |f(n_1, \ldots, n_s)| < \infty$,

(ii) *$f(n_1, \ldots, n_s)$ is multiplicative in $n_1, \ldots, n_s$, i.e.*

$$f(n_1, \ldots, n_s)\, f(n_1', \ldots, n_s') = f(n_1 n_1', \ldots, n_s n_s')$$

for

$$(n_i, n_j') = 1 \quad (1 \leqslant i, j \leqslant s),$$

then

$$\sum_{n_1, \ldots, n_s \geqslant 1} f(n_1, \ldots, n_s) = \prod_p \chi_p,$$

where

$$\chi_p = \sum_{\delta_1, \ldots, \delta_s \geqslant 0} f(p^{\delta_1}, \ldots, p^{\delta_s}).$$

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* See, for example, A. Scholz, *Einführung in die Zahlentheorie* (Göschen), Satz 31.

The case $s = 1$ is Euler's identity, and the generalization to any $s > 1$ is immediately obvious.

LEMMA 5. *Let $t, q, r, \rho, k_1, \ldots, k_t$ be given integers; $t \geqslant 1$, $q \geqslant 1$, $r \geqslant 2$, and let p be a given prime.*

If $k_1 \equiv \ldots \equiv k_t \equiv \rho \pmod{p^r}$, then

$$\sum_{\substack{\eta_1, \ldots, \eta_t = 0, 1\\ \eta_1 + \ldots + \eta_t > 0}} (-1)^{\eta_1 + \ldots + \eta_t} E\begin{pmatrix} q, & p^{r\eta_1}, & \ldots, & p^{r\eta_t} \\ 0, & k_1, & \ldots, & k_t \end{pmatrix} = \begin{cases} -1 & \text{if } (p^r, q) \mid \rho, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $k_1 \equiv \ldots \equiv k_t \equiv \rho \pmod{p^r}$ and $\eta_i = 1$ $[1 \leqslant i \leqslant t]$, it follows by the definition of the E-symbol and lemma 1 that

$$E\begin{pmatrix} q, & p^{r\eta_1}, & \ldots, & p^{r\eta_t} \\ 0, & k_1, & \ldots, & k_t \end{pmatrix} = E\begin{pmatrix} q, & p^r \\ 0, & \rho \end{pmatrix} = \begin{cases} 1 & \text{if } (p^r, q) \mid \rho, \\ 0 & \text{otherwise.} \end{cases}$$

Also,
$$\sum_{\substack{\eta_1, \ldots, \eta_t = 0, 1\\ \eta_1 + \ldots + \eta_t > 0}} (-1)^{\eta_1 + \ldots + \eta_t} = -1,$$

and the lemma is therefore proved.

LEMMA 6. *Let $r, q, s, c_1, \ldots, c_s$ be given integers; $r \geqslant 2$, $q \geqslant 1$, $s \geqslant 1$, and*

$$S = S_{r,q}(c_1, \ldots, c_s) = \sum_{n_1, \ldots, n_s \geqslant 1} \frac{\mu(n_1) \ldots \mu(n_s)}{\{q, n_1^r, \ldots, n_s^r\}} E\begin{pmatrix} q, & n_1^r, & \ldots, & n_s^r \\ 0, & c_1, & \ldots, & c_s \end{pmatrix}.$$

Then
$$S = \begin{cases} 0 & \text{if } D(p^r, q \mid c_1, \ldots, c_s) = (p^r, q)^{-1} p^r \quad [p]. \\ q^{-1} \prod_{p^r \nmid q} \left\{ 1 - \dfrac{(p^r, q)}{p^r} D(p^r, q \mid c_1, \ldots, c_s) \right\} > 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose that
$$(n_i, n_j') = 1 \quad (1 \leqslant i, j \leqslant s). \tag{3}$$

Then
$$q^{-1}\{q, n_1, \ldots, n_s\} \times q^{-1}\{q, n_1', \ldots, n_s'\} = q^{-1}\{q, n_1 n_1', \ldots, n_s n_s'\}, \tag{4}$$

$$E\begin{pmatrix} q, & n_1, & \ldots, & n_s \\ 0, & c_1, & \ldots, & c_s \end{pmatrix} E\begin{pmatrix} q, & n_1', & \ldots, & n_s' \\ 0, & c_1, & \ldots, & c_s \end{pmatrix} = E\begin{pmatrix} q, & n_1 n_1', & \ldots, & n_s n_s' \\ 0, & c_1, & \ldots, & c_s \end{pmatrix}. \tag{5}$$

To prove (4) consider any prime p, and let $p^\lambda, p^{\nu_i}, p^{\nu_i'}$ be the highest powers of p which divide q, n_i, n_i' respectively $(1 \leqslant i \leqslant s)$. It is clearly sufficient to show that

$$\max(\lambda, \nu_1, \ldots, \nu_s) + \max(\lambda, \nu_1', \ldots, \nu_s') - 2\lambda = \max(\lambda, \nu_1 + \nu_1', \ldots, \nu_s + \nu_s') - \lambda.$$

But since, by (3), $\nu_1 = \ldots = \nu_s = 0$ or $\nu_1' = \ldots = \nu_s' = 0$, this is evidently true, and so (4) is established.

To prove (5) we note that, if at least one factor on the left vanishes, then the right-hand side vanishes also. If, however, both factors on the left are equal to 1, then, by lemma 1,

$$\left.\begin{matrix} (n_i, n_j) \mid (c_i - c_j); \; (n_i, q) \mid c_i \\ (n_i', n_j') \mid (c_i - c_j); \; (n_i', q) \mid c_i \end{matrix}\right\} \quad (1 \leqslant i, j \leqslant s).$$

Therefore, by (3),

$$(n_i n_i', n_j n_j') \mid (c_i - c_j), \quad (n_i n_i', q) \mid c_i \quad (1 \leqslant i, j \leqslant s),$$

and so, by lemma 1, the right-hand side of (5) is also equal to 1. Thus

$$qS = \sum_{n_1, \ldots, n_s \geqslant 1} \mu(n_1) \ldots \mu(n_s) \frac{q}{\{q, n_1^r, \ldots, n_s^r\}} E\begin{pmatrix} q, & n_1^r, & \ldots, & n_s^r \\ 0, & c_1, & \ldots, & c_s \end{pmatrix} = \sum_{n_1, \ldots, n_s \geqslant 1} f(n_1, \ldots, n_s)$$

where $f(n_1, \ldots, n_s)$ is multiplicative in $n_1, \ldots, n_s$. Furthermore, the series defining qS converges absolutely, for it is majorized by

$$\sum_{n_1, \ldots, n_s \geqslant 1} q\{n_1^r, \ldots, n_s^r\}^{-1} = q \sum_{n \geqslant 1} n^{-r} \sum_{\substack{\{n_1, \ldots, n_s\} = n \\ n_1, \ldots, n_s \geqslant 1}} 1 \leqslant q \sum_{n \geqslant 1} (d(n))^s n^{-r} < \infty.$$

Hence, by lemma 4,

$$qS = \prod_p \chi_p, \tag{6}$$

where
$$\chi_p = \sum_{\delta_1, \ldots, \delta_s = 0, 1} (-1)^{\delta_1 + \ldots + \delta_s} \frac{q}{\{q, p^{r\delta_1}, \ldots, p^{r\delta_s}\}} E\begin{pmatrix} q, & p^{r\delta_1}, & \ldots, & p^{r\delta_s} \\ 0, & c_1, & \ldots, & c_s \end{pmatrix}$$

$$= 1 + \frac{q}{\{p^r, q\}} \psi_p = 1 + \frac{(p^r, q)}{p^r} \psi_p, \tag{7}$$

and
$$\psi_p = \sum_{\substack{\delta_1, \ldots, \delta_s = 0, 1 \\ \delta_1 + \ldots + \delta_s > 0}} (-1)^{\delta_1 + \ldots + \delta_s} E\begin{pmatrix} q, & p^{r\delta_1}, & \ldots, & p^{r\delta_s} \\ 0, & c_1, & \ldots, & c_s \end{pmatrix}.$$

To evaluate ψ_p we first observe that, if

$$1 \leqslant i \leqslant s, \quad 1 \leqslant \nu_1 < \ldots < \nu_i \leqslant s, \quad \delta_{\nu_1} = \delta_{\nu_2} = \ldots = \delta_{\nu_i} = 1 \quad \text{and} \quad c_{\nu_1}, \ldots, c_{\nu_i}$$

do not all belong to the same residue class $(\operatorname{mod} p^r)$, then

$$E\begin{pmatrix} q, & p^{r\delta_1}, & \ldots, & p^{r\delta_s} \\ 0, & c_1, & \ldots, & c_s \end{pmatrix} = 0.$$

Now for $1 \leqslant \rho \leqslant p^r$, let t_ρ denote the number of numbers amongst $c_1, \ldots, c_s$ which are congruent to ρ $(\operatorname{mod} p^r)$. If $t_\rho > 0$, denote these numbers by $k_1^{(\rho)}, \ldots, k_{t_\rho}^{(\rho)}$. We then have

$$\psi_p = \sum_{\substack{1 \leqslant \rho \leqslant p^r \\ t_\rho > 0}} \sum_{\substack{\eta_1, \ldots, \eta_{t_\rho} = 0, 1 \\ \eta_1 + \ldots + \eta_{t_\rho} > 0}} (-1)^{\eta_1 + \ldots + \eta_{t_\rho}} E\begin{pmatrix} q, & p^{r\eta_1}, & \ldots, & p^{r\eta_{t_\rho}} \\ 0, & k_1^{(\rho)}, & \ldots, & k_{t_\rho}^{(\rho)} \end{pmatrix}$$

$$= - \sum_{\substack{1 \leqslant \rho \leqslant p^r \\ t_\rho > 0 \\ (p^r, q) \mid \rho}} 1 \quad \text{(by lemma 5)}$$

$$= -D(p^r, q \mid c_1, \ldots, c_s).$$

Hence, by (6) and (7),

$$qS = \prod_p \left\{ 1 - \frac{(p^r, q)}{p^r} D(p^r, q \mid c_1, \ldots, c_s) \right\}.$$

The first assertion is now evident. To prove the second we suppose that

$$D(p^r,q\mid c_1,\dots,c_s)<(p^r,q)^{-1}p^r \quad (p).$$

Then

$$D(p^r,q\mid c_1,\dots,c_s)=0 \quad \text{if} \quad p^r\mid q.$$

We therefore have

$$qS=\prod_{p^r\nmid q}\left\{1-\frac{(p^r,q)}{p^r}D(p^r,q\mid c_1,\dots,c_s)\right\}.$$

Furthermore, the value of the infinite product is positive, since each factor is positive and since the factor corresponding to p is at least $1-sp^{-r}$ for all sufficiently large values of p.

The general pattern

THEOREM 1. *Let $r, q, s, c_1, \dots, c_s$ be given integers; $r\geqslant 2$; $q, s, c_1,\dots,c_s\geqslant 1$.*

(i) *If $D(p^r,q\mid c_1,\dots,c_s)=(p^r,q)^{-1}p^r\,[p]$, then $M_r(x;q;c_1,\dots,c_s)=0$.*

(ii) *Otherwise,**

$$M_r(x;q;c_1,\dots,c_s)=S_{r,q}(c_1,\dots,c_s)\,x+O(x^{s/(r+s-1)+\epsilon}),$$

where

$$S_{r,q}(c_1,\dots,c_s)=q^{-1}\prod_{p^r\nmid q}\left\{1-\frac{(p^r,q)}{p^r}D(p^r,q\mid c_1,\dots,c_s)\right\}>0.$$

Proof. (i) Suppose that, for some p,

$$D(p^r,q\mid c_1,\dots,c_s)=(p^r,q)^{-1}p^r.$$

This means that for every k in the range

$$1\leqslant k\leqslant (p^r,q)^{-1}p^r$$

we have

$$k(p^r,q)\equiv c_i\,(\operatorname{mod} p^r) \quad [1\leqslant i\leqslant s]. \tag{8}$$

Next we write $\nu=q/(p^r,q)$. We then have, for every $n\geqslant 1$,

$$(p^r,q)^{-1}p^r\mid \nu n+k \quad [1\leqslant k\leqslant (p^r,q)^{-1}p^r].$$

Hence

$$p^r\mid nq+k(p^r,q) \quad [1\leqslant k\leqslant (p^r,q)^{-1}p^r],$$

and so, by (8),

$$p^r\mid nq+c_i \quad [1\leqslant i\leqslant s].$$

Therefore

$$M_r(x;q;c_1,\dots,c_s)=0.$$

(ii) Now suppose that

$$D(p^r,q\mid c_1,\dots,c_s)<(p^r,q)^{-1}p^r \quad (p).$$

For the remainder of this proof it will be understood that summation indices range over positive integers.

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* The method used in the proof below does, in fact, show that for $s=1$ the error term is $O(x^{s/(r+s-1)})=O(x^{1/r})$.

Let $0 < \alpha \leqslant r^{-1}$. Using the relation

$$\mu_r(n) = \sum_{c^r \mid n} \mu(c),$$

we have

$$M_r(x; q; c_1, \ldots, c_s) = \sum_{nq \leqslant x} \sum_{\substack{n_\lambda^r \mid nq + c_\lambda \\ (1 \leqslant \lambda \leqslant s)}} \mu(n_1) \ldots \mu(n_s) = \sum_{\substack{n_\lambda^r t_\lambda - nq = c_\lambda \\ (1 \leqslant \lambda \leqslant s) \\ nq \leqslant x}} \mu(n_1) \ldots \mu(n_s)$$

$$= \sum_{\substack{n_\lambda \leqslant (x + c_\lambda)^{1/r} \\ (1 \leqslant \lambda \leqslant s)}} \mu(n_1) \ldots \mu(n_s) \sum_{\substack{n_\lambda^r t_\lambda - nq = c_\lambda \\ (1 \leqslant \lambda \leqslant s) \\ nq \leqslant x}} 1 = \Sigma_1 + \Sigma_2, \tag{9}$$

where in Σ_1 the outer summation extends over $n_1, \ldots, n_s \leqslant x^\alpha$, whilst Σ_2 consists of the remaining terms.

By lemma 3,

$$\Sigma_1 = \sum_{n_1, \ldots, n_s \leqslant x^\alpha} \mu(n_1), \ldots \mu(n_s) \left\{ x \frac{E\begin{pmatrix} q, & n_1^r, & \ldots, & n_s^r \\ 0, & c_1, & \ldots, & c_s \end{pmatrix}}{\{q, n_1^r, \ldots, n_s^r\}} + O(1) \right\}$$

$$= xq^{-1} \prod_{p^r \nmid q} \left\{ 1 - \frac{(p^r, q)}{p^r} D(p^r, q \mid c_1, \ldots, c_s) \right\}$$

$$+ O\left(x \sum_{k=1}^{s} \sum_{\substack{n_k > x^\alpha \\ n_1, \ldots, n_s \geqslant 1}} \{n_1^r, \ldots, n_s^r\}^{-1} \right) + O(x^{\alpha s})$$

by lemma 6. But

$$\sum_{\substack{n_k > x^\alpha \\ n_1, \ldots, n_s \geqslant 1}} \{n_1^r, \ldots, n_s^r\}^{-1} \leqslant \sum_{n > x^\alpha} n^{-r} \sum_{\{n_1, \ldots, n_s\} = n} 1 \leqslant \sum_{n > x^\alpha} (d(n))^s n^{-r} = O(x^{-\alpha(r-1)+\epsilon}),$$

since $d(n) = O(n^\epsilon)$. Hence

$$\Sigma_1 = xq^{-1} \prod_{p^r \nmid q} \left\{ 1 - \frac{(p^r, q)}{p^r} D(p^r, q \mid c_1, \ldots, c_s) \right\} + O(x^{1-\alpha(r-1)+\epsilon}) + O(x^{\alpha s}). \tag{10}$$

Again,

$$|\Sigma_2| \leqslant \sum_{k=1}^{s} f_k,$$

where

$$f_k = \sum_{\substack{n_\lambda^r t_\lambda - nq = c_\lambda \\ (1 \leqslant \lambda \leqslant s) \\ nq \leqslant x \\ n_k > x^\alpha}} 1 = \sum_{\substack{n_k^r t_k - nq = c_k \\ nq \leqslant x \\ n_k > x^\alpha}} \prod_{\substack{1 \leqslant \lambda \leqslant s \\ \lambda \neq k}} \left(\sum_{n_\lambda^r t_\lambda = nq + c_\lambda} 1 \right)$$

$$\leqslant \sum_{\substack{n_k^r t_k - nq = c_k \\ nq \leqslant x \\ n_k > x^\alpha}} \prod_{\substack{1 \leqslant \lambda \leqslant s \\ \lambda \neq k}} d(nq + c_\lambda)$$

$$= O\left(x^\epsilon \sum_{\substack{n_k^r t_k - nq = c_k \\ n_k^r t_k \leqslant x + c_k \\ n_k > x^\alpha}} 1 \right) = O\left(x^\epsilon \sum_{\substack{n_k^r t_k \leqslant x + c_k \\ n_k > x^\alpha}} 1 \right) = O(x^{1-\alpha(r-1)+\epsilon}),$$

so that

$$\Sigma_2 = O(x^{1-\alpha(r-1)+\epsilon}). \tag{11}$$

Putting $\alpha = 1/(r+s-1)$, we have, by (9), (10) and (11),

$$M_r(x; q; c_1, \ldots, c_s) = xq^{-1} \prod_{p^r \nmid q} \left\{1 - \frac{(p^r, q)}{p^r} D(p^r, q \mid c_1, \ldots, c_s\right\} + O(x^{s/(r+s-1)+\epsilon}).$$

The proof is completed by noting that the value of the infinite product appearing in this formula is positive by lemma 6.

THEOREM 2. (i) *If* $D(p^r \mid a_1, \ldots, a_l) = p^r \quad [p]$, *then* $H(x) = 0$.

(ii) *Otherwise,** $\quad H(x) = hx + O(x^{(l+m)/(r+l+m-1)+\epsilon}),$

where†

$$h = h_r(a_1, \ldots, a_l; b_1, \ldots, b_m)$$
$$= \sum_{k=0}^{m} (-1)^k \sum_{1 \leqslant \nu_1 < \ldots < \nu_k \leqslant m} \prod_p \left\{1 - \frac{D(p^r \mid a_1, \ldots, a_l, b_{\nu_1}, \ldots, b_{\nu_k})}{p^r}\right\}$$

and $h > 0$.

Proof. (i) Suppose that

$$D(p^r \mid a_1, \ldots, a_l) = p^r \quad [p].$$

Let $n \geqslant 1$. Then

$$D(p^r \mid n+a_1, \ldots, n+a_l) = p^r \quad [p],$$

and so

$$p^r \mid n+a_i \quad [1 \leqslant i \leqslant l].$$

Hence

$$H(x) = 0.$$

(ii) Next suppose that

$$D(p^r \mid a_1, \ldots, a_l) < p^r \quad (p).$$

Write $t = \max(a_1, \ldots, a_l, b_1, \ldots, b_m)$. Putting $q = 1$ in theorem 1, we obtain

$$M_r(x; c_1, \ldots, c_s) = x \prod_p \left\{1 - \frac{D(p^r \mid c_1, \ldots, c_s)}{p^r}\right\} + O(x^{s/(r+s-1)+\epsilon}). \tag{12}$$

We have

$$H(x) = \sum_{0 \leqslant n \leqslant x-t} \mu_r(n+a_1) \ldots \mu_r(n+a_l)\{1-\mu_r(n+b_1)\} \ldots \{1-\mu_r(n+b_m)\} + O(1)$$
$$= \sum_{k=0}^{m} (-1)^k \sum_{1 \leqslant \nu_1 < \ldots < \nu_k \leqslant m} M_r(x-t; a_1, \ldots, a_l, b_{\nu_1}, \ldots, b_{\nu_k}) + O(1)$$
$$= x \sum_{k=0}^{m} (-1)^k \sum_{1 \leqslant \nu_1 < \ldots < \nu_k \leqslant m} \prod_p \left\{1 - \frac{D(p^r \mid a_1, \ldots, a_l, b_{\nu_1}, \ldots, b_{\nu_k})}{p^r}\right\}$$
$$+ O(x^{(l+m)/(r+l+m-1)+\epsilon}), \quad \text{by (12)},$$
$$= hx + O(x^{(l+m)/(r+l+m-1)+\epsilon}).$$

* If $l = 1$, $m = 0$ or $l = 0$, $m = 1$, then it is seen immediately from (1) that the error term is $O(x^{1/r})$.

† When $l = 0$ the term corresponding to $k = 0$, in the sum expressing h, is interpreted as 1.

It remains to prove that $h > 0$, and this will clearly follow from

$$H(x) > Ax \quad (x > x_0),$$

where A, x_0 are some positive numbers independent of x.

Assume, without loss of generality, that $l > 0$, $m > 0$. Let $p_1, \ldots, p_m$ be any m distinct primes exceeding $t^{1/r}$, and write $q = p_1^r \ldots p_m^r$. Let n_0 be the least positive solution (known to exist by lemma 1) of the system of congruences in n,

$$n + b_k \equiv 0 \pmod{p_k^r} \quad (1 \leqslant k \leqslant m). \tag{13}$$

Then every positive solution of this system is given by

$$n = n_0 + q\nu, \quad \nu = 0, 1, 2, \ldots.$$

Therefore, by (13) and our choice of p_k's,

$$p_k^r \nmid n_0 + a_i \quad (1 \leqslant k \leqslant m,\ 1 \leqslant i \leqslant l). \tag{14}$$

If $p = p_k$ $[1 \leqslant k \leqslant m]$, then

$$D(p^r, q \mid n_0 + a_1, \ldots, n_0 + a_l) = D(p^r, p^r \mid n_0 + a_1, \ldots, n_0 + a_l) = 0 \quad \text{by (14)}.$$

If $p \neq p_k$ $(1 \leqslant k \leqslant m)$, then

$$\begin{aligned} D(p^r, q \mid n_0 + a_1, \ldots, n_0 + a_l) &= D(p^r \mid n_0 + a_1, \ldots, n_0 + a_l) \\ &= D(p^r \mid a_1, \ldots, a_l) < p^r \quad \text{by hypothesis.} \end{aligned}$$

Thus, in any case,

$$D(p^r, q \mid n_0 + a_1, \ldots, n_0 + a_l) < (p^r, q)^{-1} p^r \quad (p). \tag{15}$$

We have

$$\begin{aligned} H(x) &\geqslant \sum_{\substack{0 \leqslant n \leqslant x - t \\ p_k^r \mid n + b_k \\ (1 \leqslant k \leqslant m)}} \mu_r(n + a_1) \ldots \mu_r(n + a_l) \\ &= \sum_{0 \leqslant \nu \leqslant (x - t - n_0)/q} \mu_r(\nu q + n_0 + a_1) \ldots \mu_r(\nu q + n_0 + a_l) \\ &= M_r(x; q; n_0 + a_1, \ldots, n_0 + a_l) + O(1) \\ &> Ax \quad (x > x_0), \end{aligned}$$

by theorem 1 and (15). This completes the proof.

Two special cases

Let C be a specified class of integers. By a *block* of s integers with respect to C we shall understand a sequence of s consecutive positive integers, say n, $n+1$, $\ldots$, $n+s-1$, which are in C, whilst $n-1$ and $n+s$ are not in C. By an easy application of theorem 2 we are able to investigate the frequency of blocks of r-free numbers and of r-numbers.

Let $Q_{r,s}(x)$ denote the number of blocks of s r-free numbers not exceeding x, and $V_{r,s}(x)$ the number of blocks of s r-numbers not exceeding x.

THEOREM 3. (i) *For* $r \geqslant 2$, $s \geqslant 2^r$, $Q_{r,s}(x) = 0$.

(ii) *For* $r \geqslant 2$, $1 \leqslant s \leqslant 2^r - 1$,

$$Q_{r,s}(x) = q_{r,s}x + O(x^{(s+2)/(r+s+1)+\epsilon}),$$

where

$$q_{r,s} = \begin{cases} \prod_p \left(1 - \frac{s}{p^r}\right) - 2\prod_p \left(1 - \frac{s+1}{p^r}\right) + \prod_p \left(1 - \frac{s+2}{p^r}\right) & (1 \leqslant s \leqslant 2^r - 2) \\ \prod_p \left(1 - \frac{2^r - 1}{p^r}\right) & (s = 2^r - 1) \end{cases}$$

and $q_{r,s} > 0$.

Proof. (i) $r \geqslant 2$, $s \geqslant 2^r$ imply $Q_{r,s}(x) = 0$, since of 2^r consecutive integers one must be divisible by 2^r.

(ii) Suppose that $r \geqslant 2$, $1 \leqslant s \leqslant 2^r - 1$, so that

$$D(p^r \mid 2, 3, \ldots, s+1) \leqslant s < 2^r \leqslant p^r \quad (p).$$

Since

$$Q_{r,s}(x) = H_r(x;\, 2, 3, \ldots, s+1;\, 1, s+2) + O(1),$$

it follows by theorem 2 that

$$Q_{r,s}(x) = q_{r,s}x + O(x^{(s+2)/(r+s+1)+\epsilon}),$$

where $q_{r,s} > 0$ and

$$\begin{aligned} q_{r,s} &= \prod_p \left\{1 - \frac{D(p^r \mid 2, 3, \ldots, s+1)}{p^r}\right\} - \prod_p \left\{1 - \frac{D(p^r \mid 1, 2, \ldots, s+1)}{p^r}\right\} \\ &\qquad - \prod_p \left\{1 - \frac{D(p^r \mid 2, 3, \ldots, s+2)}{p^r}\right\} + \prod_p \left\{1 - \frac{D(p^r \mid 1, 2, \ldots, s+2)}{p^r}\right\} \\ &= \begin{cases} \prod_p \left(1 - \frac{s}{p^r}\right) - 2\prod_p \left(1 - \frac{s+1}{p^r}\right) + \prod_p \left(1 - \frac{s+2}{p^r}\right) & (1 \leqslant s \leqslant 2^r - 2) \\ \prod_p \left(1 - \frac{2^r - 1}{p^r}\right) & (s = 2^r - 1). \end{cases} \end{aligned}$$

THEOREM 4. *For* $r \geqslant 2$, $s \geqslant 1$,

$$V_{r,s}(x) = v_{r,s}x + O(x^{(s+2)/(r+s+1)+\epsilon}),$$

where

$$v_{r,s} = \sum_{k=0}^{s} (-1)^k g(k) \prod_{p > (s+1)^{1/r}} \left(1 - \frac{k+2}{p^r}\right),$$

$$g(k) = g_{r,s}(k) = \sum_{1 \leqslant \nu_1 < \ldots < \nu_k \leqslant s} \; \prod_{p \leqslant (s+1)^{1/r}} \left\{1 - \frac{D(p^r \mid 0, \nu_1, \ldots, \nu_k, s+1)}{p^r}\right\},$$

and $v_{r,s} > 0$.

Proof.
$$V_{r,s}(x) = H_r(x;\, 1, s+2;\, 2, 3, \ldots, s+1) + O(1)$$
$$= v_{r,s}\,x + O(x^{(s+2)/(r+s+1)+\epsilon}) \quad \text{by theorem 2.}$$

Here $v_{r,s} > 0$, and

$$v_{r,s} = \sum_{k=0}^{s} (-1)^k \sum_{2 \leqslant \nu_1 < \ldots < \nu_k \leqslant s+1} \prod_p \left\{1 - \frac{D(p^r \mid 1, s+2, \nu_1, \ldots, \nu_k)}{p^r}\right\}$$
$$= \sum_{k=0}^{s} (-1)^k \sum_{1 \leqslant \nu_1 < \ldots < \nu_k \leqslant s} \prod_{p \leqslant (s+1)^{1/r}} \left\{1 - \frac{D(p^r \mid 0, \nu_1, \ldots, \nu_k, s+1)}{p^r}\right\} \prod_{p > (s+1)^{1/r}} \left(1 - \frac{k+2}{p^r}\right).$$

COROLLARY. *For* $r \geqslant 2$, $1 \leqslant s \leqslant 2^r - 2$,

$$v_{r,s} = \sum_{k=0}^{s} (-1)^k \binom{s}{k} \prod_p \left(1 - \frac{k+2}{p^r}\right).$$

Furthermore,
$$v_{r,2^r-1} = (1 - 2^{-r}) \sum_{k=0}^{2^r-2} (-1)^k \binom{2^r-2}{k} \prod_{p>2} \left(1 - \frac{k+2}{p^r}\right),$$
$$v_{r,2^r} = (1 - 2^{1-r}) \sum_{k=0}^{2^r-1} (-1)^k \binom{2^r-1}{k} \prod_{p>2} \left(1 - \frac{k+2}{p^r}\right).$$

These results follow almost immediately from theorem 4.

Added in proof (2 September 1948). Since writing this paper I have pursued the subject a little further, and have been able to reduce the order of the error terms in all the asymptotic formulae given above. For a brief exposition of the improved method see my paper "Note on an asymptotic formula connected with r-free integers", *Quart. J. of Math.* (Oxford), 18 (1947), 178–182.

Department of Mathematics
University of Sheffield

COVERING THEOREMS FOR ORDERED SETS

By R. RADO

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Introduction

1. Let S be an abstract set. The elements of S will be called *points*. S is supposed to be ordered by means of a relation $a<b$. This means that, given any points a and b, exactly one of the three relations

$$a<b, \quad a=b, \quad b<a$$

holds, and the relations

$$a<b, \quad b<c$$

imply

$$a<c.$$

A subset I of S is called an *interval of* S if it satisfies the following condition. Whenever†

$$a<c<b, \quad a,b \prec I$$

then

$$c \prec I.$$

There need not exist any points which play the roles of end-points of I, even in the case of "bounded" intervals. For let S be the set of all rational numbers, ordered according to magnitude, and let I consist of all rational numbers x satisfying

$$2<x^3<3.$$

This fact introduces a certain amount of complication into our arguments.

Let N be an abstract set. The elements of N will be called *indices*. Suppose that with every index ν is associated an interval I_ν of S. The possibility is not excluded that there are pairs of distinct indices ν, ν' satisfying

$$I_\nu = I_{\nu'}.$$

In theorem 1 of this note the following problem is solved. Under what circumstances is it possible to divide the system of intervals I_ν, i.e. the index

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† A relation $A \prec B$ denotes the fact that A is an element of the set B; $B' \subset B$ denotes the fact that B' is a subset of the set B; $B' \subsetneq B$ is equivalent to saying that $B' \subset B$ and $B' \neq B$. The relations $A \not\prec B$, $B' \not\subset B$ are the negations of $A \prec B$, $B' \subset B$ respectively. O denotes the number zero as well as the empty set.

set N, into a given number k of parts in such a way that no two intervals I_ν belonging to the same part have a point in common? An obviously necessary condition is that no $k+1$ intervals I_ν belonging to $k+1$ distinct indices ν should have a point in common. Theorem 1 asserts that this condition is also sufficient. It will be shown that this result is equivalent to the following proposition:

If any system of $k+1$ distinct indices $\nu_1, \nu_2, \ldots, \nu_{k+1}$ can be subdivided into k groups in such a way that no two intervals I_{ν_κ} belonging to indices of the same group have a point in common, then the same kind of subdivision is possible for the whole system N of indices.

Theorem 2 deals with the following problem. Under what circumstances is it possible to find a finite number of subsystems of a given system of intervals I_ν which have the property that no two intervals belonging to the same subsystem have a point in common and, at the same time, the set of points covered by the intervals of the subsystems coincides with the set of points covered by all intervals of the given system I_ν? A necessary and sufficient condition will be shown to be the following. It should be possible to define a well-ordering of the index set N which is such that the set $N(x)$ of all indices ν whose corresponding intervals I_ν contain any given point x possesses a last element. Furthermore, it will be shown that, if any finite number k of subsystems of the given system I_ν can be found which possess the above-mentioned property, then this is always possible with $k = 2$. The main idea of the proof of theorem 2 is closely connected with a method used by Denjoy† for proving a more special covering theorem.

Theorems 3 and 4 are covering theorems for the set of all real numbers. Theorem 3 is an extension of the one-dimensional case of Lindelöf's covering theorem.‡ Theorems 4 and 5 are quantitative versions of theorem 2 in which a "small" set of uncovered points is admitted.

Definitions; statements of theorems 1 *and* 2

2. Sets occurring in this note may have any arbitrary cardinal number. Zermelo's axiom will be used freely. If

$$A_1, A_2, \ldots, A_k$$

are sets, then the symbols

$$A_1 + A_2 + \ldots + A_k, \quad \sum_{\kappa=1}^{k} A_\kappa$$

† Reproduced in Hobson, *The Theory of Functions of a Real Variable*, 1 (2nd edition), 638.

‡ Carathéodory, *Vorlesungen über reelle Funktionen* (2nd edition), 46.

denote their sum-set and the symbols

$$A_1 A_2 \dots A_k, \quad \prod_{\kappa=1}^{k} A_\kappa$$

denote their intersection. A relation

$$(\alpha_1, \alpha_2, \dots, \alpha_r)_{\neq}$$

expresses, by definition, the fact that the objects

$$\alpha_1, \alpha_2, \dots, \alpha_r$$

are different from each other. Unless the contrary is stated, any set is allowed to be empty, and summation indices are allowed to range over the empty set, in which case the value of the sum is taken to be 0.

THEOREM 1. *Suppose that with every element ν of a set N is associated an interval I_ν of an ordered set S. Then it is possible, for a given positive integer k, to find k sets N_κ satisfying*

$$N_1 + N_2 + \dots + N_k = N, \quad I_\nu I_{\nu'} = 0 \quad (\nu \neq \nu';\ \nu, \nu' \prec N_\kappa;\ 1 \leqslant \kappa \leqslant k),$$

if, and only if, the following condition holds. For any choice of $k+1$ distinct indices ν_κ we have

$$I_{\nu_1} I_{\nu_2} \dots I_{\nu_{k+1}} = 0.$$

This condition is equivalent to postulating that every set N' consisting of $k+1$ indices ν can be written in the form

$$N'_1 + N'_2 + \dots + N'_k,$$

where
$$I_\nu I_{\nu'} = 0 \quad (\nu \neq \nu';\ \nu, \nu' \prec N'_\kappa;\ 1 \leqslant \kappa \leqslant k).$$

THEOREM 2. *Suppose that with every element ν of a set N is associated an interval I_ν of an ordered set S. Then it is possible to find a positive number k and to determine k sets N_κ satisfying*

$$N_1 + N_2 + \dots + N_k \subset N,$$

$$\sum_{\nu \prec N} I_\nu = \sum_{\kappa=1}^{k} \sum_{\nu \prec N_\kappa} I_\nu,$$

$$I_\nu I_{\nu'} = 0 \quad (\nu \neq \nu';\ \nu, \nu' \prec N_\kappa;\ 1 \leqslant \kappa \leqslant k),$$

if, and only if, the following condition holds. The set N can be normally ordered† *in such a way that, given any point x of $\sum_{\nu \prec N} I_\nu$, the set $N(x)$ of those indices ν for which x lies in I_ν possesses, in the sense of this ordering, a last element. Moreover, if this condition holds then $k = 2$ is a possible value.*

† I.e. well-ordered (*wohlgeordnet*), see Hobson, *loc. cit.* 211.

The condition in theorem 2 is, in particular, satisfied if every point occurs in only a finite number of intervals I_ν. In the following example no finite number of sets N_κ of the described type can be found:†

$$S = N = \{1, 2, 3, \ldots\}, \qquad I_\nu = \{1, 2, \ldots, \nu\} \quad (\nu \prec N).$$

Lemmas

3. LEMMA 1. *Let U be a set, V be a normally ordered set, with ordering relation "$<$". Let $F(u)$ be a function which associates with every element u of U an element $F(u)$ of V. Then U can be normally ordered (ordering relation "$\lessdot$") in such a way that*

$$F(u) \leqslant F(u') \quad (u \lessdot u').$$

Proof. Define any normal order of U (ordering relation "$\rightarrow$"). Define an order relation "$\lessdot$" in U by putting

$$u \lessdot u'$$

if, and only if, either

$$F(u) < F(u')$$

or

$$F(u) = F(u'), \quad u \rightarrow u'.$$

Then, obviously, for any elements u, u' of U exactly one of the three relations

$$u \lessdot u', \quad u = u', \quad u' \lessdot u$$

is satisfied. Also, the relation $\lessdot$ is transitive. It remains to show that "$\lessdot$" defines a normal order. Assume that there is a sequence $u_1, u_2, \ldots$ of elements of U such that

$$u_1 \gtrdot u_2 \gtrdot u_3 \gtrdot \ldots. \tag{1}$$

Then

$$F(u_1) \geqslant F(u_2) \geqslant \ldots.$$

Since $F(u_n) \prec V$, and since "$<$" defines a normal order, this implies that

$$F(u_{n_0}) = F(u_{n_0+1}) = \ldots$$

for some suitable index n_0. Then, by definition of "$\lessdot$",

$$u_{n_0} \leftarrow u_{n_0+1} \leftarrow u_{n_0+2} \leftarrow \ldots.$$

This is a contradiction against the fact that "$\rightarrow$" defines a normal order. Hence (1) is impossible, and the lemma is proved.

4. For lemmas 2–6 we assume that S is some fixed ordered set (ordering relation "$<$"). Corresponding to any interval I of S we define the set I^+ by putting

$$I^+ = \sum_{x \leqslant y \prec I} \{x\}.$$

Clearly I^+ is an interval.

† The symbol $\{A, B, \ldots, K\}$ denotes the set which has as elements the objects $A, B, \ldots, K$.

LEMMA 2. *Given any intervals I, J, at least one of the relations*

$$I^+ \subset J^+, \quad J^+ \subset I^+ \tag{2}$$

holds.

Proof. If both relations (2) were false then we could find points a, b such that

$$a \prec I^+ - I^+J^+, \quad b \prec J^+ - I^+J^+. \tag{3}$$

Then, by definition of I^+,

$$a \leqslant a' \prec I, \quad b \leqslant b' \prec J,$$

where a', b' are suitable points. Then $a \leqslant b$ would lead to

$$a \leqslant b \leqslant b' \prec J, \quad a \prec J^+,$$

which contradicts (3). Similarly, $b \leqslant a$ is impossible.

5. LEMMA 3. *Let* $r > 0$,

$$I_\rho I_r \neq 0, \quad I_r^+ \subset I_\rho^+ \quad (1 \leqslant \rho \leqslant r).$$

Then

$$I_1 I_2 \ldots I_r \neq 0.$$

Proof. Choose points i_ρ such that

$$i_\rho \prec I_\rho I_r \quad (1 \leqslant \rho \leqslant r).$$

Then, for $1 \leqslant \sigma \leqslant r$,

$$i_\rho \prec I_r \subset I_r^+ \subset I_\sigma^+.$$

Hence

$$i_\rho \leqslant i_{\rho\sigma} \prec I_\sigma.$$

There is an index ρ_0 satisfying $1 \leqslant \rho_0 \leqslant r$,

$$i_\sigma \leqslant i_{\rho_0} \quad (1 \leqslant \sigma \leqslant r).$$

Then

$$i_\sigma \leqslant i_{\rho_0} \leqslant i_{\rho_0\sigma}, \quad i_\sigma, i_{\rho_0\sigma} \prec I_\sigma,$$

and therefore, by definition of intervals,

$$i_{\rho_0} \prec I_\sigma \quad (1 \leqslant \sigma \leqslant r), \qquad I_1 I_2 \ldots I_r \neq 0.$$

6. For lemmas 4–6 we assume that there is a set N and, associated with every element ν of N, an interval I_ν. The letters α, β, γ, ν denote typical "indices", i.e. elements of N. We shall assume, without loss of generality, that

$$I_\nu \neq 0 \quad (\nu \prec N).$$

This is justified by the consideration that, for the purpose of proving theorem 1, it is immaterial to which of the sets N_κ any ν belongs for which $I_\nu = 0$. It is, however, important to bear in mind that $\alpha \neq \beta$ does not imply $I_\alpha \neq I_\beta$.

A relation
$$\alpha \Lambda \beta$$
expresses, by definition, the fact that either
$$I_\alpha = I_\beta$$
or
$$I_\alpha I_\beta \neq 0,$$
$$I_\alpha \not\subset I_\beta, \quad I_\beta \not\subset I_\alpha.$$

LEMMA 4. *Suppose that* $r > 0$,
$$\nu_\rho \Lambda \nu_r \quad (1 \leqslant \rho \leqslant r).$$
Put
$$I_{\nu_\rho} = J_\rho.$$
Then the set $\{1, 2, \ldots, r-1\}$ *can be written in the form*
$$\{s_1, s_2, \ldots, s_p\} + \{t_1, t_2, \ldots, t_q\},$$
where $p, q \geqslant 0$,
$$p + q = r - 1 \tag{4}$$
and†
$$J_{s_1} J_{s_2} \ldots J_{s_p} J_r \neq 0, \tag{5}$$
$$J_{t_1} J_{t_2} \ldots J_{t_q} J_r \neq 0. \tag{6}$$

Proof. By renumbering the indices $\nu_1, \nu_2, \ldots, \nu_r$ it is possible to obtain a case in which, for some r_1 in the range $0 \leqslant r_1 < r$,
$$J_\rho \neq J_r \quad (1 \leqslant \rho \leqslant r_1), \qquad J_\rho = J_r \quad (r_1 < \rho \leqslant r). \tag{7}$$
The hypothesis implies the existence of points a_ρ, b_ρ, c_ρ such that
$$a_\rho \prec J_r - J_\rho J_r, \quad b_\rho \prec J_\rho - J_\rho J_r, \qquad c_\rho \prec J_\rho J_r \quad (1 \leqslant \rho \leqslant r_1). \tag{8}$$
A further rearrangement of the indices $\nu_1, \ldots, \nu_{r_1}$ leads to a case in which, in addition to (7) and (8),
$$b_\rho \leqslant a_\rho \quad (1 \leqslant \rho \leqslant r_2), \qquad a_\rho < b_\rho \quad (r_2 < \rho \leqslant r_1).$$
Here r_2 is a number in the range $0 \leqslant r_2 \leqslant r_1$. Then
$$b_\rho < c_\rho < a_\rho \quad (1 \leqslant \rho \leqslant r_2), \tag{9}$$
$$a_\rho < c_\rho < b_\rho \quad (r_2 < \rho \leqslant r_1). \tag{10}$$
For if (9) does not hold then, for some $\rho \leqslant r_2$, either (i) $c_\rho \leqslant b_\rho \leqslant a_\rho$ or (ii) $b_\rho \leqslant a_\rho \leqslant c_\rho$. By (8) and by definition of intervals (i) implies $b_\rho \prec J_r$ and (ii) implies $a_\rho \prec J_\rho$. Hence in either case a contradiction follows. Similarly (10) is deduced. We have, furthermore,
$$b_\rho < c_\sigma \quad (1 \leqslant \rho, \sigma \leqslant r_2), \tag{11}$$

† The cases $p = 0$ or $q = 0$ are to be interpreted in the obvious way. The same holds throughout this paper.

since otherwise, for some pair ρ, σ satisfying $1 \leqslant \rho, \sigma \leqslant r_2$, we would have $c_\sigma \leqslant b_\rho < c_\rho$ and therefore, by (8), $b_\rho \prec J_r$, which is false. Similarly,

$$c_\rho < b_\sigma \quad (r_2 < \rho, \sigma \leqslant r_1), \tag{12}$$

since otherwise, for some ρ, σ satisfying $r_2 < \rho$, $\sigma \leqslant r_1$, we would have $c_\sigma < b_\sigma \leqslant c_\rho$, $b_\sigma \prec J_r$, which is a contradiction.

We now show that the assertion of the lemma holds when

$$\{s_1, s_2, \ldots, s_p\} = \{1, 2, \ldots, r_2\}, \quad \{t_1, t_2, \ldots, t_q\} = \{r_2 + 1, \ldots, r-1\}.$$

If, first of all, $r_2 > 0$ then there exists ρ_0 such that

$$1 \leqslant \rho_0 \leqslant r_2, \qquad c_{\rho_0} \leqslant c_\rho \quad (1 \leqslant \rho \leqslant r_2).$$

Then, by (11),

$$b_\rho < c_{\rho_0} \leqslant c_\rho, \qquad b_\rho, c_\rho \prec J_\rho, \qquad c_{\rho_0} \prec J_\rho \quad (1 \leqslant \rho \leqslant r_2), \qquad c_{\rho_0} \prec J_r,$$
$$J_1 J_2 \ldots J_{r_2} J_r \neq 0. \tag{13}$$

This inequality also holds if $r_2 = 0$.

Similarly, if $r_2 < r_1$ then there is ρ_1 such that $r_2 < \rho_1 \leqslant r_1$, $c_\rho \leqslant c_{\rho_1}$ $(r_2 < \rho \leqslant r_1)$. Then, by (12),

$$c_\rho \leqslant c_{\rho_1} < b_\rho, \qquad c_{\rho_1} \prec J_\rho \quad (r_2 < \rho \leqslant r_1),$$
$$J_{r_2+1} \ldots J_{r_1} J_{r_1+1} \ldots J_r \neq 0. \tag{14}$$

This inequality also holds if $r_2 = r_1$. Relations (13) and (14) prove the lemma.

7. Define a relation

$$\alpha \sim \beta \tag{15}$$

as follows. (15) holds if, and only if, there exists an integer $r \geqslant 0$ and indices $\nu_0, \nu_1, \ldots, \nu_r$ satisfying

$$\nu_0 = \alpha, \qquad \nu_r = \beta, \qquad \nu_{\rho-1} \Lambda \nu_\rho \quad (1 \leqslant \rho \leqslant r).$$

In particular, $\alpha \sim \alpha$ for every α. The relation (15) is symmetrical and transitive and therefore defines a distribution of all elements of N into non-overlapping classes, called Λ-classes or simply classes. The class to which α_0 belongs is denoted by $K(\alpha_0)$, and any typical non-empty classes are denoted by K, K'. Thus, for every K,

$$K = K(\alpha_0) = \sum_{\alpha \sim \alpha_0} \{\alpha\} \quad (\alpha_0 \prec K).$$

The relation

$$\alpha \nsim \beta$$

is meant to express the fact that (15) does not hold.

8. LEMMA 5. *Suppose that*

$$\alpha_0 \nsim \beta_0, \tag{16}$$

$$I_{\alpha_0} I_{\beta_0} \neq 0. \tag{17}$$

Then either $$I_\alpha \Subset I_{\beta_0} \quad (\alpha \sim \alpha_0) \tag{18}$$

or, if (18) *is false,* $$I_\beta \Subset I_{\alpha_0} \quad (\beta \sim \beta_0). \tag{19}$$

Proof. Relations (18) and (19) cannot hold simultaneously. For if one applies (18) to $\alpha = \alpha_0$ and (19) to $\beta = \beta_0$, then the contradiction

$$I_{\alpha_0} \Subset I_{\beta_0} \Subset I_{\alpha_0}$$

follows. We now show that at least one of the relations (18), (19) is true.

By (16) we have $I_{\alpha_0} \neq I_{\beta_0}$. Therefore it is no restriction to assume that

$$I_{\beta_0} \not\subset I_{\alpha_0}. \tag{20}$$

Then $$I_{\alpha_0} \Subset I_{\beta_0}. \tag{21}$$

For otherwise we would have $$I_{\alpha_0} \not\subset I_{\beta_0},$$

which, together with (20) and (17), would imply $\alpha_0 \Lambda \beta_0$ and hence $\alpha_0 \sim \beta_0$ which contradicts (16). We shall now establish (18). Let us assume that there exists an index α satisfying

$$\alpha \sim \alpha_0, \quad I_\alpha \not\subset I_{\beta_0}. \tag{22}$$

Our aim is to deduce a contradiction.

One can find a number $r \geqslant 0$ and indices α_ρ satisfying

$$\alpha_r = \alpha, \qquad \alpha_{\rho-1} \Lambda \alpha_\rho \quad (1 \leqslant \rho \leqslant r).$$

We deduce from (21) and (22) that $\alpha_0 \neq \alpha$. Therefore $r > 0$. Let ρ_0 be the largest number ρ satisfying

$$0 \leqslant \rho \leqslant r, \quad I_{\alpha_\rho} \subset I_{\beta_0}.$$

The existence of ρ_0 follows from (21) and (22). In fact, $0 \leqslant \rho_0 < r$. Then

$$0 \neq I_{\alpha_{\rho_0}} I_{\alpha_{\rho_0+1}} \subset I_{\beta_0} I_{\alpha_{\rho_0+1}}, \quad I_{\alpha_{\rho_0+1}} \not\subset I_{\beta_0}.$$

If $$I_{\beta_0} \not\subset I_{\alpha_{\rho_0+1}},$$

then it follows that $$\alpha_{\rho_0+1} \Lambda \beta_0, \quad \alpha_0 \sim \alpha_{\rho_0+1} \sim \beta_0,$$

which contradicts (16). Hence

$$I_{\beta_0} \Subset I_{\alpha_{\rho_0+1}}, \quad I_{\alpha_{\rho_0}} \subset I_{\beta_0} \Subset I_{\alpha_{\rho_0+1}}.$$

But this is impossible in view of the fact that $\alpha_{\rho_0} \Lambda \alpha_{\rho_0+1}$. We have obtained the desired contradiction. Our argument proves that

$$I_\alpha \subset I_{\beta_0} \quad (\alpha \sim \alpha_0).$$

The proof of (18) is completed by noting that the relations

$$\alpha' \sim \alpha_0, \quad I_{\alpha'} = I_{\beta_0}$$

would imply that $\alpha' \Lambda \beta_0$, and hence $\alpha_0 \sim \alpha' \sim \beta_0$, which contradicts (16).

9. Let N' be a subset of N which has the following property. There exists a constant C such that the relations

$$\nu_1, \nu_2, \ldots, \nu_m \prec N', \quad (\nu_1, \ldots, \nu_m)_+, \qquad I_{\nu_1} I_{\nu_2} \ldots I_{\nu_m} \neq 0$$

imply $m \leqslant C$. Then we denote by

$$R(N')$$

the smallest number C of this kind. For all other sets $N' \subset N$ we put

$$R(N') = \infty.$$

Thus

$$R(0) = 0.$$

Clearly

$$R(N' + N'') \leqslant R(N') + R(N''). \tag{23}$$

The main part of theorem 1 can be expressed as follows. If

$$R(N) \leqslant k, \tag{24}$$

then there are k sets N_κ such that

$$N_1 + N_2 + \ldots + N_k = N, \qquad R(N_\kappa) \leqslant 1 \quad (1 \leqslant \kappa \leqslant k).$$

10. LEMMA 6. *If* (24) *holds then every class* K *contains at most enumerably many indices* ν.

Proof. Let ν^* be any fixed element of K. Let $r > 0$,

$$(\nu_1, \nu_2, \ldots, \nu_{r-1}, \nu^*)_+, \qquad \nu_\rho \Lambda \nu^* \quad (1 \leqslant \rho < r).$$

Apply lemma 4, with $\nu_r = \nu^*$. We see from (5) and (24) that, in the notations of lemma 4,

$$p + 1 \leqslant R(N) \leqslant k,$$

and from (6) and (24) that

$$q + 1 \leqslant R(N) \leqslant k.$$

Therefore, by adding and using (4),

$$r + 1 \leqslant 2k.$$

Hence there are only a finite number of different indices ν satisfying $\nu \Lambda \nu^*$. Therefore, for any given number s, the number of different systems $\alpha_0, \alpha_1, \ldots, \alpha_s$, such that

$$\nu^* = \alpha_0, \qquad \alpha_{\sigma-1} \Lambda \alpha_\sigma \quad (1 \leqslant \sigma \leqslant s) \tag{25}$$

holds, is finite. If now s and the α_σ are allowed to vary, subject to their respective restrictions, then α_s can be made to coincide with any element α

of K. Let $s_0(\alpha)$ denote the smallest value of s corresponding to which $s+1$ indices α_σ can be found satisfying $\alpha_s = \alpha$ and (25). Then for any number t there are only a finite number of elements α of K satisfying $s_0(\alpha) = t$. Therefore all elements of K can be arranged in a sequence

$$\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \ldots$$

such that $$s_0(\alpha^{(1)}) \leqslant s_0(\alpha^{(2)}) \leqslant \ldots .$$

Proof of theorem 1

11. We begin by proving the trivial part of theorem 1. Let us suppose that

$$N_1 + N_2 + \ldots + N_k = N, \qquad R(N_\kappa) \leqslant 1 \quad (1 \leqslant \kappa \leqslant k).$$

Then, by (23), $$R(N) \leqslant R(N_1) + \ldots + R(N_k) \leqslant k.$$

Hence $$I_{\nu_1} I_{\nu_2} \ldots I_{\nu_{k+1}} = 0,$$

whenever $$(\nu_1, \ldots, \nu_{k+1})_+.$$

12. Let N' be a set consisting of $k+1$ indices ν. Our next aim is to show the equivalence of the following two statements:

(A) There exist k sets N'_κ satisfying

$$N'_1 + N'_2 + \ldots + N'_k = N', \tag{26}$$

$$R(N'_\kappa) \leqslant 1 \quad (1 \leqslant \kappa \leqslant k). \tag{27}$$

(B) $$\prod_{\nu \prec N'} I_\nu = 0.$$

Let $$N' = \{\nu_1, \ldots, \nu_{k+1}\}.$$

Then $$(\nu_1, \ldots, \nu_{k+1})_+.$$

First of all, suppose that (A) holds. Then there are two of the $k+1$ indices ν_κ which lie in the same set $N'_{\kappa'}$, say

$$\nu_\lambda, \nu_\mu \prec N'_{\kappa'},$$

where $$1 \leqslant \kappa' \leqslant k, \quad 1 \leqslant \lambda < \mu \leqslant k+1.$$

Then, by (27), $$\prod_{\nu \prec N'} I_\nu \subset I_{\nu_\lambda} I_{\nu_\mu} = 0,$$

which proves (B).

Next suppose that (A) is false. By lemma 2, we may assume that our notation is such that

$$I^+_{\nu_{k+1}} \subset I^+_{\nu_k} \subset \ldots \subset I^+_{\nu_1}.$$

If $$I_{\nu_\kappa} I_{\nu_{k+1}} \neq 0 \quad (1 \leqslant \kappa \leqslant k+1)$$

then we conclude, by means of lemma 3, that

$$I_{\nu_1} I_{\nu_2} \ldots I_{\nu_{k+1}} \neq 0,$$

i.e. that (B) is false. If, on the other hand,

$$I_{\nu_{\kappa'}} I_{\nu_{k+1}} = 0$$

for some index κ' in the range $1 \leqslant \kappa' \leqslant k+1$, then $\kappa' \leqslant k$. Call the k sets

$$\{\nu_1\}, \{\nu_2\}, \ldots, \{\nu_{\kappa'-1}\}, \{\nu_{\kappa'}, \nu_{k+1}\}, \{\nu_{\kappa'+1}\}, \ldots, \{\nu_k\},$$

in any order $$N'_1, N'_2, \ldots, N'_k.$$

Then (26) and (27) hold. But this is a contradiction since we assumed (A) to be false.

We have proved the equivalence of (A) and (B), and we now come to the proof of the main part of theorem 1.

13. *Case* 1. Suppose that N is a finite set. Then we may assume that

$$N = \{1, 2, \ldots, n\}.$$

We are given that $$R(N) \leqslant k.$$

In view of lemma 2 we can choose our notation in such a way that

$$I_n^+ \subset I_{n-1}^+ \subset \ldots \subset I_1^+.$$

We define a function $f(\nu)$ $(1 \leqslant \nu \leqslant n)$ as follows. Let $\nu_0 \prec N$. Suppose† that $f(\nu)$ has already been defined for all $\nu < \nu_0$, and furthermore, that this function has the property that

$$f(\nu) \prec \{1, 2, \ldots, k\} \quad (\nu < \nu_0),$$

$$I_\alpha I_\beta = 0 \quad (\alpha < \beta < \nu_0, \quad f(\alpha) = f(\beta)).$$

Then let $f(\nu_0)$, by definition, be the smallest integer κ such that

$$1 \leqslant \kappa \leqslant k+1,$$

$$I_\nu I_{\nu_0} = 0 \quad (\nu < \nu_0, \quad f(\nu) = \kappa).$$

$f(\nu_0)$ exists because $k+1$ is a number κ, not necessarily the smallest, of the required type. Put $f(\nu_0) = k_0$. Then the definition of k_0 implies the existence of $k_0 - 1$ indices ν_κ such that

$$\nu_\kappa < \nu_0, \quad f(\nu_\kappa) = \kappa, \qquad I_{\nu_\kappa} I_{\nu_0} \neq 0 \quad (1 \leqslant \kappa < k_0).$$

Also $$I_{\nu_0}^+ \subset I_{\nu_\kappa}^+ \quad (0 \leqslant \kappa < k_0).$$

Therefore, by lemma 3,

$$I_{\nu_0} I_{\nu_1} \ldots I_{\nu_{k_0-1}} \neq 0. \tag{28}$$

† If $\nu_0 = 1$ then no assumption is made.

Furthermore, $$(f(\nu_0), f(\nu_1), \ldots, f(\nu_{k_0-1}))_{\neq},$$

and hence $$(\nu_0, \nu_1, \ldots, \nu_{k_0-1})_{\neq}. \tag{29}$$

We deduce from (28) and (29) that

$$k_0 \leqslant R(N) \leqslant k.$$

Thus the definition of the function $f(\nu)$ has been extended to the range $\nu \leqslant \nu_0$, and

$$f(\nu) \prec \{1, 2, \ldots, k\} \quad (\nu \leqslant \nu_0),$$

$$I_\alpha I_\beta = 0 \quad (\alpha < \beta \leqslant \nu_0, \quad f(\alpha) = f(\beta)).$$

In this way the inductive definition of $f(\nu)$ for all ν of N has been accomplished. The resulting function satisfies the conditions

$$f(\nu) \prec \{1, 2, \ldots, k\} \quad (\nu \prec N),$$

$$I_\alpha I_\beta = 0 \quad (\alpha \neq \beta, \quad f(\alpha) = f(\beta)).$$

Let N_κ be the set of all ν such that

$$f(\nu) = \kappa. \tag{30}$$

Then $$N_1 + N_2 + \ldots + N_k = N, \tag{31}$$

$$R(N_\kappa) \leqslant 1 \quad (1 \leqslant \kappa \leqslant k). \tag{32}$$

Theorem 1 is proved for finite sets N.

14. *Case* 2. Suppose that N is enumerable. We may assume that

$$N = \{1, 2, 3, \ldots\}.$$

By § 13 there exists, for every $n \prec N$, a function $f_n(\nu)$, defined for $\nu \leqslant n$ and satisfying

$$f_n(\nu) \prec \{1, 2, \ldots, k\} \quad (\nu \leqslant n),$$

$$I_\alpha I_\beta = 0 \quad (\alpha < \beta \leqslant n, \quad f_n(\alpha) = f_n(\beta)). \tag{33}$$

By means of Cantor's diagonal process one can find a sequence $n' \to \infty$ which has the property that the limit

$$\lim_{n' \to \infty} f_{n'}(\nu) = f(\nu),$$

say, exists for every ν. Then there are numbers m_ν such that

$$m_\nu \geqslant \nu, \qquad f_{n'}(\nu) = f(\nu) \quad (n' \geqslant m_\nu).$$

Let N_κ be the set of all ν satisfying (30). Then (31) holds. If

$$\alpha, \beta \prec N_\kappa, \quad \alpha < \beta,$$

then, for any $n' \geqslant \min(m_\alpha, m_\beta)$,

$$f_{n'}(\alpha) = f(\alpha) = \kappa = f(\beta) = f_{n'}(\beta),$$

and therefore, by (33), $\qquad I_\alpha I_\beta = 0.$

This proves (32).

15. *Case* 3. Suppose that N is an arbitrary set. If K, K' are any Λ-classes then we introduce the symbol

$$K \triangle K' \tag{34}$$

as an expression of the fact that there exists an index α' of K' satisfying

$$I_\alpha \Subset I_{\alpha'} \quad (\alpha \prec K).$$

At present we shall not require any general properties of the relation (34).

Let r be a positive integer, and suppose that the classes $K_0, K_1, \ldots, K_r$ satisfy

$$K_0 \triangle K_1 \triangle K_2 \triangle \ldots \triangle K_r. \tag{35}$$

Then there are indices $\alpha_1, \alpha_2, \ldots, \alpha_r$ satisfying

$$\alpha_\rho \prec K_\rho, \qquad I_\alpha \Subset I_{\alpha_\rho} \quad (1 \leqslant \rho \leqslant r,\ \alpha \prec K_{\rho-1}).$$

Since K_0 is not empty we can choose an element α_0 of K_0. Then, by definition of (34),

$$I_{\alpha_0} \Subset I_{\alpha_1} \Subset I_{\alpha_2} \Subset \ldots \Subset I_{\alpha_r}.$$

Hence $\qquad (\alpha_0, \alpha_1, \ldots, \alpha_r)_+, \quad I_{\alpha_0} I_{\alpha_1} I_{\alpha_2} \ldots I_{\alpha_r} = I_{\alpha_0} \neq 0,$

and thus $\qquad r + 1 \leqslant R(N) \leqslant k.$

It follows that, given any class K_0, there exists a largest number r, denoted, say, by

$$r = F(K_0),$$

which has the property that (35) is satisfied for suitable classes $K_1, \ldots, K_r$. Then

$$0 \leqslant F(K_0) < k.$$

If for a given class K there is no class K' satisfying (34) then we put $F(K) = 0$.

Apply lemma 1 to the sets

$$U = \Sigma\{K\}, \quad V = \{0, 1, \ldots, k-1\}.$$

We obtain the existence of a normal order of U (ordering relation "$\lessdot$") which has the property that

$$F(K') \leqslant F(K'') \quad (K' \lessdot K''). \tag{36}$$

I shall now define, by means of transfinite induction, a certain function $f(\nu)$. Let K_0 be any fixed class, and assume† that

(i) $f(\nu)$ has already been defined for every index ν satisfying, for some suitable class K, the relation

$$\nu \prec K \lessdot K_0;$$

(ii) $$f(\nu) \prec \{1, 2, \ldots, k\} \quad (\nu \prec K \lessdot K_0);$$

(iii) $$I_\alpha I_\beta = 0$$

whenever $$\alpha \prec K' \lessdot K_0, \quad \beta \prec K'' \lessdot K_0, \quad \alpha \neq \beta, \quad f(\alpha) = f(\beta).$$

I shall then extend the range of definition of $f(\nu)$ by defining $f(\nu)$ for every ν of K_0, and the extended function will satisfy

(ii$a$) $$f(\nu) \prec \{1, 2, \ldots, k\} \quad (\nu \prec K \leqslant K_0);$$

(iii$a$) $$I_\alpha I_\beta = 0$$

whenever $$\alpha \prec K' \leqslant K_0, \quad \beta \prec K'' \leqslant K_0, \quad \alpha \neq \beta, \quad f(\alpha) = f(\beta).$$

This definition of a function $f(\nu)$ by transfinite induction proceeds by means of an extension of the region of definition not by an individual new index ν_0, as is commonly the case, but by all indices ν belonging to some non-empty class K_0. No two classes K overlap, and the set of all these classes has been normally ordered. This procedure constitutes a valid definition of a function. The function $f(\nu)$ thus obtained will be defined for all ν and will satisfy

$$f(\nu) \prec \{1, 2, \ldots, k\} \quad (\nu \prec N), \qquad I_\alpha I_\beta = 0 \quad (\alpha \neq \beta, f(\alpha) = f(\beta)).$$

Hence the sets N_κ, again defined as consisting of all indices satisfying (30), will have the properties (31) and (32), and the proof of theorem 1 will be completed.

We now proceed to describe the extension of a function $f(\nu)$ which is supposed to have properties (i), (ii) and (iii) above. Let α_0 be any fixed element of K_0. Suppose that an index ν_1 and a class K_1 are such that

$$\nu_1 \prec K_1 \lessdot K_0, \tag{37}$$

$$I_{\alpha_0} \subset I_{\nu_1}. \tag{38}$$

Then $$\alpha_0 \not\prec K_1, \quad \alpha_0 \nsim \nu_1, \quad I_{\alpha_0} I_{\nu_1} = I_{\alpha_0} \neq 0.$$

Hence, by lemma 5, either

$$I_\alpha \subset I_{\nu_1} \quad (\alpha \prec K_0), \tag{39}$$

or $$I_\nu \subset I_{\alpha_0} \quad (\nu \prec K_1).$$

The last relation is false, in view of (38).

† If, in the normal order of U, K_0 is the first element then no assumption is made.

Let $N^{(1)}$ be the set of all indices ν_1 satisfying (38) corresponding to which there exists a class K_1 satisfying (37). Then, by (39),

$$I_\alpha \subset I_\nu \quad (\alpha \prec K_0,\ \nu \prec N^{(1)}). \tag{40}$$

The set $N^{(1)}$ may be empty. I assert that $N^{(1)}$ is a finite set containing, say, $n^{(1)}$ elements, and that

$$R(K_0) \leqslant k - n^{(1)}. \tag{41}$$

In order to see this, choose any positive integer m, and suppose that the indices $\alpha_1, \alpha_2, \ldots, \alpha_m$ satisfy

$$\alpha_1, \alpha_2, \ldots, \alpha_m \prec K_0, \tag{42}$$

$$(\alpha_1, \alpha_2, \ldots, \alpha_m)_{\neq}, \tag{43}$$

$$I_{\alpha_1} I_{\alpha_2} \ldots I_{\alpha_m} \neq 0. \tag{44}$$

Then, by (42) and (40),

$$I_{\alpha_\mu} \subset I_\nu \quad (1 \leqslant \mu \leqslant m,\ \nu \prec N^{(1)}).$$

Hence, whenever $r > 0$ and

$$\nu_\rho \prec N^{(1)} \quad (1 \leqslant \rho \leqslant r), \tag{45}$$

$$(\nu_1, \nu_2, \ldots, \nu_r)_{\neq}, \tag{46}$$

we have

$$I_{\alpha_1} I_{\alpha_2} \ldots I_{\alpha_m} I_{\nu_1} \ldots I_{\nu_r} = I_{\alpha_1} \ldots I_{\alpha_m} \neq 0. \tag{47}$$

Also, by (45) and the definition of $N^{(1)}$, there are classes K_ρ satisfying

$$\nu_\rho \prec K_\rho \lessdot K_0 \quad (1 \leqslant \rho \leqslant r).$$

Hence

$$\nu_\rho \neq \alpha_\mu \quad (1 \leqslant \rho \leqslant r,\ 1 \leqslant \mu \leqslant m).$$

Thus

$$(\alpha_1, \ldots, \alpha_m, \nu_1, \ldots, \nu_r)_{\neq},$$

and therefore, by (47),

$$m + r \leqslant R(N) \leqslant k.$$

This holds for any choice of any number of indices ν_ρ satisfying (45) and (46). Therefore $N^{(1)}$ is a finite set containing, say, exactly $n^{(1)}$ elements, and

$$m + n^{(1)} \leqslant k, \quad m \leqslant k - n^{(1)}.$$

This is true for every choice of any number of indices α_μ satisfying (42), (43) and (44). Hence (41).

According to (37), $f(\nu)$ is defined for every element ν of $N^{(1)}$. The set

$$A = \sum_{\nu \prec N^{(1)}} \{f(\nu)\} \tag{48}$$

contains at most† $n^{(1)}$ elements. Hence the set

$$B = \{1, 2, \ldots, k\} - A$$

† In fact, A contains exactly $n^{(1)}$ elements, as is easily seen from (38) and (iii).

contains at least $k-n^{(1)}$, and therefore, by (41), at least $R(K_0)$ elements. Put

$$R(K_0) = k_0.$$

By lemma 6, K_0 contains at most enumerably many elements. Hence, by case 1 or case 2 of the present proof, there exist k_0 sets $N_\kappa^{(0)}$ satisfying

$$N_1^{(0)} + N_2^{(0)} + \ldots + N_{k_0}^{(0)} = K_0, \tag{49}$$

$$R(N_\kappa^{(0)}) \leqslant 1 \quad (1 \leqslant \kappa \leqslant k_0). \tag{50}$$

Let $$B = \{b_1, b_2, \ldots, b_t\},$$

where $$(b_1, b_2, \ldots, b_t)_{\neq}.$$

Then $t \geqslant k_0$. We extend the range of definition of the function $f(\nu)$ by putting

$$f(\nu) = b_\kappa \quad (\nu \prec N_\kappa^{(0)},\ 1 \leqslant \kappa \leqslant k_0). \tag{51}$$

We have to show that (ii$a$) and (iii$a$) are satisfied. It is obvious that (ii$a$) holds. In order to prove (iii$a$), assume that there are indices α_1, β_1 and classes K', K'' such that

$$\alpha_1 \prec K' \leqslant K_0, \quad \beta_1 \prec K'' \leqslant K_0, \quad \alpha_1 \neq \beta_1, \quad f(\alpha_1) = f(\beta_1), \quad I_{\alpha_1} I_{\beta_1} \neq 0. \tag{52}$$

We have to deduce a contradiction. There is no loss of generality in assuming that

$$K' \leqslant K''.$$

If, in addition, $$K'' < K_0,$$

then (52) would contradict (iii). Therefore

$$K' \leqslant K'' = K_0.$$

If $$K' = K'' = K_0, \tag{53}$$

then, by (52) and (51), $$f(\alpha_1) = f(\beta_1) = b_{\kappa_1},$$

where κ_1 is some number in the range $1 \leqslant \kappa_1 \leqslant k_0$. Then

$$\alpha_1, \beta_1 \prec N_{\kappa_1}^{(0)},$$

and therefore, by (50), $$I_{\alpha_1} I_{\beta_1} = 0.$$

This is a contradiction against (52). Hence (53) is false, and

$$K' < K'' = K_0. \tag{54}$$

Then $$\alpha_1 \nsim \beta_1, \quad I_{\alpha_1} I_{\beta_1} \neq 0.$$

Therefore, by lemma 5, either

$$I_\alpha \subset I_{\beta_1} \quad (\alpha \prec K') \tag{55}$$

or $$I_\beta \subset I_{\alpha_1} \quad (\beta \prec K''). \tag{56}$$

Case A. Suppose that (55) holds. Then, by definition of (34),

$$K' \triangle K''.$$

Put $F(K'') = s$. Then, by definition of $F(K'')$, there are s classes K_σ satisfying

$$K' \triangle K'' \triangle K_1 \triangle K_2 \triangle \dots \triangle K_s.$$

Then, by definition of $F(K')$,

$$F(K') \geqslant s+1 > F(K'').$$

On the other hand, (54) and (36) lead to

$$F(K') \leqslant F(K''),$$

which is the desired contradiction.

Case B. Suppose that (56) holds. Then we may put, in (56), $\beta = \alpha_0$ and obtain

$$I_{\alpha_0} \Subset I_{\alpha_1}.$$

Also, by (54), $$\alpha_1 \prec K' \lessdot K_0.$$

Therefore, by definition of $N^{(1)}$, $\alpha_1 \prec N^{(1)}$. Hence, by (48),

$$f(\alpha_1) \prec A. \tag{57}$$

On the other hand, $$\beta_1 \prec K'' = K_0,$$

and therefore, by (49) and (51),

$$f(\beta_1) \prec B = \{1, 2, \dots, k\} - A, \quad f(\beta_1) \not\prec A. \tag{58}$$

We deduce from (57) and (58) that

$$f(\alpha_1) \neq f(\beta_1),$$

which contradicts (52). This completes the proof of theorem 1.

Proof of theorem 2

16. As in theorem 1, we are given a set N and, corresponding to every element ν of N, an interval I_ν of the ordered set S. We introduce the notation

$$[N'] = \sum_{\nu \prec N'} I_\nu \quad (N' \subset N).$$

In order to prove one part of theorem 2, we assume the existence of a finite number of sets $N_1, \dots, N_k$ satisfying

$$R(N_\kappa) \leqslant 1 \quad (1 \leqslant \kappa \leqslant k), \tag{59}$$

$$[N_1 + N_2 + \dots + N_k] = [N]. \tag{60}$$

We want to prove that N can be normally ordered in such a way that, given any element x of $[N]$, the set

$$N(x) = \sum_{x \prec I_\nu} \{\nu\}$$

contains, in the sense of this normal order, a last element.

Define any normal order of the set

$$N' = N_1 + \ldots + N_k$$

and any normal order of the set

$$N'' = N - N',$$

the ordering relation in both cases being denoted by "$<$". Extend this ordering relation so as to obtain an ordering of N, by putting, in addition,

$$a < b \quad (a \prec N'', b \prec N'). \tag{61}$$

It is well known that this process leads to a normal ordering of the set $N = N' + N''$. If now

$$x \prec [N],$$

then, by (60),

$$x \prec [N_\kappa] \tag{62}$$

for at least one index κ in the range $1 \leqslant \kappa \leqslant k$. Let $\kappa_1, \kappa_2, \ldots, \kappa_m$ be all distinct indices κ satisfying (62). Then we have $1 \leqslant m \leqslant k$. If $1 \leqslant \mu \leqslant m$ then there exists, by (59), exactly one index ν_μ such that

$$x \prec I_{\nu_\mu}, \quad \nu_\mu \prec N_{\kappa_\mu}.$$

Hence the set $N'N(x)$ is finite and not empty. It possesses, in the sense of the normal order defined above, a last element ν^*. Then ν^* is the last element of $N(x)$. For let $\nu \prec N(x) = N'N(x) + N''N(x)$. If $\nu \prec N'N(x)$, then, by definition of ν^*, $\nu \leqslant \nu^*$. If $\nu \prec N''N(x) \subset N''$ then, by (61), $\nu < \nu^*$. Hence the condition stated in theorem 2 is necessary.

17. In this section we assume that N has been normally ordered, ordering relation "$<$", in such a way that, given any point x of $[N]$, the set $N(x)$ contains a last element. For convenience we define a symbol

$$t * M,$$

where t is either the number 0 or the number 1 and M is any set, by putting

$$t * M = \begin{cases} \text{the empty set} & (t = 0) \\ M & (t = 1). \end{cases}$$

I define, by means of transfinite induction, a function $f(\nu)$. Let $\nu_0 \prec N$, and suppose that for every $\nu < \nu_0$ a number $f(\nu)$ has already been defined satisfying

$$f(\nu) \prec \{0, 1\} \quad (\nu < \nu_0).$$

Then put
$$f(\nu_0) = 0,$$
if
$$I_{\nu_0} \subset \sum_{\nu<\nu_0} f(\nu) * I_\nu + \sum_{\nu>\nu_0} I_\nu, \tag{63}$$
and put
$$f(\nu_0) = 1$$
otherwise. This defines $f(\nu)$ for every ν, and we have
$$f(\nu) \prec \{0, 1\} \quad (\nu \prec N).$$
Put
$$N' = \sum_{\nu \prec N} f(\nu) * \{\nu\}.$$
Then
$$[N'] = [N]. \tag{64}$$
For it is obvious that
$$[N'] \subset [N]. \tag{65}$$
Now let x_0 be any element of $[N]$. Then there exists, by hypothesis, a last element ν_0 of the set $N(x_0)$.

Case 1. Suppose that $f(\nu_0) = 0$. Then (63) holds, and hence
$$x_0 \prec I_{\nu_0} \subset \sum_{\nu<\nu_0} f(\nu) * I_\nu + \sum_{\nu>\nu_0} I_\nu.$$
Then, by definition of ν_0,
$$x_0 \not\prec I_\nu \quad (\nu > \nu_0).$$
Therefore
$$x_0 \prec \sum_{\nu<\nu_0} f(\nu) * I_\nu, \quad x_0 \prec I_{\nu_1}$$
for some ν_1 satisfying
$$\nu_1 < \nu_0, \quad f(\nu_1) = 1.$$
Then
$$\nu_1 \prec N', \quad x_0 \prec I_{\nu_1} \subset [N'].$$

Case 2. Suppose that $f(\nu_0) = 1$. Then
$$\nu_0 \prec N', \quad x_0 \prec I_{\nu_0} \subset [N'].$$
Since x_0 was any point of $[N]$ we conclude that
$$[N] \subset [N'].$$
This result, in conjunction with (65), proves (64).

The proof of theorem 2 will be completed when I have shown that
$$R(N') \leqslant 2. \tag{66}$$
For then, by theorem 1, there are sets N_1, N_2 satisfying
$$N_1 + N_2 = N' \subset N, \qquad R(N_\kappa) \leqslant 1 \quad (1 \leqslant \kappa \leqslant 2).$$
I prove (66) indirectly. Assume the existence of indices α, β, γ and of a point z satisfying
$$(\alpha, \beta, \gamma)_{+}, \quad \alpha, \beta, \gamma \prec N', \quad z \prec I_\alpha I_\beta I_\gamma.$$

We shall deduce a contradiction. We have

$$f(\alpha) = f(\beta) = f(\gamma) = 1,$$

$$I_\alpha \not\subset \sum_{\nu<\alpha} f(\nu) * I_\nu + \sum_{\nu>\alpha} I_\nu = A, \tag{67}$$

say. Then

$$I_\beta \subset A. \tag{68}$$

For if $\beta > \alpha$ then

$$I_\beta \subset \sum_{\nu>\alpha} I_\nu \subset A,$$

and if $\beta < \alpha$ then

$$I_\beta = f(\beta) * I_\beta \subset A.$$

By reasons of symmetry, we have

$$I_\gamma \subset A. \tag{69}$$

By (67), (68) and (69),

$$I_\alpha \not\subset I_\beta + I_\gamma.$$

One can find a point x_α such that

$$x_\alpha \prec I_\alpha - I_\alpha(I_\beta + I_\gamma).$$

By reasons of symmetry, one can find points x_β, x_γ satisfying

$$x_\beta \prec I_\beta - I_\beta(I_\alpha + I_\gamma), \quad x_\gamma \prec I_\gamma - I_\gamma(I_\alpha + I_\beta). \tag{70}$$

Without loss of generality, we may assume that

$$x_\alpha \leqslant x_\beta \leqslant x_\gamma.$$

If

$$z \leqslant x_\beta$$

then

$$z \leqslant x_\beta \leqslant x_\gamma, \quad z, x_\gamma \prec I_\gamma,$$

and therefore, by definition of intervals,

$$x_\beta \prec I_\gamma.$$

This contradicts (70). Hence $x_\beta < z$.

Then, similarly, $x_\alpha \leqslant x_\beta < z, \quad x_\alpha, z \prec I_\alpha, \quad x_\beta \prec I_\alpha.$

This contradicts (70), and theorem 2 is proved.

Special covering theorems for sets of real numbers

18. In this last section S denotes the set of all real numbers, ordered according to their magnitude. The letter T stands for subsets of S. The letter I denotes, as before, intervals of S. When S is the set of real numbers the meaning of the term "interval", as defined in this note, coincides with its common meaning. It denotes a set of numbers x defined by any of the following four relations:

$$a < x < b, \quad a < x \leqslant b, \quad a \leqslant x < b, \quad a \leqslant x \leqslant b.$$

Here a and b are either real numbers or any of the symbols $\pm\infty$.

A set T is called *semi-open* if $x \prec T$ implies the existence of a number $y = y(x, T) \neq 0$ satisfying

$$x+sy \prec T \quad (0<s<1).$$

Every open set of real numbers is semi-open. The sum-set of semi-open sets is semi-open. The intersection of a finite number of semi-open sets is semi-open. Every interval, with the exception of those which consist of a single point, is semi-open.

A quantitative note is introduced into our covering problems by assuming that a *measure* $|T|$ has been defined which possesses certain properties. Let M be a set of subsets of S. The set M plays the part of the system of all "measurable" sets. We suppose that M satisfies the following condition:

If $$J, J' \prec M$$

then $$J+J' \prec M, \quad JJ' \prec M, \quad J-JJ' \prec M. \tag{71}$$

The letter J will always denote elements of M. With every J is associated a non-negative number $|J|$, the "measure" of J, which satisfies the condition

$$|J+J'| = |J|+|J'| \quad \text{if} \quad JJ' = 0. \tag{72}$$

This implies that, for any J_1, J_2,

$$\begin{aligned}|J_1+J_2|+|J_1J_2| &= |J_1|+|J_2-J_1J_2|+|J_1J_2| \\ &= |J_1|+|J_2|,\end{aligned}$$

and hence $$|J_1| \leqslant |J_1+J_2| \leqslant |J_1|+|J_2|. \tag{73}$$

19. THEOREM 3. *Suppose that with every element ν of a set N is associated a semi-open set T_ν of real numbers. Then there exists an at most enumerable subset N' of N satisfying*

$$\sum_{\nu \prec N'} T_\nu = \sum_{\nu \prec N} T_\nu. \tag{74}$$

THEOREM 4. *Let M be a set of sets of real numbers satisfying* (71). *Let $|J|$ be a non-negative measure defined for all sets of M and satisfying* (72). *Suppose that with every element ν of a set N is associated an interval I_ν of real numbers, and let I_ν belong to M and contain more than one number. Finally, suppose that there is a constant C such that*

$$|I_{\nu_1}+I_{\nu_2}+\ldots+I_{\nu_r}| \leqslant C \tag{75}$$

for every positive number r and every choice of r elements ν_ρ of N. Then, given $\epsilon>0$, there exists a finite number of elements α_ρ of N and a sequence of sets $J^{(\lambda)}$ of M satisfying

$$\sum_{\nu \prec N} I_\nu = \sum_{\rho=1}^{r} I_{\alpha_\rho} + \sum_{\lambda=1}^{\infty} J^{(\lambda)}, \tag{76}$$

$$I_{\alpha_p} I_{\alpha_q} = 0 \quad (1 \leqslant p \leqslant q-2 \leqslant r-2), \tag{77}$$

$$|J^{(1)}|+|J^{(2)}|+\ldots \leqslant \epsilon. \tag{78}$$

THEOREM 5. *Suppose that the hypotheses of theorem 4 hold and, in addition, that the set*

$$\sum_{\nu \prec N} I_\nu$$

belongs to M. Suppose that the measure $|J|$ satisfies the condition

$$\left|\sum_{\lambda=1}^{\infty} J_\lambda\right| \leqslant \sum_{\lambda=1}^{\infty} \left|J_\lambda\right| \tag{79}$$

whenever the set $\sum_\lambda J_\lambda$ belongs to M and the series $\sum_\lambda |J_\lambda|$ converges.

Then there are a finite number of elements $\beta_1, \ldots, \beta_m$ of N satisfying

$$\left|\sum_{\mu=1}^{m} I_{\beta_\mu}\right| \geqslant \frac{1}{2}\left|\sum_{\nu \prec N} I_\nu\right| - \epsilon, \tag{80}$$

$$I_{\beta_p} I_{\beta_q} = 0 \quad (1 \leqslant p < q \leqslant m). \tag{81}$$

The special case of theorem 3 when the sets T_ν are open coincides with the one-dimensional case of Lindelöf's covering theorem.† Without any restrictions on the sets T_ν theorem 3 is, of course, not valid, as is shown by the example in which N is the set of all real numbers ν, and $T_\nu = \{\nu\}$.

Theorem 4 implies that there exist, under the hypotheses stated, two finite systems of non-overlapping intervals, namely

$$\text{(i)} \quad I_{\alpha_1}, I_{\alpha_3}, I_{\alpha_5}, \ldots,$$

$$\text{(ii)} \quad I_{\alpha_2}, I_{\alpha_4}, I_{\alpha_6}, \ldots,$$

which between them cover "approximately" the same set as the whole system I_ν $(\nu \prec N)$. The following example shows that (79) does not follow from the hypotheses of theorem 3. Take as M the set of all sets T. Put

$$|J| = 1$$

if there are numbers a and b such that $a < 0 < b$,

$$x \prec J \quad (a < x < b,\ x \neq 0).$$

Put
$$|J| = \tfrac{1}{2}$$

if no such pair of numbers a, b exists but if there is a number $c \neq 0$ satisfying

$$x \prec J \quad (0 < cx < 1).$$

Put
$$|J| = 0$$

for all remaining sets J. Then (72) holds. Let J_λ be the closed interval $\langle \lambda^{-1}, 1 \rangle$ $(\lambda = 1, 2, \ldots)$. Then (79) is false, since

$$\left|\sum_\lambda J_\lambda\right| = \tfrac{1}{2}, \quad \sum_\lambda |J_\lambda| = 0.$$

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† *Loc. cit.*

Theorem 5 generalizes a well-known lemma in the theory of real variables. In (80) the constant $\frac{1}{2}$ is best possible, as is shown by the following example. The system M consists of all sets of real numbers which are Lebesgue-measurable, and $|J|$ is the Lebesgue-measure of J. Let $0<t<1$, and put

$$I_\nu = \langle t^\nu, t^{\nu-1}\rangle \quad (\nu = 1, 2, \ldots).$$

Then, for any finite number of indices β_μ satisfying (81),

$$|I_{\beta_1}+I_{\beta_2}+\ldots+I_{\beta_m}| \leqslant |I_1+I_3+I_5+\ldots| = (1+t)^{-1}\left|\sum_{\nu=1}^{\infty} I_\nu\right|.$$

The constant $\frac{1}{2}$ in (80) cannot be improved because $(1+t)^{-1}\to\frac{1}{2}$ as $t\to 1$.

20. We require one more lemma, which may, on account of its generality, possess some interest of its own. It generalizes a known theorem on sets of intervals.†

LEMMA 7. *Suppose that with every element ν of a set N is associated an arbitrary set T_ν of real numbers. Then every real number x, except those contained in an at most enumerable set X, satisfies exactly one of the following conditions:*

(i) *There exists an element ν such that x is an interior point of T_ν.*

(ii) *Given any element ν, there exist two sequences a_m, b_m of numbers such that*

$$a_m, b_m \not\prec T_\nu, \tag{82}$$

$$a_m < x < b_m \quad (m = 1, 2, \ldots), \tag{83}$$

$$\lim_{m\to\infty} a_m = \lim_{m\to\infty} b_m = x. \tag{84}$$

Proof. Let x be a number which satisfies neither (i) nor (ii). Then there exists an element $\nu=\nu(x)$ such that, corresponding to these x and ν, no sequences a_m, b_m can be found satisfying (82), (83), (84). Then one can find a number $c(x)\neq 0$ such that

$$x+sc(x)\prec T_{\nu(x)} \quad (0<s<1). \tag{85}$$

Let X_+ be the set of those numbers x of X for which $c(x)>0$, and put $X_- = X - X_+$. Suppose that the numbers x_1, x_2 satisfy

$$x_1, x_2 \prec X_+, \quad x_1 < x_2.$$

Then
$$x_1 + c(x_1) \leqslant x_2. \tag{86}$$

For otherwise
$$x_1 < x_2 < x_1 + c(x_1).$$

† Hobson, *loc. cit.*, last proposition of §73.

Then one could find numbers s_1, s_2 such that

$$0 < s_1 < s_2 < 1, \quad x_1 + s_1 c(x_1) < x_2 < x_1 + s_2 c(x_1).$$

Then we deduce, by applying (85) to $x = x_1$, that x_2 is an interior point of T_{ν_1}, i.e. that x_2 satisfies (i). This contradicts the fact that

$$x_2 \prec X_+ \subset X.$$

Hence (86) is established. Thus

$$x_1 < x_1 + c(x_1) \leqslant x_2 < x_2 + c(x_2)$$

whenever $\qquad x_1, x_2 \prec X_+, \quad x_1 < x_2.$

No two of the open intervals $(x, x + c(x))$ overlap when x ranges through X_+, and therefore X_+ is at most enumerable. By reasons of symmetry this proves at the same time that X_- is at most enumerable. The lemma is proved.

Proofs of theorems 3, 4 *and* 5

21. Suppose that the hypotheses of theorem 3 hold. Consider any bounded open interval I which has rational end-points and which, moreover, satisfies $I \subset T_\nu$ for at least one element ν. All these intervals I can be arranged in a sequence, say as

$$I^{(1)}, I^{(2)}, \ldots.$$

Choose ν_ρ such that $\qquad I^{(\rho)} \subset T_{\nu_\rho} \quad (\rho = 1, 2, \ldots).$

Let X be the at most enumerable set defined in lemma 7, belonging to the semi-open sets T_ν of theorem 3. Then

$$X \sum_{\nu \prec N} T_\nu = \{x_1, x_2, \ldots\},$$

where the system of numbers x_μ is either finite or enumerably infinite. Corresponding to every number x_μ, choose an element ν'_μ satisfying

$$x_\mu \prec T_{\nu'_\mu} \quad (\mu = 1, 2, \ldots).$$

Then (74) holds when

$$N' = \{\nu_1, \nu_2, \ldots\} + \{\nu'_1, \nu'_2, \ldots\}.$$

For assume that a number x satisfies

$$x \prec T_{\nu_0} \tag{87}$$

for some ν_0 of N, but at the same time

$$x \not\prec T_\nu \quad (\nu \prec N'). \tag{88}$$

We shall deduce a contradiction. (88) implies that $x \not\prec X$. Hence, by lemma 7, either (i) x is interior point of some set T_ν, and therefore

$$x \prec I^{(\rho)} \subset T_\nu \tag{89}$$

for some $\rho > 0$, or (ii) there are sequences a_m, b_m satisfying (83), (84) and

$$a_m,\, b_m \not\prec T_{\nu_0} \quad (m = 1, 2, \ldots). \tag{90}$$

The relation (89) is a contradiction against (88). On the other hand, the simultaneous validity of (87), (83), (84) and (90) contradicts the fact that T_{ν_0} is semi-open. Thus a contradiction follows in either case, and theorem 3 is proved.

22. Suppose that the hypotheses of theorem 4 hold. We may assume, without loss of generality, that

$$I_\nu \neq 0 \quad (\nu \prec N).$$

Since the sets I_ν are semi-open, there exists, by theorem 3, an at most enumerable set N' such that

$$\sum_{\nu \prec N'} I_\nu = \sum_{\nu \prec N} I_\nu. \tag{91}$$

We may assume that $\quad N' = \{1, 2, 3, \ldots\}.$

Unless the contrary is stated the letters ν, λ, n denote typical elements of N'. Put

$$J_\lambda = I_\lambda - I_\lambda \sum_{\nu < \lambda} I_\nu.$$

Then

$$J_\lambda J_{\lambda'} = 0 \quad (\lambda \neq \lambda').$$

It follows from (73) and (75) that there exists the limit

$$\lim_{n\to\infty} \Big| \sum_{\nu \leqslant n} I_\nu \Big| = L.$$

Hence a positive number n_0 can be found satisfying

$$\Big| \sum_{\nu \leqslant n_0} I_\nu \Big| \geqslant L - \epsilon.$$

Then, for $n > n_0$,

$$\sum_{n_0+1}^{n} |J_\nu| = \left| \sum_{n_0+1}^{n} J_\nu \right| = \left| \sum_{1}^{n} I_\nu - \sum_{1}^{n_0} I_\nu \right|$$
$$= \left| \sum_{1}^{n} I_\nu \right| - \left| \sum_{1}^{n_0} I_\nu \right| \leqslant L - (L - \epsilon) = \epsilon.$$

Hence, making $n \to \infty$,

$$\sum_{n_0+1}^{\infty} |J_\nu| \leqslant \epsilon. \tag{92}$$

Also

$$\sum_{1}^{\infty} I_\nu = \sum_{1}^{n_0} I_\nu + \sum_{n_0+1}^{\infty} J_\nu. \tag{93}$$

Let r be the smallest number such that there are r indices α_ρ satisfying

$$1 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_r \leqslant n_0, \quad \sum_{\rho=1}^{r} I_{\alpha_\rho} = \sum_{1}^{n_0} I_\nu. \tag{94}$$

Then $1 \leqslant r \leqslant n_0$. Furthermore,

$$I_{\alpha_\rho} \not\subset \sum_{\rho' \neq \rho} I_{\alpha_{\rho'}} \quad (1 \leqslant \rho \leqslant r),$$

since otherwise r could be reduced to $r-1$ by the omission of some index α_ρ. Hence one can choose numbers x_ρ satisfying

$$x_\rho \prec I_{\alpha_\rho} - I_{\alpha_\rho} \sum_{\rho' \neq \rho} I_{\alpha_{\rho'}} \quad (1 \leqslant \rho \leqslant r). \tag{95}$$

By numbering suitably one can obtain a case in which

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_r.$$

Then the numbers $\alpha_1, \alpha_2, \ldots, \alpha_r$ have the property required in theorem 4. For we have, by (91), (93) and (94),

$$\sum_{\nu \prec N} I_\nu = \sum_{\nu \prec N'} I_\nu = \sum_{\rho=1}^{r} I_{\alpha_\rho} + \sum_{\nu=n_0+1}^{\infty} J_\nu. \tag{96}$$

We have to deduce a contradiction from the existence of indices p, q satisfying

$$1 \leqslant p \leqslant q-2 \leqslant r-2, \quad I_{\alpha_p} I_{\alpha_q} \neq 0.$$

Choose a number x such that

$$x \prec I_{\alpha_p} I_{\alpha_q}.$$

Then an argument similar to one used at the end of § 17 will lead to a contradiction.

Case 1. Suppose that $x \leqslant x_{p+1}$. Then

$$x \leqslant x_{p+1} \leqslant x_q, \quad x, x_q \prec I_{\alpha_q}, \quad x_{p+1} \prec I_{\alpha_q},$$

which contradicts (95).

Case 2. Suppose that $x_{p+1} < x$. Then

$$x_p \leqslant x_{p+1} < x, \quad x_p, x \prec I_{\alpha_p}, \quad x_{p+1} \prec I_{\alpha_p}$$

which contradicts (95). This contradiction establishes (77). Relations (76) and (78) follow from (96) and (92) respectively if one puts

$$J^{(\lambda)} = J_{n_0+\lambda} \quad (\lambda = 1, 2, \ldots).$$

This completes the proof of theorem 4.

23. It is easy to deduce theorem 5 from theorem 4. Suppose that the hypotheses of theorem 5 hold. By theorem 4, there are r indices α_ρ and sets $J^{(\lambda)}$ satisfying (76), (77) and (78). A suitable numbering of the indices α_ρ will ensure that

$$\Big| \sum_{\rho \text{ even}} I_{\alpha_\rho} \Big| \leqslant \Big| \sum_{\rho \text{ odd}} I_{\alpha_\rho} \Big|.$$

By using property (71) of M and the hypotheses of theorem 5, we see that the set

$$\sum_{\lambda=1}^{\infty} J^{(\lambda)} = \sum_{\nu \prec N} I_\nu - \sum_{\rho=1}^{r} I_{\alpha_\rho}$$

belongs to M. Hence, by (78) and (79),

$$\begin{aligned} \Big| \sum_{\nu \prec N} I_\nu \Big| &= \Big| \sum_{\rho \text{ even}} I_{\alpha_\rho} + \sum_{\rho \text{ odd}} I_{\alpha_\rho} + \sum_{\lambda} J^{(\lambda)} \Big| \\ &\leqslant \Big| \sum_{\rho \text{ even}} I_{\alpha_\rho} \Big| + \Big| \sum_{\rho \text{ odd}} I_{\alpha_\rho} \Big| + \Big| \sum_{\lambda} J^{(\lambda)} \Big| \\ &\leqslant 2 \Big| \sum_{\rho \text{ odd}} I_{\alpha_\rho} \Big| + \sum_{\lambda} \big| J^{(\lambda)} \big| \\ &< 2 \Big| \sum_{\rho \text{ odd}} I_{\alpha_\rho} \Big| + 2\epsilon, \end{aligned}$$

and so

$$\Big| \sum_{\rho \text{ odd}} I_{\alpha_\rho} \Big| > \tfrac{1}{2} \Big| \sum_{\nu \prec N} I_\nu \Big| - \epsilon.$$

Hence theorem 5 follows if one puts

$$\beta_\mu = \alpha_{2\mu-1} \quad \left(1 \leqslant \mu \leqslant \frac{r+1}{2}\right).$$

For (81) is a consequence of (77).

The University
Sheffield

THE CENTRE OF FLEXURE OF A HOLLOW SHAFT

By A. C. STEVENSON

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1. *History of the problem*

The flexure problem for a hollow tube of circular cross-section with a cavity of circular cross-section placed eccentrically, was unsolved when it was suggested in 1906 by Love† that the classical Saint-Venant flexure functions could be found as series expansions in suitable orthogonal curvilinear coordinates, and, taking note of this suggestion, Young, Elderton, and Pearson‡ in 1918 wrote down the form of the solution in series for the case where the load is at right angles to the axis of symmetry of the cross-section. They did not proceed further, however, to evaluate explicitly the coefficients in these series, and to complete the solution by finding the mean twist of the cross-section when the centroid is taken as the load-point of the cross-section. It was not until 1936 that Seth§ gave the solution for the Saint-Venant flexure functions for the cases where the load is resolved along and perpendicular to the axis of symmetry, and the associated twist due to the latter resolute when the centroid is the load-point.

In this paper the problem is solved by finding the canonical flexure functions and moment integrals introduced by the author||, to facilitate a systematic investigation of such problems, and all coefficients of the series solutions are found explicitly and the associated twist is determined. The solution of the problem when any other point of the cross-section is the load-point depends upon the superposition of the solution for the torsion problem, with an appropriate additional twist proportional to the distance of

† Love, *Mathematical Theory of Elasticity*, 2nd edition (1906), 325.

‡ Young, Elderton and Pearson, *Draper's Co. Research Memoirs, Tech. series*, no. VII (1918), 69.

§ Seth, *Proc. Ind. Acad. Sci.* 4 (1936), 531, and 5 (1937), 23.

|| Stevenson, *Phil. Trans. Roy. Soc.* 237 (1938), 161.

the load from the centroid. The torsion solution for this cross-section was given in 1893 by Macdonald†.

An alternative way of relating the constants of the mathematical solution with the external force system is to give, in addition to the torsion solution, the solution when the load-point is chosen so that the mean twist of the cross-section is zero. This load-point is termed the centre of flexure, and Timoshenko ‡ remarks that its position is a matter of practical importance in thin-walled sections, and that its exact theoretical determination has only been carried out for one or two cross-sections. Another load-point of interest is the centre of least strain, defined as the load-point which results in minimum strain energy of the beam. It can be shown that the coordinates of this centre of least strain can be obtained from those of the centre of flexure merely by setting the elastic constant equal to zero. In compact sections the two points are in practice barely distinguishable. General formulae have been given by the author for the associated twist and these centres of flexure and least strain in terms of the canonical moment integrals, and in this paper the numerical values of the associated twist and coordinates of the centres of flexure and least strain are given for a comprehensive range of values of the radius of the inner cavity and the displacement of its centre from the centre of the outer boundary. These numerical results are especially interesting as the section weakens by the cavity becoming larger or more eccentrically placed or both.

Seth did not consider the centre of flexure, and gave numerical results for the associated twist in two cases only. Neither Macdonald nor Seth considered especially the limiting case when the cavity just reaches the surface of the cylinder. In this case it will be shown that the moment integrals can be expressed in terms of trigamma and tetragamma functions, which have been extensively tabulated. The results of this paper are accordingly almost entirely supplementary to those of Seth§.

2. *Introductory analysis*

For these cross-sections we employ orthogonal curvilinear coordinates ξ, η defined by

$$z = c\tan\tfrac{1}{2}\zeta \quad (z = x+iy,\ \zeta = \xi+i\eta). \tag{2·1}$$

† Macdonald, *Proc. Camb. Phil. Soc.* 8 (1893), 62.

‡ Timoshenko, *Theory of Elasticity* (1934), 301; *Strength of Materials*, 1 (1931), 195.

§ Since this paper was written (in 1938), complex variable treatments of these problems, of considerable interest, have been devised. See, for example, R. Capildeo, "On the torsion function in Saint-Venant flexure with shear," Ph.D. thesis, London University, 1948.

Then the curves given by ξ = const. are a set of coaxial circles with $z = \pm ic$ as common points, and the orthogonal family given by η = const. are a set of coaxial circles having real limiting points S_1, S_2 at $z = \pm ic$, corresponding to $\eta = \pm\infty$. The y-axis between the limiting points is given by $\xi = 0$, and outside the segment of the y-axis connecting these two points $\xi = \pm\pi$, and there is a discontinuity of 2π in crossing the y-axis outside this segment. We shall take the boundary of our cross-section to be given by $\eta = \alpha$, $\eta = \beta$, with $\beta > \alpha$. These two non-intersecting circles will be assumed to have radii a, b with $a > b$, and to have their centres C_1, C_2 at a distance f apart (see Fig. 1).

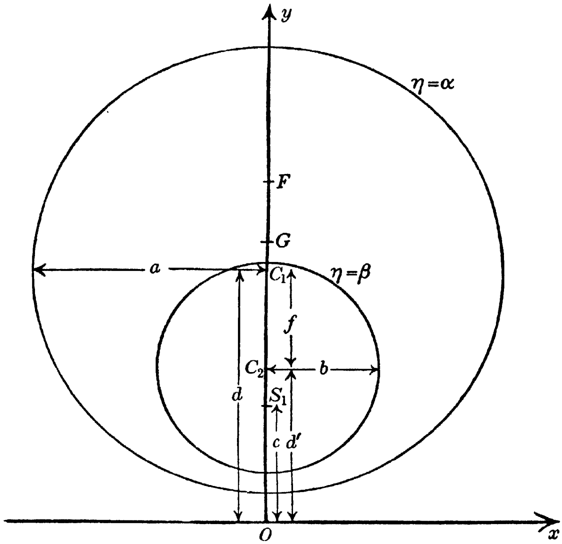

Figure 1

The radii are given by

$$a = c \operatorname{cosech} \alpha, \quad b = c \operatorname{cosech} \beta, \tag{2·2}$$

and if the centres of the circles are at the points $(0, d)$, $(0, d')$ respectively, then

$$d = c \coth \alpha, \quad d' = c \coth \beta. \tag{2·3}$$

Hence
$$f = d - d' = c(\coth \alpha - \coth \beta), \tag{2·4}$$

and we write
$$f' = d + d' = c(\coth \alpha + \coth \beta). \tag{2·5}$$

From these we deduce that $$ff' = a^2 - b^2, \tag{2·6}$$

$$4c^2f^2 = (f+a+b)(f+a-b)(f-a-b)(f-a+b), \tag{2·7}$$

and d and d' are given in terms of f by the equations

$$d = (a^2-b^2+f^2)/2f, \quad d' = (a^2-b^2-f^2)/2f. \tag{2·8}$$

In practice we are given a, b and f, and c is then calculated from (2·7). Since α and β appear in the combinations $\alpha \pm \beta$, there is some advantage in numerical calculations in using the equations

$$\sinh(\beta-\alpha) = cf/ab, \quad \sinh(\beta+\alpha) = c(a^2-b^2)/abf. \tag{2·9}$$

Now equation (2·1) gives

$$x+iy = c(\sin\xi + i\sinh\eta)/(\cos\xi+\cosh\eta), \tag{2·10}$$

and this can also be rewritten as

$$x+iy = -ic\{1-2/(1+e^{i\xi-\eta})\},$$

whence $$x = -2c\sum_{n=1}^{\infty}(-1)^n e^{-n\eta}\sin n\xi, \tag{2·11}$$

$$y = c\left\{1+2\sum_{n=1}^{\infty}(-1)^n e^{-n\eta}\cos n\xi\right\}. \tag{2·12}$$

From (2·10) we find easily that

$$x^3 = -\tfrac{1}{2}c^2\frac{\partial^2 x}{\partial\eta^2} - \tfrac{3}{2}c^2\coth\eta\frac{\partial x}{\partial\eta} - c^2x,$$

$$y^3 = \tfrac{1}{2}c^2\frac{\partial^2 y}{\partial\eta^2} - \tfrac{3}{2}c^2\coth\eta\frac{\partial y}{\partial\eta} - \tfrac{1}{2}c^2(1-3\coth^2\eta)\,y,$$

$$x^2+y^2 = 2cy\coth\eta - c^2,$$

and then, using (2·11) and (2·12), we have

$$\tfrac{1}{3}x^3 = c^3\sum_{n=1}^{\infty}(-1)^n\{\tfrac{1}{3}(n^2+2) - n\coth\eta\}\,e^{-n\eta}\sin n\xi, \tag{2·13}$$

$$\tfrac{1}{3}y^3 = c^3(\tfrac{1}{3}+\tfrac{1}{2}\operatorname{cosech}^2\eta)$$
$$+c^3\sum_{n=1}^{\infty}(-1)^n\{\tfrac{1}{3}(n^2+2)+n\coth\eta+\operatorname{cosech}^2\eta\}\,e^{-n\eta}\cos n\xi, \tag{2·14}$$

$$\tfrac{1}{2}(x^2+y^2) = c^2(\coth\eta - \tfrac{1}{2}) + c^2\sum_{n=1}^{\infty}(-1)^n\,2\coth\eta\, e^{-n\eta}\cos n\xi. \tag{2·15}$$

We then derive also the following useful results

$$c\sum_{n=1}^{\infty}(-1)^n e^{-n\eta}\sin n\xi = -\tfrac{1}{2}x, \tag{2·16}$$

$$c^2\sum_{n=1}^{\infty}(-1)^n n e^{-n\eta}\sin n\xi = \tfrac{1}{2}c\frac{\partial x}{\partial \eta} = -\tfrac{1}{2}xy, \tag{2·17}$$

$$c^3\sum_{n=1}^{\infty}(-1)^n n^2 e^{-n\eta}\sin n\xi = -\tfrac{1}{2}c^2\frac{\partial^2 x}{\partial \eta^2} = \tfrac{1}{4}c^2x+\tfrac{1}{4}(x^3-3xy^2), \tag{2·18}$$

$$c\sum_{n=1}^{\infty}(-1)^n e^{-n\eta}\cos n\xi = \tfrac{1}{2}(y-c), \tag{2·19}$$

$$c^2\sum_{n=1}^{\infty}(-1)^n n e^{-n\eta}\cos n\xi = -\tfrac{1}{2}c\frac{\partial y}{\partial \eta} = -\tfrac{1}{2}\{cy\coth\eta-y^2\} = -\tfrac{1}{4}(x^2-y^2+c^2), \tag{2·20}$$

$$c^3\sum_{n=1}^{\infty}(-1)^n n^2 e^{-n\eta}\cos n\xi = \tfrac{1}{2}c^2\frac{\partial^2 y}{\partial \eta^2} = \tfrac{1}{4}(y^3-3x^2y)-\tfrac{1}{4}c^2y. \tag{2·21}$$

3. *The canonical flexure functions*

For the solution of the problem of the flexure of an elastic beam, possessing a cross-section having the y-axis as an axis of symmetry and under the action of a load W parallel to the x-axis, the author has shown that we need three complex functions of z $(= x+iy)$, $\Omega_1, \omega_2, \omega_3$, where $\Omega_1 = \chi_1+i\chi_1^*$, $\omega_r = \phi_r+i\psi_r$ $(r = 2, 3)$, such that the plane harmonic functions χ_1, ψ_2, ψ_3 satisfy the equations

$$\frac{\partial}{\partial n}(\chi_1-\tfrac{1}{3}x^3) = 0, \tag{3·1}$$

$$\psi_2-\tfrac{1}{3}y^3 = \text{const.}, \tag{3·2}$$

$$\psi_3-\tfrac{1}{2}(x^2+y^2) = \text{const.}, \tag{3·3}$$

round the boundary of the cross-section, n denoting the normal to the boundary.

In virtue of (2·13)–(2·15) these boundary conditions become

$$\frac{\partial \chi_1}{\partial \eta} = -c^3\sum_{n=1}^{\infty}(-1)^n n\{\tfrac{1}{3}(n^2+2)-n\coth\eta-\operatorname{cosech}^2\eta\}e^{-n\eta}\sin n\xi,$$

$$\psi_2 = c^3\sum_{n=1}^{\infty}(-1)^n\{\tfrac{1}{3}(n^2+2)+n\coth\eta+\operatorname{cosech}^2\eta\}e^{-n\eta}\cos n\xi+\text{const.},$$

$$\psi_3 = c^2\sum_{n=1}^{\infty}(-1)^n 2\coth\eta\, e^{-n\eta}\cos n\xi+\text{const.}$$

We assume suitable plane harmonic forms for χ_1, ψ_2, ψ_3, namely,

$$\chi_1 = c^3 \sum_{n=1}^{\infty} (-1)^n \{A_n + A'_n e^{2n\eta}\} e^{-n\eta} \sin n\xi,$$

so that
$$\frac{\partial \chi_1}{\partial \eta} = -c^3 \sum_{n=1}^{\infty} (-1)^n n\{A_n - A'_n e^{2n\eta}\} e^{-n\eta} \sin n\xi;$$

also
$$\psi_2 = c^3 \sum_{n=1}^{\infty} (-1)^n \{B_n + B'_n e^{2n\eta}\} e^{-n\eta} \cos n\xi,$$

$$\psi_3 = c^2 \sum_{n=1}^{\infty} (-1)^n \{C_n + C'_n e^{2n\eta}\} e^{-n\eta} \cos n\xi.$$

The boundary conditions will be satisfied along $\eta = \alpha$ and $\eta = \beta$ if

$$\begin{aligned}
A_n - A'_n e^{2n\alpha} &= \tfrac{1}{3}(n^2+2) - n \coth\alpha - \operatorname{cosech}^2\alpha,\\
A_n - A'_n e^{2n\beta} &= \tfrac{1}{3}(n^2+2) - n \coth\beta - \operatorname{cosech}^2\beta,\\
B_n + B'_n e^{2n\alpha} &= \tfrac{1}{3}(n^2+2) + n \coth\alpha + \operatorname{cosech}^2\alpha,\\
B_n + B'_n e^{2n\beta} &= \tfrac{1}{3}(n^2+2) + n \coth\beta + \operatorname{cosech}^2\beta,\\
C_n + C'_n e^{2n\alpha} &= 2\coth\alpha,\\
C_n + C'_n e^{2n\beta} &= 2\coth\beta.
\end{aligned}$$

Solving these equations for the constants A_n, A'_n, B_n, B'_n, C_n, C'_n we find, on making use of (2·2)–(2·5), that

$$A_n = \tfrac{1}{3}(n^2+2) - \frac{1}{2}\frac{n}{c}f' - \frac{1}{2}\frac{(a^2+b^2)}{c^2} - \tfrac{1}{2}f\frac{a_n}{c} - \tfrac{1}{2}ff'\frac{b_n}{c^2},$$

$$B_n = \tfrac{1}{3}(n^2+2) + \frac{1}{2}\frac{n}{c}f' + \frac{1}{2}\frac{(a^2+b^2)}{c^2} + \tfrac{1}{2}f\frac{a_n}{c} + \tfrac{1}{2}ff'\frac{b_n}{c^2},$$

$$-A'_n = -B'_n = \tfrac{1}{2}f\frac{a'_n}{c} + \tfrac{1}{2}ff'\frac{b'_n}{c^2},$$

$$C_n = \frac{f'}{c} + f\frac{b_n}{c}, \quad C'_n = -f\frac{b'_n}{c},$$

where
$$a_n = n\coth n(\beta-\alpha), \quad a'_n = ne^{-n(\alpha+\beta)}\operatorname{cosech} n(\beta-\alpha), \tag{3·4}$$
$$b_n = \coth n(\beta-\alpha), \quad b'_n = e^{-n(\alpha+\beta)}\operatorname{cosech} n(\beta-\alpha). \tag{3·5}$$

Then with the aid of (2·16)–(2·21) we find that

$$\chi_1 = \chi_{11} + \chi_{12}, \quad \psi_2 = \psi_{21} + \psi_{22}, \quad \psi_3 = \psi_{31} + \psi_{32},$$

where, dropping irrelevant additional constants,

$$\chi_{11} = \tfrac{1}{12}(x^3-3xy^2)+\tfrac{1}{4}f'xy+\tfrac{1}{4}x(a^2+b^2-c^2), \tag{3·6}$$

$$\chi_{12} = -\tfrac{1}{2}fc^2\sum_{n=1}^{\infty}(-1)^n\{a_n e^{-n\eta}+a'_n e^{n\eta}\}\sin n\xi - \tfrac{1}{2}ff'c\sum_{n=1}^{\infty}(-1)^n\{b_n e^{-n\eta}+b'_n e^{n\eta}\}\sin n\xi, \tag{3·7}$$

$$\psi_{21} = \tfrac{1}{2}fc^2\sum_{n=1}^{\infty}(-1)^n\{a_n e^{-n\eta}-a'_n e^{n\eta}\}\cos n\xi + \tfrac{1}{2}ff'c\sum_{n=1}^{\infty}(-1)^n\{b_n e^{-n\eta}-b'_n e^{n\eta}\}\cos n\xi, \tag{3·8}$$

$$\psi_{22} = \tfrac{1}{12}(y^3-3x^2y)-\tfrac{1}{8}f'(x^2-y^2)+\tfrac{1}{4}y(a^2+b^2+c^2), \tag{3·9}$$

$$\psi_{31} = fc\sum_{n=1}^{\infty}(-1)^n\{b_n e^{-n\eta}-b'_n e^{n\eta}\}\cos n\xi, \tag{3·10}$$

$$\psi_{32} = \tfrac{1}{2}f'y, \tag{3·11}$$

and hence $\Omega_1 = \Omega_{11}+\Omega_{12}$, $\omega_2 = \omega_{21}+\omega_{22}$, $\omega_3 = \omega_{31}+\omega_{32}$,

where

$$\Omega_{11} = \chi_{11}+i\chi_{11}^* = \tfrac{1}{12}z^3-\tfrac{1}{8}if'z^2+\tfrac{1}{4}z(a^2+b^2-c^2), \tag{3·12}$$

$$\omega_{22} = \phi_{22}+i\psi_{22} = -\tfrac{1}{12}z^3-\tfrac{1}{8}if'z^2+\tfrac{1}{4}z(a^2+b^2+c^2), \tag{3·13}$$

$$\omega_{32} = \phi_{32}+i\psi_{32} = \tfrac{1}{2}f'z, \tag{3·14}$$

and

$$\phi_{21} = \chi_{12} = \tfrac{1}{2}f'\phi_{31}-\tfrac{1}{2}fc^2\sum_{n=1}^{\infty}(-1)^n\{a_n e^{-n\eta}+a'_n e^{n\eta}\}\sin n\xi, \tag{3·15}$$

$$\phi_{31} = -fc\sum_{n=1}^{\infty}(-1)^n\{b_n e^{-n\eta}+b'_n e^{n\eta}\}\sin n\xi. \tag{3·16}$$

4. *The canonical moment integrals*

We next find certain moment integrals taken over the cross-section and obtained from the canonical flexure functions by the following formulae:

$$L_1 = L'_1+\int x^2y\,dS, \tag{4·1}$$

$$M_2 = M'_2+\int y^3\,dS, \tag{4·2}$$

$$M_3 = M'_3+\int (x^2+y^2)\,dS, \tag{4·3}$$

where

$$L'_1 = \text{real part of} \int iz\frac{d\Omega_1}{dz}\,dS, \tag{4·4}$$

$$M'_r = \text{real part of} \int iz\frac{d\omega_r}{dz}\,dS \quad (r = 1, 2), \tag{4·5}$$

which can be expressed alternatively in the form of line integrals round the boundary of the cross-section as

$$L_1' = \int \tfrac{1}{2}(x^2+y^2)\frac{\partial \chi_1}{\partial s}\,ds, \tag{4·6}$$

$$M_r' = \int \tfrac{1}{2}(x^2+y^2)\frac{\partial \phi_r}{\partial s}\,ds. \tag{4·7}$$

For this cross-section there is no difficulty in obtaining the following results:

$$\int y\,dS = kS = \pi(a^2d-b^2d'), \qquad S = \pi(a^2-b^2), \tag{4·8}$$

$$\int x^2\,dS = \tfrac{1}{4}\pi(a^4-b^4), \qquad \int (x^2-y^2)\,dS = -\pi(a^2d^2-b^2d'^2), \tag{4·9}$$

$$\int x^2y\,dS = \tfrac{1}{4}\pi(a^4d-b^4d'), \quad \int (y^3-3x^2y)\,dS = \pi(a^2d^3-b^2d'^3). \tag{4·10}$$

We write $L_1' = L_{11}'+L_{12}'$, $M_2' = M_{21}'+M_{22}'$, $M_3' = M_{31}'+M_{32}'$ to correspond with the splitting up of χ_1, ψ_2 and ψ_3, and then use (4·4) and (4·5) for the evaluation of L_{11}', M_{22}', M_{32}', and (4·6) and (4·7) for the evaluation of L_{12}', M_{21}', M_{31}'. We find readily, making use of (4·8)–(4·10), that

$$\begin{aligned}
L_{11}' &= \tfrac{1}{4}\pi(a^2d^3-b^2d'^3)-\tfrac{1}{4}\pi f'(a^2d^2-b^2d'^2)-\tfrac{1}{4}\pi(a^2+b^2-c^2)(a^2d-b^2d'),\\
M_{22}' &= -\tfrac{1}{4}\pi(a^2d^3-b^2d'^3)-\tfrac{1}{4}\pi f'(a^2d^2-b^2d'^2)-\tfrac{1}{4}\pi(a^2+b^2+c^2)(a^2d-b^2d'),\\
M_{32}' &= -\tfrac{1}{2}\pi f'(a^2d-b^2d').
\end{aligned}$$

Now from (4·7), using (2·15) and (3·16),

$$\begin{aligned}
M_{31}' &= \left[-2\pi fc^3\sum_{n=1}^{\infty}\coth\eta(a_n e^{-2n\eta}+a_n')\right]_{\eta=\beta}^{\eta=\alpha}\\
&= -4\pi f^2c^2\sum_{n=1}^{\infty}a_n' - 2\pi fc^2d\sum_{n=1}^{\infty}ne^{-2n\alpha} - 2\pi fc^2d'\sum_{n=1}^{\infty}ne^{-2n\beta}.
\end{aligned}$$

But

$$\sum_{n=1}^{\infty} ne^{-2nx} = \tfrac{1}{4}\operatorname{cosech}^2 x, \tag{4·11}$$

whence, making use of (4·11) and (2·2),

$$M_{31}' = -4\pi f^2c^2\sum_{n=1}^{\infty}a_n' - \tfrac{1}{2}\pi f(a^2d+b^2d').$$

Hence, adding M_{31}, M_{32}, and using (2·9), (4·3) and (4·9), we have

$$M_3 = \tfrac{1}{2}\pi(a^4-b^4)-4\pi a^2b^2S_1, \tag{4·12}$$

where

$$S_1 = \sinh^2(\beta-\alpha)\sum_{n=1}^{\infty} ne^{-n(\alpha+\beta)}\operatorname{cosech} n(\beta-\alpha). \tag{4·13}$$

This agrees with Macdonald's result for the torsion moment ($\mu\tau M_3$).

In similar fashion we readily find from (4·6) and (4·7), using (2·15) and (3·15), that

$$M'_{21} = L'_{12} = \tfrac{1}{2}f'M'_{31} - \pi f c^4 \sum_{n=1}^{\infty} n\left[\coth\eta(a_n e^{-2n\eta} + a'_n)\right]_{\eta=\beta}^{\eta=\alpha}$$

$$= \tfrac{1}{2}f'M'_{31} - 2\pi f^2c^3 \sum_{n=1}^{\infty} na'_n - \pi dfc^3 \sum_{n=1}^{\infty} n^2e^{-2n\alpha} - \pi d'f'c^3 \sum_{n=1}^{\infty} n^2 e^{-2n\beta}.$$

But from (4·11) we deduce that

$$\sum_{n=1}^{\infty} n^2 e^{-2nx} = \tfrac{1}{4}\coth x\,\mathrm{cosech}^2 x; \tag{4·14}$$

hence, substituting for M'_{31}, using (4·14) and adding M'_{21}, M'_{22}, we now find

$$-M'_2 = \tfrac{1}{4}\pi c^2(a^2d - b^2d') + \tfrac{1}{2}\pi(a^4d - b^4d') + \tfrac{3}{4}\pi(a^2d^3 - b^2d'^3)$$

$$+ 2\pi f'f^2c^2 \sum_{n=1}^{\infty} a'_n + 2\pi f^2c^3 \sum_{n=1}^{\infty} na'_n,$$

and so from (4·2) and (4·10), using (2·2) and (2·3), we obtain

$$M_2 = \tfrac{1}{2}\pi(a^4d - b^4d') - 2\pi a^2b^2(a^2 - b^2)\frac{S_1}{f} - 2\pi a^3b^3\frac{S_2}{f}, \tag{4·15}$$

where S_1 is given by (4·13) and S_2 by

$$S_2 = \sinh^3(\beta - \alpha) \sum_{n=1}^{\infty} n^2 e^{-n(\alpha+\beta)}\,\mathrm{cosech}\,n(\beta - \alpha). \tag{4·16}$$

Similarly, adding L'_{11} and L'_{12}, and making use of (4·1), (4·10) and (4·15), we find that

$$L_1 = M_2 - \pi(a^4d - b^4d'). \tag{4·17}$$

We can find alternative expressions for the series S_1 and S_2 of positive terms, since

$$\sum_{n=1}^{\infty} ne^{-n(\alpha+\beta)}\,\mathrm{cosech}\,n(\beta-\alpha) = 2\sum_{n=1}^{\infty} ne^{-2n\beta}/(1 - e^{-2n(\beta-\alpha)}),$$

$$= 2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty} ne^{-2n[(r-1)(\beta-\alpha)+\beta]},$$

or, from (4·11),

$$S_1 = \tfrac{1}{2}\sinh^2(\beta-\alpha)\sum_{n=0}^{\infty}\mathrm{cosech}^2[n(\beta-\alpha)+\beta]. \tag{4·18}$$

In similar fashion we find, with the aid of (4·14), that

$$S_2 = \tfrac{1}{2}\sinh^3(\beta-\alpha)\sum_{n=0}^{\infty}\frac{\cosh[n(\beta-\alpha)+\beta]}{\sinh^3[n(\beta-\alpha)+\beta]}. \tag{4·19}$$

To express a^4d-b^4d' in terms of a, b, f, we find from (2·8)

$$a^4d-b^4d' = \tfrac{1}{2}f(a^4+b^4)+\frac{1}{2f}(a^2+b^2)(a^2-b^2)^2. \tag{4·20}$$

It is also possible to express the terms of the series S_1 and S_2 explicitly in terms of a, b and f, since we can show that

$$\frac{\sinh[n(\beta-\alpha)+\beta]}{\sinh(\beta-\alpha)} = \frac{1}{f}(aQ_n-bQ_{n-1}), \tag{4·21}$$

$$\cosh[n(\beta-\alpha)+\beta] = \frac{d'}{b}Q_n-\frac{d}{a}Q_{n-1}, \tag{4·22}$$

in which Q_n is a polynomial in x, where

$$x = 2\cosh(\beta-\alpha) = (a^2+b^2-f^2)/ab, \tag{4·23}$$

and Q_n is given by

$$Q_n = x^n-\frac{(n-1)}{1!}x^{n-2}+\frac{(n-2)(n-3)}{2!}x^{n-4}\dots(-1)^r\frac{(n-r)\dots(n-2r+1)}{r!}x^{n-2r}\dots. \tag{4·24}$$

We replace d and d' in (4·22) by their explicit values from (2·8). After the first few terms of the series the explicit expressions, in terms of a, b and f only, rapidly become unwieldy, however, and computation of S_1 and S_2 is best effected by finding c from (2·7), given a, b and f, and then deducing $\beta\pm\alpha$ from (2·9).

5. *Special case—circular boundaries in contact*

When the cylindrical cavity just reaches to the boundary of the cylinder, the solutions found for the moment integrals require special treatment before they can be of direct numerical use. This case is obtained by letting c, α and β all tend to zero in such a way that

$$\lim c \operatorname{cosech}\alpha = a, \quad \lim c\operatorname{cosech}\beta = b.$$

In this case $f = a-b$, and from (4·23) and (4·24)

$$\lim x = 2, \quad \lim_{x\to 2} Q_n = n+1,$$

so that (4·21) and (4·22) give

$$\lim_{x\to 2}\cosh[n(\beta-\alpha)+\beta] = 1, \quad \lim_{x\to 2}\frac{\sinh[n(\beta-\alpha)+\beta]}{\sinh(\beta-\alpha)} = n+\frac{a}{a-b}.$$

We can make the term-by-term passage to the limit for the series S_1 and S_2, since they can be shown to be uniformly convergent with respect to x. We have then

$$\lim_{x\to 2} S_1 = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{\left(n+\dfrac{a}{a-b}\right)^2} = \tfrac{1}{2}\psi'\left(\frac{a}{a-b}\right), \tag{5·1}$$

where $\psi'(z)$ is the trigamma function, and

$$\lim_{x\to 2} S_2 = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{\left(n+\dfrac{a}{a-b}\right)^3} = -\tfrac{1}{4}\psi''\left(\frac{a}{a-b}\right), \tag{5·2}$$

where $\psi''(z)$ is the tetragamma function.

Hence putting $b = \lambda a$ $(\lambda < 1)$, we find from (4·12), (4·15) and (4·17), using (5·1) and (5·2), that

$$M_3 = \tfrac{1}{2}\pi a^4\left\{1-\lambda^4-4\lambda^2\psi'\left(\frac{1}{1-\lambda}\right)\right\}, \tag{5·3}$$

$$M_2 - L_1 = \pi a^5(1-\lambda^5), \tag{5·4}$$

$$M_2 + L_1 = \pi a^5\left\{\frac{\lambda^3}{1-\lambda}\psi''\left(\frac{1}{1-\lambda}\right) - 2\lambda^2(1+\lambda)\,\psi'\left(\frac{1}{1-\lambda}\right)\right\}, \tag{5·5}$$

which are readily calculated at once from the tables† for these functions.

6. *The associated twist*

The author has previously shown‡ that the associated twist per unit length (τ) for a uni-axial cross-section is given by

$$\tau = (W/EI)\{A+B\eta\}, \tag{6·1}$$

where E, η are Young's modulus and Poisson's ratio respectively, I is the second moment of the cross-section about the axis of symmetry, W being the load parallel to the x-axis, the load-point being the centroid $(0, k)$ of the cross-section, and A and B are given in terms of the canonical moment integrals by the formulae

$$A = -(L_1 + 2kI)/M_3, \tag{6·2}$$

$$B = -(L_1 - M_2 + kM_3 + 2kI)/M_3. \tag{6·3}$$

We express all lengths in terms of the radius a of the outer boundary, putting $b = \lambda a$, $f = \mu a$, and construct tables in terms of λ and μ in steps

† Davis, *Tables of the Higher Mathematical Functions*, 2 (1935), Principia Press (U.S.A.). British Association for the Advancement of Science, *Mathematical Tables*, 1 (1931).

‡ Stevenson, *loc. cit.*, 173.

of 0·1. Table I gives the torsion moment, Tables II (a) and (b) give A and B for the associated twist.

7. *The centres of flexure and least strain*

The load-point when the mean twist taken over the cross-section is zero, i.e. the centre of flexure, has been shown† to have coordinates $(0, g_0)$ given by

$$g_0 = -[L_1 - \sigma(M_2 - kM_3)]/2I, \tag{7·1}$$

where σ is the modified Poisson's ratio given by $(1+\eta)(1-\sigma) = 1$.

For these cross-sections we shall, however, compute the distance δ of the centre of flexure from the centre of the outer circular boundary, measured positively away from the centre of the inner boundary, so that

$$\delta = g_0 - d, \quad \text{or} \quad \delta = p + \sigma q, \tag{7·2}$$

where

$$p = -(L_1/2I) - (f^2 + a^2 - b^2)/2f, \tag{7·3}$$

$$q = (M_2 - kM_3)/2I. \tag{7·4}$$

The point $(0, p)$ is the load-point which results in the strain-energy of the elastic cylinder being a minimum (the centre of least strain). Table III (a) gives the position of this centre of least strain. Table III (b) shows the possible difference in position between this point and the centre of flexure. Table IV gives the position of the centroid of the cross-section for comparison, and $\bar{y}$ is measured in a fashion similar to δ, i.e. $\bar{y} = k - d$.

Changes are rapid in the last interval in row or column for A, B, p and q, and computation becomes laborious. In Table III (b) the last entries in a row or column change sign, except in the column $\lambda = 0·1$, where we have evidence of the change of sign occurring before $\mu = 0·8$. To show the decrease in the tabulated quantity $-q/a$ for $\mu = 0·1$ between $\lambda = 0·8$ and $0·9$, we calculate for the case $\mu = 0·1$, $\lambda = 0·875$, finding

$$M_3 = 0·39448, \quad p/a = 0·62072, \quad -q/a = 0·05530,$$

showing that $(-q/a)$ has passed its maximum for variation with respect to λ.

The limiting case as λ and μ tend to unity and zero respectively is of interest, and leads to the value $\delta = 1·5a$, which may be contrasted with the known result for the limiting case for the split ring section of uniform thickness (tending to zero) and of radius a, which leads to $\delta = 2·0a$. A difference is to be expected, of course; for example, the infinitely thin straight line segment can be approached alternatively from a rectangle or from a circular sector, in the first case the centre of flexure is at the midpoint, in the second it is at a distance $0·8a + \sigma 0·26a$ from the apex end.

† Stevenson, *loc. cit.*, 176.

The torsion moment: $\mu\tau M_3$

TABLE I. M_3/a^4

$\mu \backslash \lambda$	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0·1	1·57000	1·56566	1·55178	1·51857	1·45152	1·33135	1·13166	0·80794	0·00497
0·2	1·56810	1·55777	1·53272	1·48190	1·38619	1·21728	0·91949	0·03741	—
0·3	1·56492	1·54452	1·49974	1·41824	1·26839	0·99003	0·11786	—	—
0·4	1·56044	1·52565	1·45309	1·32140	1·07101	0·25806	—	—	—
0·5	1·55465	1·50063	1·38732	1·17162	0·45956	—	—	—	—
0·6	1·54745	1·46793	1·29065	0·71239	—	—	—	—	—
0·7	1·53861	1·42201	0·99382	—	—	—	—	—	—
0·8	1·52703	1·26736	—	—	—	—	—	—	—
0·9	1·48180	—	—	—	—	—	—	—	—

The associated twist: $\tau = (W/EI)(A + B\eta)$

TABLE II (*a*). A/a

$\mu \backslash \lambda$	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0·1	0·00199	0·00799	0·01848	0·03309	0·05429	0·08610	0·13923	0·26539	102·2613
0·2	0·00393	0·01588	0·03716	0·06783	0·11511	0·19437	0·37569	21·2836	—
0·3	0·00577	0·02357	0·05724	0·10682	0·19468	0·39658	7·64248	—	—
0·4	0·00734	0·03101	0·07695	0·15594	0·33327	3·33836	—	—	—
0·5	0·00894	0·03835	0·10110	0·22729	1·57137	—	—	—	—
0·6	0·01020	0·04597	0·13513	0·74160	—	—	—	—	—
0·7	0·01129	0·05641	0·32672	—	—	—	—	—	—
0·8	0·01254	0·11926	—	—	—	—	—	—	—
0·9	0·02557	—	—	—	—	—	—	—	—

TABLE II (*b*). B/a

$\mu \backslash \lambda$	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0·1	0·00002	0·00010	0·00048	0·00068	0·00161	0·00413	0·01166	0·05097	102·4973
0·2	0·00017	0·00079	0·00272	0·00569	0·01400	0·03744	0·13243	21·5023	—
0·3	0·00057	0·00271	0·00942	0·02098	0·05576	0·18512	7·84056	—	—
0·4	0·00137	0·00664	0·02166	0·05778	0·18321	3·51214	—	—	—
0·5	0·00270	0·01367	0·04696	0·13318	1·71730	—	—	—	—
0·6	0·00477	0·02554	0·10014	0·85643	—	—	—	—	—
0·7	0·00786	0·04767	0·40792	—	—	—	—	—	—
0·8	0·01286	0·16603	—	—	—	—	—	—	—
0·9	0·04142	—	—	—	—	—	—	—	—

The centre of flexure: $\delta = p + q\sigma$

TABLE III(*a*). p/a

$\mu \backslash \lambda$	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0·1	0·00300	0·01215	0·02829	0·05188	0·08684	0·14009	0·22808	0·40898	1·36628
0·2	0·00594	0·02411	0·05633	0·10377	0·17502	0·28555	0·48156	1·21401	—
0·3	0·00878	0·03571	0·08477	0·15612	0·26768	0·45592	1·04282	—	—
0·4	0·01133	0·04684	0·11133	0·21082	0·37572	0·85510	—	—	—
0·5	0·01390	0·05749	0·13947	0·26922	0·65704	—	—	—	—
0·6	0·01611	0·06803	0·17128	0·45945	—	—	—	—	—
0·7	0·01813	0·08031	0·27763	—	—	—	—	—	—
0·8	0·02027	0·12971	—	—	—	—	—	—	—
0·9	0·03321	—	—	—	—	—	—	—	—

TABLE III(*b*). $-q/a$

$\mu \backslash \lambda$	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0·1	0·00197	0·00784	0·01792	0·03216	0·05193	0·07981	0·12095	0·18680	−0·00389
0·2	0·00376	0·01499	0·03388	0’06016	0·09519	0·13972	0·18739	−0·00825	—
0·3	0·00518	0·02054	0·04603	0·07954	0·11965	0·15312	−0·01956	—	—
0·4	0·00593	0·02371	0·05157	0·08474	0·19697	−0·03280	—	—	—
0·5	0·00617	0·02362	0·04821	0·07204	−0·04554	—	—	—	—
0·6	0·00536	0·01912	0·02898	−0·05345	—	—	—	—	—
0·7	0·00336	0·00792	−0·05179	—	—	—	—	—	—
0·8	−0·00032	−0·03780	—	—	—	—	—	—	—
0·9	−0·01495	—	—	—	—	—	—	—	—

The centroid: $\bar{y} = k - d$

TABLE IV. $\bar{y}/a$

$\mu \backslash \lambda$	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
0·1	0·00101	0·00417	0·00989	0·01905	0·03333	0·05625	0·09608	0·17778	0·42632
0·2	0·00202	0·00833	0·01978	0·03810	0·06667	0·11250	0·19216	0·35556	—
0·3	0·00303	0·01250	0·02967	0·05714	0·10000	0·16875	0·28824	—	—
0·4	0·00404	0·01667	0·03956	0·07619	0·13333	0·22500	—	—	—
0·5	0·00505	0·02083	0·04945	0·09524	0·16667	—	—	—	—
0·6	0·00606	0·02500	0·05934	0·11429	—	—	—	—	—
0·7	0·00707	0·02917	0·06923	—	—	—	—	—	—
0·8	0·00808	0·03333	—	—	—	—	—	—	—
0·9	0·00909	—	—	—	—	—	—	—	—

University College
London, W.C. 1

ON DIOPHANTINE APPROXIMATION TO CERTAIN EXPONENTIAL AND BESSEL FUNCTIONS

By H. TODD

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1. The problem of simultaneous rational approximations to several irrational numbers is one of considerable interest and importance. The fundamental existence theorem is due to Dirichlet, who proved (1) that, if $\theta_1, \theta_2, \ldots, \theta_s$ are any irrational numbers, there exists an infinity of integer sets $(p_1, p_2, \ldots, p_s, q)$ satisfying

$$| p_i - q\theta_i | < cq^{-1/s} \quad (i = 1, \ldots, s),$$

with $c = 1$. It may be shown that the index $1/s$ is the best possible for arbitrary θ_i, but the best possible value of c is not known for $s \geqslant 2$.

No simple algorithm is known for finding integers q which satisfy the above inequality. However, the theorem quoted implies the weaker result that there exists a sequence of integers (t_n) such that the difference between $t_n \theta_i$ and the nearest integer to it, for $i = 1, \ldots, s$, tends to zero as $n \to \infty$. For some particular irrationals, we may find such sequences (t_n) by special methods. Jacobi showed (2), by a generalization of the continued fraction algorithm, how this problem could be solved for the numbers $\theta_1 = \sqrt[3]{p}$, $\theta_2 = \sqrt[3]{p^2}$ $(p = 2, 3, 5)$. For example, he proved that for the numbers $\theta_1 = \sqrt[3]{3}$, $\theta_2 = \sqrt[3]{9}$ we may take the sequence (t_n) defined by the difference equation

$$t_n = 12t_{n-1} + 6t_{n-2} + t_{n-3},$$

with $t_{-2} = -4$, $t_{-1} = 1$, $t_0 = 0$ as initial values for t_n.

In this paper it is shown that solutions of a certain type of linear difference equation of order $s+1$ provide simultaneous rational approximations to s irrationals, which are determined by the coefficients of the difference equation and the initial values chosen for the formation of the solutions. In particular, the irrationals thus defined may be simple combinations of exponential or of Bessel functions, and some examples of this type are given to illustrate the general theory.

The author would like to express his thanks to the referees who have greatly helped in the presentation of the results.

2. The difference equations considered in this paper are of the form

$$t_n = b_{n,0}t_{n-1} + b_{n,1}t_{n-2} + \dots + b_{n,s}t_{n-s-1}, \tag{2·1}$$

in which the coefficients $b_{n\,r}$ are polynomials in n defined by

$$b_{n,r} = \Delta^r a_{n-r}/r!\,\alpha^r \quad (r = 0, \dots, s), \tag{2·2}$$

where $\Delta a_n = a_{n+1} - a_n$, and

$$a_n = \alpha^s n^s + \beta_1 n^{s-1} + \dots + \beta_s, \tag{2·3}$$

where α, β_1, ..., β_s are integers, $\alpha > 0$ and $a_n \neq 0$ for any integer $n \geqslant 0$. Clearly $b_{n,0} = a_n$ and $b_{n,s} = 1$.

It will be shown that, if p_n, t_n are any solutions of (2·1), then $|p_n - t_n\theta| \to 0$ as $n \to \infty$, where θ is a number depending on the choice of a_n and on the initial conditions defining the solutions considered. Since the difference equation (2·1) will have $s+1$ linearly independent solutions, corresponding to a given solution t_n there will be s sequences (p_n) and so s numbers θ. In order to prove this result and to determine the numbers θ, we need to introduce some auxiliary functions defined as follows:

$$f_{n-s} = 1 + \sum_{r=1}^{\infty} 1/(a_{n+1}a_{n+2}\dots a_{n+r}\,r!\,\alpha^r), \tag{2·4}$$

$$\psi(n,r) = a_n \dots a_{n+r}\left(f_{n-s-1} - \sum_{m=0}^{r} \frac{b_{n+m,m}}{a_n a_{n+1}\dots a_{n+m}} f_{n-s+m}\right), \tag{2·5}$$

for $r = 0, \dots, s$, with $\psi(n, -1) = f_{n-s-1}$,

$$T_n = \sum_{r=-1}^{s-1} t_{n-r-1}\psi(n-r, r), \tag{2·6}$$

with the same definition of P_n in terms of (p_n), where (p_n) is any other sequence of integers satisfying (2·1).

We may now state the principal result of this paper in the form:

THEOREM. *The difference equation* (2·1) *has* $s+1$ *linearly independent solutions* $t_n, p_n, q_n, \dots$, *in integers such that*

$$|p_n - t_nP_0/T_0| < A(\epsilon)\, t_n^{-\frac{1}{s}+\epsilon}, \tag{2·7}$$

where ϵ *is an arbitrary positive number, and* $A(\epsilon)$ *is independent of* n *and depends only on* ϵ; *with similar relations for the other solutions* $q_n, \dots$.

3. The proof of the theorem may conveniently be made to depend on a number of lemmas. We first collect together some simple results which are required later.

LEMMA 1.

$$\psi(n-r,r) = a_n\psi(n-r,r-1)-b_{n,r}f_{n-s} \quad (r=0,\dots,s), \tag{3·1}$$

$$\psi(n,s)=0, \tag{3·2}$$

$$\psi(n-s,s-1)=f_{n-s}/a_n, \tag{3·3}$$

$$\psi(n-r,r)=\sum_{m=r+1}^{s}\frac{b_{n-r+m,m}}{a_{n+1}\dots a_{n+m-r}}f_{n-r-s+m}. \tag{3·4}$$

The relation (3·1) follows immediately from the definition. To prove (3·2) we note that

$$f_{n-s-1}-f_{n-s}=\sum_{r=1}^{\infty}(a_{n+r}-a_n)/(a_n\dots a_{n+r}r!\,\alpha^r),$$

whilst
$$a_{n+r}=a_n+r\Delta a_n+\binom{r}{2}\Delta^2 a_n+\dots+\Delta^r a_n,$$

so that
$$f_{n-s-1}=\sum_{m=0}^{s}\frac{b_{n+m,m}}{a_n\dots a_{n+m}}f_{n-s+m},$$

which is simply (3·2) written in full. The relation (3·3) follows at once from (3·1) and (3·2), and finally (3·4) is derived from (2·5) and (3·2).

LEMMA 2.
$$T_n=a_nT_{n-1}. \tag{3·5}$$

For

$$\begin{aligned}T_n-a_nT_{n-1}&=t_n\psi(n+1,\,-1)+\sum_{r=0}^{s-1}t_{n-r-1}\{\psi(n-r,r)-a_n\psi(n-r,r-1)\}\\&\qquad -t_{n-s-1}a_n\psi(n-s,s-1)\\&=t_nf_{n-s}-\sum_{r=0}^{s-1}t_{n-r-1}b_{n,r}f_{n-s}-t_{n-s-1}f_{n-s}\\&=0,\end{aligned}$$

since t_n satisfies (2·1).

LEMMA 3. *There exists an integer n_0 such that, for appropriate initial values $t_0, t_{-1}, \dots, t_{-s}$, we have*
$$T_n>t_n>0$$
for all $n\geqslant n_0$.

We may show by considering the difference equation that t_n is ultimately of fixed sign, and this will be positive if the initial values are suitably selected. Again, we may clearly take n large enough to make $\psi(n-r,r)>0$ for $r=0,\dots,s-1$, and $\psi(n+1,-1)=f_{n-s}>1$. Then for $n\geqslant n_0$, say, all the terms of the expression (2·6) for T_n are positive, so that $T_n>t_n>0$.

LEMMA 4.
$$\frac{p_n}{t_n}-\frac{P_0}{T_0}=\frac{1}{t_nT_n}\sum_{r=0}^{s-1}u_{r+1}\psi(n-r,r), \tag{3·6}$$

where
$$u_r=p_nt_{n-r}-p_{n-r}t_n.$$

From Lemma 2 we have

$$P_n/T_n = P_{n-1}/T_{n-1} = \dots = P_0/T_0,$$

and so

$$\begin{aligned} p_n/t_n - P_0/T_0 &= p_n/t_n - P_n/T_n \\ &= (p_n T_n - t_n P_n)/t_n T_n \\ &= (1/t_n T_n) \sum_{r=0}^{s-1} u_{r+1} \psi(n-r, r). \end{aligned}$$

We must now investigate the behaviour of the right-hand side of (3·6) for large n, and in particular that of u_r. The estimate for u_r is deduced from one for $u_1 = p_n t_{n-1} - p_{n-1} t_n$, and to obtain the latter we now require a general theorem from the theory of difference equations; this we shall state as a lemma, referring the reader to the literature for the proof (**3**).

LEMMA 5. *Let*

$$y_n + \alpha_1(n) y_{n-1} + \dots + \alpha_s(n) y_{n-s} = 0 \tag{3·7}$$

be an equation of Poincaré's type, that is, let $\lim_{n\to\infty} \alpha_i(n) = \alpha_i$ *exist for* $i = 1, \dots, s$. *Let* $q_1, \dots, q_\sigma$ *be the distinct moduli of the roots of the associated characteristic equation*

$$t^s + \alpha_1 t^{s-1} + \dots + \alpha_s = 0,$$

and let l_λ *be the number of roots whose modulus is* q_λ, *multiple roots being counted multiply. Then provided* $\alpha_s(n) \neq 0$ *for integral* n, *the difference equation* (3·7) *has a fundamental system of solutions which fall into* σ *classes, such that, for the solutions of the* λ*th class and their linear combinations,*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|y_n|} = q_\lambda.$$

The number of solutions of the λ*th class is* l_λ.

LEMMA 6.

$$|u_1| = \{b_{n,1}\}(1+\epsilon_n)^n/s^n, \tag{3·8}$$

where

$$\{b_{n,1}\} = b_{n,1} b_{n-1,1} \cdots b_{k,1},$$

k being any integer independent of n, and

$$\overline{\lim_{n\to\infty}} \epsilon_n = 0.$$

Since p_n, t_n satisfy (2·1), it follows (4) that $v_n = p_n/t_n - p_{n-1}/t_{n-1}$ satisfies the equation

$$\begin{aligned} t_n v_n + (b_{n,1} t_{n-2} + \dots + t_{n-s-1}) v_{n-1} \\ + (b_{n,2} t_{n-3} + \dots + t_{n-s-1}) v_{n-2} + \dots + t_{n-s-1} v_{n-s} = 0. \end{aligned}$$

If we substitute

$$v_n = \{b_{n,1}\} y_n / t_n t_{n-1},$$

we obtain

$$y_n + \frac{b_{n,1} t_{n-2} + \dots}{b_{n,1} t_{n-2}} y_{n-1} + \frac{(b_{n,2} t_{n-3} + \dots) t_{n-1}}{b_{n,1} b_{n-1,1} t_{n-3} t_{n-2}} y_{n-2} + \dots + \frac{\{b_{n-s,1}\} t_{n-1}}{\{b_{n,1}\} t_{n-s}} y_{n-s} = 0. \tag{3·9}$$

Now $$b_{n,1} \sim b_{n-1,1} \sim \dots \sim \binom{s}{1} (\alpha n)^{s-1},$$

$$b_{n,2} \sim b_{n-1,2} \sim \dots \sim \binom{s}{2} (\alpha n)^{s-2},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

and $$t_{n-1} \sim (\alpha n)^s t_{n-2} \sim (\alpha n)^{2s} t_{n-3} \sim \dots;$$

it follows that (3·9) is of Poincaré's type, with associated characteristic equation

$$\gamma^s + \gamma^{s-1} + \binom{s}{2} \gamma^{s-2} s^{-2} + \dots + s^{-s} = 0,$$

or $$\left(\gamma + \frac{1}{s}\right)^s = 0.$$

Hence, by Lemma 5, $$\overline{\lim_{n\to\infty}} \sqrt[n]{| y_n |} = 1/s,$$

so that we may write $$| y_n | = (1+\epsilon_n)^n / s^n,$$

where $$\overline{\lim_{n\to\infty}} \epsilon_n = 0.$$

It follows that $$\begin{aligned} | u_1 | &= | p_n t_{n-1} - p_{n-1} t_n | \\ &= \{b_{n,1}\} | y_n | \\ &= \{b_{n,1}\} (1+\epsilon_n)^n / s^n, \end{aligned}$$

and the lemma is proved.

LEMMA 7. $\quad | u_r | < A\{b_{n,1}\} (1+\eta_n)^n n^{r-1} s^{-n} \quad (r = 1, \dots, s),$

where A is independent of n and $\eta_n \to 0$ as $n \to \infty$.

Writing $\omega_n = p_n t_{n-1} - p_{n-1} t_n = u_1$, we have

$$\frac{u_r}{t_n t_{n-r}} = \frac{\omega_n}{t_n t_{n-1}} + \frac{\omega_{n-1}}{t_{n-1} t_{n-2}} + \dots + \frac{\omega_{n-r+1}}{t_{n-r+1} t_{n-r}}.$$

If we define $\eta_n = \max(0, \epsilon_n, \dots, \epsilon_{n-s+1})$, and put $\pi_n = \{b_{n,1}\} s^{-n}$, we have $\eta_n \to 0$ as $n \to \infty$, and

$$\frac{| u_r |}{t_n t_{n-r}} \leqslant \left(\frac{\pi_n}{t_n t_{n-1}} + \dots + \frac{\pi_{n-r+1}}{t_{n-r+1} t_{n-r}}\right) (1+\eta_n)^n,$$

since $$| \omega_n | = \pi_n (1+\epsilon_n)^n.$$

Further, $$\pi_{n-1} = s\pi_n / b_{n,1} < A_1 n^{1-s} \pi_n,$$

where $A_1, A_2, \dots$, occurring here and later denote numbers independent of n. Hence $\pi_{n-m} < A_2 n^{m(1-s)} \pi_n$ for any fixed m. Recalling that $t_n \sim (\alpha n)^{ms} t_{n-m}$ it follows that $| u_r | < A n^{r-1} \pi_n (1+\eta_n)^n$, which is the required result.

LEMMA 8. $$\{b_{n,1}\} < A(k)\, s^n n^{\frac{1}{2}(1-s)} t_n^{(s-1)/s}$$
for any fixed integer $k > n_0 + s$, where $A(k)$ is independent of n.

Direct calculation gives

$$a_{n-1}a_{n-2}\dots a_{n-s+1} = \alpha^{s(s-1)} n^{(s-1)^2}(n^{s-1} + n^{s-2}[(s-1)\beta_1/\alpha^s - \tfrac{1}{2}\alpha s^2(s-1)] + \dots),$$
$$b_{n,1}b_{n-1,1}\dots b_{n-s+1,1} = s^s\alpha^{s(s-1)} n^{(s-1)^2}(n^{s-1} + n^{s-2}[(s-1)\beta_1/\alpha^s - \tfrac{1}{2}s^2(s-1)] + \dots),$$

so that $$\frac{b_{n,1}b_{n-1,1}\dots b_{n-s+1,1}}{a_{n-1}\dots a_{n-s+1}} < s^s(1 + A_3/n^2).$$

Writing ϕ_n for the expression on the left-hand side of this inequality, we have

$$\{\phi_n\} < s^{s(n-k+1)} \prod_{m=k}^{n} (1 + A_3/m^2)$$
$$< A_4 s^{ns},$$

since $\prod^{\infty} (1 + A_3/m^2)$ converges. This may be written

$$\frac{b_{n,1}b_{n-1,1}^2 b_{n-2,1}^3 \dots b_{n-s+1,1}^s b_{n-s,1}^s \dots b_{k,1}^s}{a_{n-1}a_{n-2}^2 \dots a_{n-s+1}^{s-1} a_{n-s}^{s-1} \dots a_k^{s-1}} < A_5 s^{ns},$$

or $$\frac{\{b_{n,1}\}^s}{\{a_n\}^{s-1}} < A_5 s^{ns} \frac{b_{n,1}^{s-1} b_{n-1,1}^{s-2} \dots b_{n-s+2,1}}{a_n^{s-1} a_{n-2}^{s-2} \dots a_{n-s+2}}$$
$$< A_6 s^{ns} n^{-\frac{1}{2}s(s-1)},$$

since $$b_{n,1}/a_n \sim s/\alpha n.$$

Now let us fix $k > n_0 + s$, and so, in particular, $b_{n,r} > 0$ $(r = 0, \dots, s)$ and $t_n > 0$ for $n \geqslant k - s - 1$. Hence

$$t_n > a_n t_{n-1} > a_n a_{n-1} t_{n-2} > \dots > \{a_n\} t_{k-1},$$

so that $$\{a_n\} < A_7 t_n.$$

It follows then that $$\{b_{n\,1}\} < A(k)\, s^n n^{\frac{1}{2}(1-s)} t_n^{(s-1)/s}.$$

LEMMA 9. $$\psi(n-r, r) \sim A_8 n^{-r-1}.$$

From equation (3·4) we have

$$\psi(n-r, r) \sim \binom{s}{r+1} (\alpha n)^{s-r-1}/(\alpha n)^s = A_8 n^{-r-1}.$$

LEMMA 10. $$t_n^{1/s} > A_n^B \alpha^n n!,$$
where A, B are independent of n.

In Lemma 8 we proved that $t_n > A(k)\{\alpha_n\}$. Now

$$a_n = \alpha^s n^s + \beta_1 n^{s-1} + \dots = \alpha^s n^s(1 + \beta_1/n\alpha^s + \dots),$$

giving $$\log\{a_n\} = \sum_{r=k}^{n} \log a_r = \sum_{r=k}^{n} s \log \alpha r + \sum_{r=k}^{n} \log(1 + \beta_1/r\alpha^s + \dots).$$

But $$\sum_{r=k}^{n} \log(1+\beta_1/r\alpha^s+\ldots) \sim \frac{\beta_1}{\alpha^s}\log n, \qquad \text{if } \beta_1 \neq 0$$
$$\sim \text{constant}, \qquad \text{if } \beta_1 = 0,$$
and the result follows.

Proof of the theorem. Using Lemmas 3, 4, 7, 8 and 9, we have

$$\begin{aligned} |p_n/t_n - P_0/T_0| &= \left|(1/t_n T_n)\sum_{r=0}^{s-1} u_{r+1}\psi(n-r,r)\right| \\ &< t_n^{-2} sAs^n n^{\frac{1}{2}(1-s)} t_n^{(s-1)/s} s^{-n}(1+\eta_n)^n n^r n^{-r-1} \\ &= An^{-\frac{1}{2}(1+s)}(1+\eta_n)^n t_n^{-1-1/s}. \end{aligned}$$

But $$n^{-\frac{1}{2}(1+s)}(1+\eta_n)^n = O(e^n) = O(t_n^{\epsilon})$$
for any positive ϵ, and the theorem follows.

4. *Special cases.* Changing the notation slightly by writing $s-1$ in place of s, and putting $\alpha = sm$, $a_n = m^{s-1}(sn-1)(sn-2)\ldots(sn-s+1)$, we see that the conditions of the theorem are satisfied. Further, the functions $f_{-1}, f_{-2}, \ldots, f_{-s}$ can all be expressed as linear homogeneous functions of $\exp(\omega^k/m)$ $(k = 0, \ldots, s-1)$, where ω is a primitive sth root of unity, though the explicit formulation is laborious for large s. We give examples of the cases $s = 2, 3, 4$.

(i) $s = 2$. Here we have

$$t_n = m(2n-1)t_{n-1} + t_{n-2},$$
$$T_0 = t_0 f_{-1} + \frac{1}{m} t_{-1} f_0 = t_0 \cosh\frac{1}{m} + t_{-1}\sinh\frac{1}{m}.$$

Thus, if we take the initial values

$$\begin{pmatrix} p_0, & p_{-1} \\ t_0, & t_{-1} \end{pmatrix} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix},$$

we have $P_0 = \cosh 1/m$, $T_0 = \sinh 1/m$, and we obtain the known result (5)

$$\coth\frac{1}{m} = m + \frac{1}{3m+}\,\frac{1}{5m+\ldots}.$$

(ii) $s = 3$. Here

$$\begin{aligned} t_n &= m^2(3n-1)(3n-2)t_{n-1} + 6m(n-1)t_{n-2} + t_{n-3}, \\ T_0 &= t_0 f_{-2} + 2m^2 t_{-1}(f_{-3} - f_{-2}) + \frac{1}{2m^2} t_{-2} f_{-1} \\ &= t_0 Y_0 + t_{-1}(Y_2 - 2mY_1) + t_{-2}Y_1, \end{aligned}$$

where $$3Y_k = \sum_{r=0}^{2} \omega^{kr}\exp\left(\frac{\omega^r}{m}\right), \quad \omega = \exp\left(\frac{2\pi i}{3}\right).$$

Taking as initial values

$$\begin{pmatrix} p_0, & p_{-1}, & p_{-2} \\ q_0, & q_{-1}, & q_{-2} \\ t_0, & t_{-1}, & t_{-2} \end{pmatrix} = \begin{pmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{pmatrix},$$

we have $$P_0 = Y_0, \quad Q_0 = Y_2 - 2mY_1, \quad T_0 = Y_1;$$

this choice of initial values ensures that Jacobi's algorithm applied to $Y_0 : Y_2 - 2mY_1 : Y_1$ will give "partial quotients"

$$a_n = m^2(3n-1)(3n-2), \quad b_n = 6m(n-1).$$

(iii) $s = 4$. In this case

$$t_n = m^3(4n-1)(4n-2)(4n-3)\,t_{n-1} + 3m^2(16n^2-32n+17)\,t_{n-2} + 6m(2n-3)\,t_{n-3} + t_{n-4},$$

and $$T_0 = t_0 Z_0 + t_{-1}(Z_3 - 3mZ_2 + 6m^2 Z_1) + t_{-2}(Z_2 - 9mZ_1) + t_{-3} Z_1,$$

where $$4Z_k = \sum_{r=0}^{3} i^{kr} \exp\left(\frac{i^r}{m}\right).$$

If we take

$$\begin{pmatrix} p_0, & p_{-1}, & p_{-2}, & p_{-3} \\ q_0, & q_{-1}, & q_{-2}, & q_{-3} \\ r_0, & r_{-1}, & r_{-2}, & r_{-3} \\ t_0, & t_{-1}, & t_{-2}, & t_{-3} \end{pmatrix} = \begin{pmatrix} 1, & 1, & 1+3m, & 1+9m+21m^2 \\ 1, & 0, & -1, & -9m \\ 0, & 1, & 3m, & -1+21m^2 \\ 1, & -1, & 1-3m, & -1+9m-21m^2 \end{pmatrix},$$

a system which yields linearly independent p_n, q_n, r_n, t_n, since its determinant $= -8 \neq 0$, we obtain

$$P_0 = \exp\frac{1}{m}, \quad Q_0 = \cos\frac{1}{m}, \quad R_0 = \sin\frac{1}{m}, \quad T_0 = \exp\left(-\frac{1}{m}\right).$$

Hence

$$p_n - t_n \exp\left(\frac{2}{m}\right), \quad q_n - t_n \exp\left(\frac{1}{m}\right)\cos\left(\frac{1}{m}\right), \quad r_n - t_n \exp\left(\frac{1}{m}\right)\sin\left(\frac{1}{m}\right)$$

all tend to zero as $n \to \infty$.

The following numerical results for $m = 2$ give some idea of the closeness of the approximation.

$$t_1 = 29, \quad t_2 = 48{,}907, \quad t_3 = 387{,}366{,}095,$$

$$t_4 = 8{,}460{,}160{,}614{,}721, \quad t_5 = 393{,}500{,}185{,}153{,}048{,}349$$

and $$|\,p_n - t_n e\,|\,t_n^{\frac{1}{3}} = 0{\cdot}522\ldots,\ 0{\cdot}343\ldots,\ 0{\cdot}262\ldots,\ 0{\cdot}215\ldots,\ 0{\cdot}184\ldots,$$

$$|\,q_n - t_n e^{\frac{1}{2}} \cos\tfrac{1}{2}\,|\,t_n^{\frac{1}{3}} = 0{\cdot}124,\ 0{\cdot}077\ldots,\ 0{\cdot}057\ldots,\ 0{\cdot}045\ldots,\ 0{\cdot}032\ldots,$$

$$|\,r_n - t_n e^{\frac{1}{2}} \sin\tfrac{1}{2}\,|\,t_n^{\frac{1}{3}} = 0{\cdot}237\ldots,\ 0{\cdot}155\ldots,\ 0{\cdot}118\ldots,\ 0{\cdot}096\ldots,\ 0{\cdot}087\ldots,$$

for $n = 1, 2, 3, 4, 5$.

(iv) We obtain results similar to the above by putting

$$\alpha = sm, \quad a_n = m^{s-1}(sn-l_1)(sn-l_2)\dots(sn-l_{s-1}),$$

where $l_1, \dots, l_{s-1}$ are integers incongruent to zero and to each other modulo s. As an example, if $\alpha = 3m$, $a_n = m^2(3n-1)(3n+1)$, we find (omitting a common factor) that

$$T_0 = t_0 Y_2 + t_{-1}(Y_1 + m^2 Y_2 - mY_0) + t_{-2}(Y_0 - mY_2),$$

where Y_k is as defined in example (ii) above. The Jacobian algorithm applied to $Y_2 : Y_1 + m^2 Y_2 - mY_0 : Y_0 - mY_2$ gives "partial quotients"

$$a_n = m^2(3n-1)(3n+1), \quad b_n = 3m(2n-1).$$

(v) If we put $s = 2$, $a_n = \alpha n + \beta$, we have

$$t_n = (\alpha n + \beta) t_{n-1} + t_{n-2},$$

$$T_0 = t_0 i^{-k} J_k\left(\frac{2i}{\alpha}\right) + t_{-1} i^{-k-1} J_{k+1}\left(\frac{2i}{\alpha}\right), \quad k = \beta/\alpha.$$

Taking
$$\begin{pmatrix} p_0, & p_{-1} \\ t_0, & t_{-1} \end{pmatrix} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix},$$

we obtain the known result (**6**)

$$iJ_k\left(\frac{2i}{\alpha}\right) \Big/ J_{k+1}\left(\frac{2i}{\alpha}\right) = \alpha + \beta + \frac{1}{2\alpha+\beta+}\,\frac{1}{3\alpha+\beta+}\dots.$$

(vi) Taking $s = 4$, $\alpha = 2m$, $a_n = m^3(2n+1)(2n+\beta)(2n+\beta+1)$, where m and $m\beta$ are positive integers, we obtain

$$\begin{aligned} t_n = {} & m^3(2n+1)(2n+\beta)(2n+\beta+1)\,t_{n-1} \\ & + m^2(12n^2 + 4n(2\beta-1) + \beta^2 - \beta + 1)\,t_{n-2} + 2m(3n+\beta-2)\,t_{n-3} + t_{n-4}. \end{aligned}$$

Selecting

$$\begin{pmatrix} p_0, & p_{-1}, & p_{-2}, & p_{-3} \\ q_0, & q_{-1}, & q_{-2}, & q_{-3} \\ r_0, & r_{-1}, & r_{-2}, & r_{-3} \\ t_0, & t_{-1}, & t_{-2}, & t_{-3} \end{pmatrix} = \begin{pmatrix} (1+\beta)m^2 - 1, & 0, & 1, & (2-\beta)m \\ (1+\beta)m^2 + 1, & 0, & 1, & (2-\beta)m \\ m, & -1, & m\beta, & 1+\beta(2-\beta)m^2 \\ m, & 1, & -m\beta, & 1-\beta(2-\beta)m^2 \end{pmatrix},$$

we find that

$$p_n - t_n i^{\beta+1} J_\beta\left(\frac{2}{m}\right) \Big/ J_{\beta+1}\left(\frac{2i}{m}\right), \quad q_n - t_n i J_\beta\left(\frac{2i}{m}\right) \Big/ J_{\beta+1}\left(\frac{2i}{m}\right),$$

$$r_n - t_n i^{\beta+1} J_{\beta+1}\left(\frac{2}{m}\right) \Big/ J_{\beta+1}\left(\frac{2i}{m}\right)$$

all tend to zero as $n \to \infty$.

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(2) C. G. J. Jacobi, "Allgemeine Theorie der kettenbruchähnlichen Algorithmen, in welchen jede Zahl aus *drei* vorhergehenden gebildet wird", *J. reine andew. Math.* 69 (1868), 29–64; *Werke*, Bd VII, 385–426.

(3) O. Perron, "Über Summengleichungen und Poincarésche Differenzengleichungen", *Math. Annalen*, 84 (1921), 1.
L. M. Milne-Thomson, *Calculus of finite differences*, 548.

(4) G. Boole, *Calculus of finite differences*, 119.
L. M. Milne-Thomson, *loc. cit.*, 367.

(5) O. Perron, *Die Lehre von den Kettenbrüchen*, 354.

(6) O. Perron, *loc. cit.*, 299–300.

The University
Bristol

ON THE TOTAL VARIATION OF A FUNCTION OF TWO VARIABLES*

CORRIGENDUM

By H. P. MULHOLLAND

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The corrections detailed below are necessitated by the discovery that a certain formula, given by Saks† for the area of a continuous surface $z = f(x, y)$, is not correct for all such surfaces without restriction.

(i) In Theorem 1, on p. 294, at the end of the last sentence but one, there should be inserted the restrictive clause "*provided that, in the case of the formula* (2·92), *the corresponding formula for the area of the surface* $\Sigma(R)$ *is valid*". An alternative, and equivalent, restriction ensuring the validity of (2·92) would be "*provided that* $f(x, y)$ *is absolutely continuous in L. C. Young's sense*‡".

In the proof of Theorem 1, at the end of § 12, on p. 310, the footnote reference to Saks should be supplemented by one to L. C. Young's paper‡.

* *Proc. London Math. Soc.* (2), 46 (1940), 290–311.

† *Theory of the integral* (2nd ed.) (Warsaw, 1937), v, (8·4). The error was pointed out by Jarník, and the requisite emendations have been made by L. C. Young, "An expression connected with the area of a surface $z = F(x, y)$", *Duke Math. Journal*, 11 (1944), 43–57, who had suggested the formula in the first place.

‡ Cf. L. C. Young, *loc. cit.*, especially (3·7).

It may be noted, further, that the formula (2·91) in Theorem 1 and the formula (12·31), p. 309, for $V_R[f]$ are both valid without the new restriction.

(ii) In the proof of Theorem 5, the part of a sentence appearing on p. 311, after the comma, should be replaced by the following:

...and the arguments employed by Radó and Tonelli* to deal with these expressions can be adapted without trouble to show that the total variation of the rectangle-function $|\eta_x(I)|$ over R is equal to the Tonelli integral $W[0; f; R]$, and similarly, for the function f^θ, cf. § 1, that

$$W[\theta; f; R] = W[0; f^\theta; R^\theta]$$

equals the total variation over R^θ of $|\eta_x(f^\theta; I)|$. But since $\eta_x(f^\theta; I)$ is a continuous and additive function of rectangles, and indeed (after an obvious extension of its definition) of general polygonal regions also, this total variation will be independent of the orientation of the rectangles used in the underlying dissections of R^θ: it must thus be equal to the total variation over R^θ of $|\eta_x(f^\theta; I^\theta)|$. Now $\eta_x(f^\theta; I^\theta)$ equals the component of the vector $\eta(I)$ in the direction inclined at angle $-\theta$ to the x-axis: let us denote this by $\eta_\xi(f; I)$. As I^θ varies in R^θ the corresponding rectangle I will vary in R, and so, altogether, $W[\theta; f; R]$ will equal the total variation over R of $|\eta_\xi(f; I)|$. On integrating with respect to θ over $[0, \pi]$, and observing that the integral of the total variation is equal to the total variation of the integral, we see that, in virtue of (1·5) and (1·6), $2V_R[f]$ must be equal to the total variation over R of

$$\int_0^\pi |\eta_\xi(f; I)| \, d\theta = 2 |\eta(I)|,$$

as required for the theorem, the last equality resulting from the definition of $\eta_\xi(f; I)$.

CORRIGENDUM

ON SIMPLY HARMONIC "KAPPA SPACES" OF FOUR DIMENSIONS

[*Proc. London Math. Soc.* (2), 50 (1948), 317–329.]

H. S. RUSE

Page 327, *equations* (6·1), (6·2). The Christoffel symbols $\left\{ {4 \atop 31} \right\}$, $\left\{ {4 \atop 32} \right\}$, $\left\{ {3 \atop 41} \right\}$, $\left\{ {3 \atop 42} \right\}$ are also not identically zero, but do vanish if (6·3), (6·4) are satisfied.

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* Cf. S. Saks, *loc. cit.*, v, §§ 5–8.